Horseshoe and entropy in a fractional-order unified system*

Li Qing-Du(李清都)[a)b)†], Chen Shu(陈 述)[a)], and Zhou Ping(周 平)[b)]

[a)] *Key Laboratory of Networked Control and Intelligent Instrument of Ministry of Education, Chongqing University of Posts and Telecommunications, Chongqing* 400065, *China*

[b)] *Institute for Nonlinear Systems, Chongqing University of Posts and Telecommunications, Chongqing* 400065, *China*

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This paper studies chaotic dynamics in a fractional-order unified system by means of topological horseshoe theory and numerical computation. First it finds four quadrilaterals in a carefully-chosen Poincaré section, then shows that the corresponding map is semiconjugate to a shift map with four symbols. By estimating the topological entropy of the map and the original time-continuous system, it provides a computer assisted verification on existence of chaos in this system, which is much more convincible than the common method of Lyapunov exponents. This new method can potentially be used in rigorous studies of chaos in such a kind of system. This paper may be a start for proving a given fractional-order differential equation to be chaotic.

Keywords: chaos, topological horseshoe, fractional-order system, generalised Lorenz system

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1. Introduction

As physical interpretation of the fractional derivative becomes clear, there has been growing interest in investigating chaotic behaviours of fractional order systems.[1] Many meaningful chaotic systems have been generalised with the fractional derivative. The unified systems are a large family of chaotic systems unified from the Lorenz's system, the Chen system and the Lü system.[2] Their fractional-order form called the fractional-order unified system has been studied extensively recent years in many aspects such as chaotic phenomena,[3,4] chaotic synchronisation,[5−8] circuits implementation[9−11] and other related studies.[12−14]

A prerequisite of all above researches or applications is that the so-called 'chaotic' dynamics is indeed chaos. However, most of them were considered to be chaotic only by numerical evidences, which usually came from phase portraits, power spectrum, bifurcation diagrams or positive Lyapunov exponents. Among them, a positive Lyapunov exponent seems the most convincible evidence of chaos, and has been used extensively. However, the numerical error may make it unreliable, especially when the computed output is close to zero. In that case we cannot be sure whether it is positive or not. The numerical accuracy often depends sensitively on initial conditions, the stepsize of integration, the computer precision, as well as the integration time for averaging and the algorithm we use. When dealing with a fractional order system, it is very difficult to achieve high accuracy like integer-order systems. First because the system has no explicit Jacobian matrix, we often have to compute the Lyapunov exponents indirectly by a time series. Second because integer-order operators are often used to approach the fractional-order operators, the integration error is unavoidable. The problem is that the error will accumulate exponentially during the integration; therefore long term computation is unreliable. In Ref. [15], Tavazoei and Haeri showed that low precision methods may numerically demonstrate chaos in non-chaotic systems, and pointed out this mistake occurred in several literatures with frequency-domain approximation.

The topological horseshoe is a powerful tool for the investigation of chaotic behaviour of dynamical systems. It not only provides much information of Lyapunov exponents, but also shows detailed dynamics of chaos. It has been successfully used to prove the existence of chaos in many integer-order systems.[16−19] The topological horseshoe is based on

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†Corresponding author. E-mail: liqd@cqupt.edu.cn

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the geometry of continuous maps on some subsets of interest in state space. For each point in the subsets, only short-term computation is used to find its image under its Poincaré map. Since we only need to solve the piece of trajectory between the point and its image, numerical error is much smaller than long term computation for Lyapunov exponents. So the topological horseshoe is very applicable for investigating chaotic behaviour of fractional-order systems.

In this paper we introduce this method to study chaos in fractional-order systems, which can give more convincible arguments on the existence of chaos than the common method by computing Lyapunov exponents, and the method is illustrated with a detailed investigation on the chaotic dynamics in a fractional-order unified system by means of topological horseshoe.

2. Preliminaries of topological horseshoe

First we recall some aspects of symbolic dynamics.[20]

Let $S_m = \{0, 1, \ldots, m-1\}$ be the set of nonnegative integers from 0 to $m-1$. Let Σ_m be the collection of all one-sided infinite sequences with their elements of Σ_m, i.e., every element s of Σ_m is of the following form:

$$s = \{s_0, s_1, \ldots, s_n, \ldots\}, \quad s_i \in S_m.$$

Now consider another sequence $\bar{s} \in \Sigma_m$,

$$\bar{s} = \{\bar{s}_0, \bar{s}_1, \ldots, \bar{s}_n, \ldots\}, \quad \bar{s}_i \in S_m.$$

The distance between s and $\bar{s}$ is defined as

$$d(s, \bar{s}) = \sum_{i=0}^{\infty} \frac{1}{2^i} \frac{|s_i - \bar{s}_i|}{1 + |s_i - \bar{s}_i|}.$$

With the distance defined above, Σ_m is a metric space with three properties: compact, totally disconnected and perfect. So Σ_m is a Cantor set, and is often used to characterise complex structure of invariant set in a chaotic dynamical system.

Furthermore, now define an m-shift map $\sigma : \Sigma_m \to \Sigma_m$ as follows:

$$\sigma(s) = \sigma(\{s_0, s_1, \ldots, s_n, \ldots\}) = \{s_1, s_2, \ldots, s_n, \ldots\}.$$

So it is not hard to see that σ is a continuous map and $\sigma(\Sigma_m) = \Sigma_m$. The shift map σ as a dynamical system defined on Σ_m has the following properties:

(i) σ has a countable infinity of periodic orbits consisting of orbits of all periods;

(ii) σ has an uncountable infinity of nonperiodic orbits;

(iii) σ has a dense orbit.

A consequence of the above properties is that the dynamics generated by the shift map σ is sensitive to initial conditions therefore is chaotic.

With the previous preliminaries, we can give a result as a horseshoe lemma that will be useful for the main theorem of this paper as well as its own interests. First let us recall some notions. Let X be a metric space, D is a compact subset of X, D_1,..., D_{m-1} and D_m are mutually disjoint compact subsets of D. Suppose $f : D_i \to X$ is a continuous map, $i = 1, 2, \ldots, m$.

Definition[21] Let D_1,..., D_{m-1} and D_m be mutually disjoint compact subsets of D, let γ be a compact connected subset of D, such that for each $1 \le i \le m$, $\gamma_i = \gamma \cap D_i$ is nonempty and compact, then γ is called a connection with respect to D_1,..., D_{m-1} and D_m. Let F be a family of connections γ with respect to D_1,..., D_{m-1} and D_m satisfying the following property:

$$\gamma \in F \Rightarrow f(\gamma_i) \in F.$$

Then F is said to be an f-connected family with respect to D_1,..., D_{m-1} and D_m.

Horseshoe lemma[21] Suppose that there exists an f-connected family F with respect to disjointed compact subsets D_1, ..., D_{m-1} and D_m. Then there exists a compact invariant set $K \subset D$, such that $f|K$ is semi-conjugate to m-shift.

3. Horseshoe map in the fractional-order unified system

The fractional-order unified system model can be described by the following fractional-order differential equations:

$$\begin{cases} \dfrac{\mathrm{d}^{q_1} x}{\mathrm{d}t} = (28\alpha + 10)(y - x), \\ \dfrac{\mathrm{d}^{q_2} y}{\mathrm{d}t} = (28 - 35\alpha)x - xz + (29\alpha - 1)y, \\ \dfrac{\mathrm{d}^{q_3} z}{\mathrm{d}t} = xy - \dfrac{\alpha + 8}{3} z, \end{cases} \tag{1}$$

where q_1, q_2, q_3 are fractional orders, $0 < q_1$, q_2, $q_3 < 1$. The parameter α takes from 0 to 1. When $\alpha = 0$, $\alpha = 1$ and $\alpha = 0.8$, Eq. (1) is the fractional-order Lorenz, Chen, and Lü system, respectively.[2]

Now we take the order $q_1 = q_2 = q_3 = 0.9$ and parameter $\alpha = 0.5$ in this paper. Since $\alpha \in [0, 0.8)$, the system belongs to the fractional-order generalised Lorenz system. After days of computation, the maximum Lyapunov exponent is a positive value about 0.00853. However, it has been shown in Ref. [22] that the maximum Lyapunov exponent is a negative value about -0.0085. Because both of the two values are very small, it is hard to tell whether the system is chaotic or not with consideration of numerical error. So the method by computing Lyapunov exponents is inapplicable here. In the following paper we are going to present a detailed study and make a convincible conclusion by means of a topological horseshoe.

At first we define a Poincaré map. As shown in Fig. 1, we choose a cross-section $P : x = -10, \dot{x} < 0$ in the state space of Eq. (1). Let Γ be a trajectory started from P, let $\boldsymbol{x} = (x, y, z)$ be a point in Γ, and then the Poincaré map $H : P \to P$ is defined as: for each $\boldsymbol{x} \in P$, $H(\boldsymbol{x})$ is taken to be its first return point in Q along the trajectory.

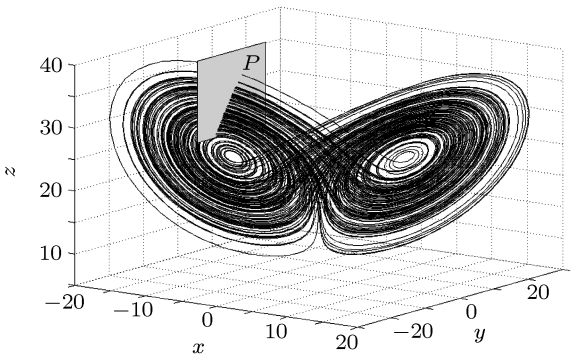


Fig. 1. Phase portrait of Eq. (1) and the cross-section.

For both Riemann–Liouville and Caputo derivatives,[1] it is easy to see that each trajectory is unique, and continuously depends on its initial condition, so it is not hard to take subsets on P while the continuity of H is guaranteed. In order to find a topological horseshoe, we use the MATLAB GUI program called 'A toolbox for finding horseshoes in 2D map' we posted in Ref. [23]. After many attempts, we take four quadrilaterals D_1, D_2, D_3 and D_4 by the method we proposed in Ref. [24]. As shown in Fig. 2, their vertices are listed in Table 1.

Table 1. The vertices of the four quadrilaterals.

subsets	sides	vertices in term of (y, z)
D_1	D_1^1	(–6.382797035, 31.058838233), (–6.296235172, 30.885022932)
	D_1^2	(–6.997841853, 30.045914585), (–7.075291942, 30.225723516)
D_2	D_2^1	(–5.640187366, 31.915927474), (–5.558181390, 31.760093066)
	D_2^2	(–6.072996683, 31.148742699), (–6.150446771, 31.316564368)
D_3	D_3^1	(–5.044517372, 32.538111053), (–4.990970947, 32.400056863)
	D_3^2	(–5.337447815, 32.031912355), (–5.390994240, 32.156161127)
D_4	D_4^1	(–4.587194838, 33.004603393), (–4.550474613, 32.879011950)
	D_4^2	(–4.818007681, 32.592506470), (–4.857350779, 32.722022646)

Unfortunately, there is no numerical method with rigorous estimation on error boundaries for initial value problems like what we have been done for integer-order systems, e.g. Lohner's algorithm.[25] Computation for the Poincaré map mentioned above still remains a harder problem. So in this paper we use the most reputed Adams–Bashforth–Moulton predictor-corrector scheme,[26] which is a time-domain approach and is more effective than other methods in most applications.[3] According to the scheme, we can compute H approximately by the following formulae:

$$x_{n+1} = x_0 + \frac{h^{q_1}}{\Gamma(q_1+2)}(28\alpha+10)(y^p_{n+1} - x^p_{n+1}) + \frac{h^{q_1}}{\Gamma(q_1+2)}\sum_{j=0}^{n} a_{1,j,n+1}(28\alpha+10)(y_j - x_j),$$

$$y_{n+1} = y_0 + \frac{h^{q_2}}{\Gamma(q_2+2)}\left[(28-35\alpha)x^p_{n+1} - x^p_{n+1}z^p_{n+1} + (29\alpha-1)y^p_{n+1}\right] + \frac{h^{q_2}}{\Gamma(q_2+2)}\sum_{j=0}^{n} a_{2,j,n+1}\left[(28-35\alpha)x_j - x_j z_j + (29\alpha-1)y_j\right],$$

$$z_{n+1} = z_0 + \frac{h^{q_3}}{\Gamma(q_3+2)}(x^p_{n+1}y^p_{n+1} - \frac{\alpha+3}{8}z^p_{n+1}) + \frac{h^{q_3}}{\Gamma(q_3+2)}\sum_{j=0}^{n} a_{3,j,n+1}(x_jy_j - \frac{\alpha+3}{8}z_j),$$

where

$$\begin{cases} x^p_{n+1} = x_0 + \dfrac{1}{\Gamma(q_1)}\sum\limits_{j=0}^{n}\beta_{1,j,n+1}(28\alpha+10)(y_j - x_j), \\ y^p_{n+1} = y_0 + \dfrac{1}{\Gamma(q_2)}\sum\limits_{j=0}^{n}\beta_{2,j,n+1}(28\alpha+10)\left[(28-35\alpha)x_j - x_jz_j + (29\alpha-1)y_j\right], \\ z^p_{n+1} = z_0 + \dfrac{1}{\Gamma(q_3)}\sum\limits_{j=0}^{n}\beta_{3,j,n+1}(x_jy_j - \dfrac{\alpha+3}{8}z_j), \end{cases}$$

$$\alpha_{i,j,n+1} = \begin{cases} n^{q_i+1} - (n-q_i)(n+1)^{q_i}, & j=0, \\ (n-j+2)^{q_i+1} + (n-j)^{q_i+1} - 2(n-j+1)^{q_i+1}, & 1\le j\le n, \\ 1, & j=n+1, \end{cases}$$

$$\beta_{i,j,n+1} = \frac{h^{q_i}}{q_i}\left[(n+1-j)^{q_i} - (n-j)^{q_i}\right], \quad 0\le j\le n, \quad \text{and} \quad i=1,2,3. \tag{2}$$

Since $q_1 = q_2 = q_3 = 0.9$, the order of this method is 1.9, that is, the truncation error is about $o(h^{1.9})$, here h is the stepsize.

By numerical computation of H for D_1, D_2, D_3 and D_4, their images are shown in Figs. 2–5. Computer simulation also suggests that H is continuous on each one of them respectively. Then it is not hard to have the following theorem.

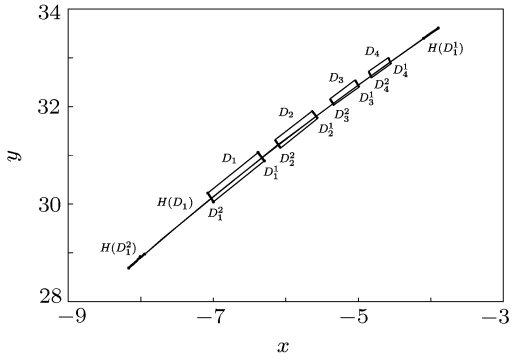


Fig. 2. The $H(D_1)$ across through D_i, $(i=1,2,3,4)$.

Theorem For the Poincaré map H corresponding to the cross section P, there exists a closed invariant set $\Lambda \subset P$ for which $H|\Lambda$ is semiconjugate to the 4-shift map.

Proof To this end, we only need to show the existence of an H-connection family F with respect to D_i $(i=1,2,3,4)$.

It is obvious in Fig. 2 that $H(D_1)$ goes through D_i with intersection points in D_i^1 and D_i^2 for each $i=1,2,3,4$. It is not hard to see that every path $\beta \subset D_1$ connecting D_1^1 and D_1^2 has a subpath $\bar{\beta} \subset \beta$ such that its H-image $H(\bar{\beta})$ goes through D_i with intersection points in D_i^1 and D_i^2. Similarly, the map H also has the same situation as D_1 for D_2, D_3 and D_4, as shown in Figs. 3–5.

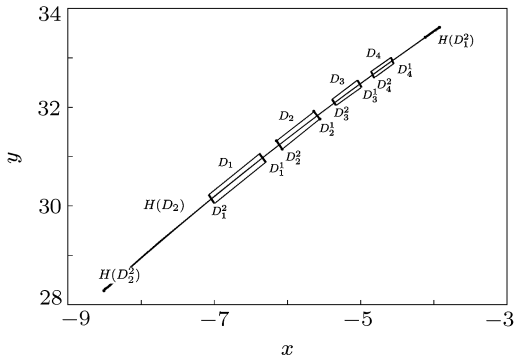


Fig. 3. The $H(D_2)$ across through D_i, $(i=1,2,3,4)$.

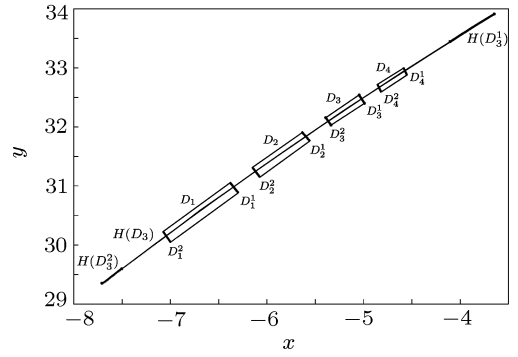


Fig. 4. The $H(D_2)$ across through D_i, $(i=1,2,3,4)$.

Let F be the family of connections γ with the following property: $\gamma \subset P$ is a path and γ goes through

D_i $(i = 1, 2, 3, 4)$ with intersection points in D_i^1 and D_i^2, that is, there is a connected subpath $\gamma_i \subset \gamma$ such that $\gamma_i \subset D_i$ and its two end points are contained in D_i^1 and D_i^2, respectively. Then from above arguments, it is easy to see that for every $\gamma_i = \gamma \cap D_i$ we have $H(\gamma_i) \in F$. Now it follows from Horseshoe lemma that there exists a compact invariant set $\Lambda \subset P$, such that $H|\Lambda$ is semiconjugate to the 4-shift map.

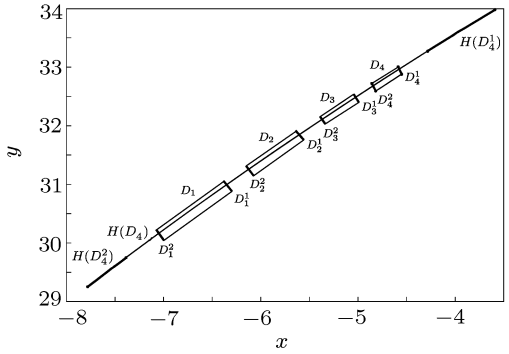


Fig. 5. The $H(D_2)$ across through D_i, $(i = 1, 2, 3, 4)$.

According to the definition of semiconjugation,[20] there exists a continuous surjective map $\eta : \Lambda \to \Sigma_4$ such that $\eta \circ H = \sigma \circ \eta$. Therefore, the topological entropy of H, i.e. $\mathrm{ent}(H)$, is not less than the topological entropy of σ. It should be emphasised here that if the topological entropy is positive then one of the Lyapunov exponents must be positive. Since σ has four symbols, we obtain $\mathrm{ent}(H) \geq \mathrm{ent}(\sigma) = \ln 4$, which means that H has a positive topological entropy. However, we are more concerned with the entropy of the original time-continuous system here. Fortunately, the topological entropy of Eq. (1) can be estimated with Abramov's formula[27]

$$\mathrm{ent}_\tau = \frac{\mathrm{ent}(H)}{\langle \tau \rangle_H},$$

where ent_τ is the topological entropy of the original dynamical system, $\mathrm{ent}(H)$ is the topological entropy of the Poincaré map H, $\langle \tau \rangle_H$ is the average time between two crossings of the Poincaré section. Numerical computation shows that $\langle \tau \rangle_H \approx 1.06$, therefore the topological entropy of Eq. (1) is not less then $\frac{\ln 4}{1.06} \approx 1.31$. This estimation provides a measure of how chaotic Eq. (1) can be for the given parameters.

4. Conclusions

We have studied a topological horseshoe in the unified system for the given parameters $q_1 = q_2 = q_3 = 0.9$ and $\alpha = 0.5$. By estimating the topological entropy, we provide a computer assisted verification on the existence of chaos in this fractional-order system, which is more convincible than the common method by computing Lyapunov exponents. We do not use the words 'a computer-assisted proof' here because the rigorous estimation on the numerical error boundaries for the Poincaré map still remains a hard problem. However, this paper can be a start of the difficult and meaningful task—proving a given fractional-order differential equations to be chaotic. According to the author's knowledge, it should be the first time for applying the topological horseshoe theory into the study of fractional dynamical systems. The method is promising in rigorous studies of chaos in such kind of systems in the future, and can potentially be used in estimating the topological entropy, proving the existence of chaos, showing the structure of chaotic attractors, revealing the mechanism inside of chaotic phenomena and so on.

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