Stochastic resonance in a time-delayed mono-stable system with correlated multiplicative and additive white noise

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This paper considers the stochastic resonance for a time-delayed mono-stable system, driven by correlated multiplicative and additive white noise. It finds that the output signal-to-noise ratio (SNR) varies non-monotonically with the delayed times. The SNR varies non-monotonically with the increase of the intensities of the multiplicative and additive noise, with the increase of the correlation strength between the two noises, as well as with the system parameter.

Keywords: stochastic resonance, time-delayed mono-stable system, signal-to-noise ratio

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1. Introduction

The concept of stochastic resonance (SR) was originally put forward in the seminal papers by Benzi *et al.*,[1,2] wherein they address the problem of the periodically recurrent ice ages. The eccentricity of the Earth's orbit varies periodically in time, and if the amplitude of this variation is too small to explain the succession of ice ages and relatively warm periods, the periodical phenomenon is amplified by some perturbations. This cooperative effect between the coherent 'signal' and the 'noise' was called as SR.

There are lots of stochastic systems with time-delayed feedback. By means of such feedback loops, signals or other key quantities (such as information, matter, energy, and so on) of the systems are fed back from output to input interfaces. The SR phenomenon in time-delayed systems has been studied by lots of researchers. Frank[3] studied nonlinear stochastic systems with time-delayed feedback using the concept of delay Fokker–Planck equations and derived an analytical expression for stationary distributions by using first-order perturbation theory. Goulding *et al.*[4] studied the classical Kramer's escape problem in a bistable system under the influence of noise consisting of the superposition of a white Gaussian noise with the same noise delayed by time τ. Zeng *et al.*[5] studied the SR phenomenon in a time-delayed gene transcriptional regulatory system with a multiplicative noise and an additive noise and a weak periodic signal. It is found that the delay time can enhance the SR of the system. Majer and Scholl[6] studied the time-delayed feedback in a tunnel diode and found much more pronounced resonant features of noise-induced oscillations in comparison with single time feedback.

There are a lot of mono-stable systems,[7−12] including chemical, electronic, physical and biological systems. Dykman *et al.*[7] and Evstigneev *et al.*[8] investigated the SR in a mono-stable over-damped system based on linear response theory. Stocks *et al.*[9,10] investigated the zero-dispersion SR in a mono-stable system, for which the dependence of eigenfrequency upon energy has an extremum. It is well known that the multiplicative noise often plays a different role on the output of a system, with respect to the additive noise. Therefore, the investigation of the response of a mono-stable system with time delay driven by multiplicative and additive noise is of great significance. In this paper, based on the adiabatic approximation theory, we study the SR phenomenon in a time-delayed mono-stable system driven by correlated multiplicative and additive noise.

2. The time-delayed mono-stable system and its signal-to-noise ratio (SNR)

We consider a stochastic differential equation with time delay

$$\frac{\mathrm{d}x}{\mathrm{d}t} = h(x, x_\tau) + g_1(x, x_\theta)\xi(t) + g_2(x, x_\tau)\eta(t). \quad (1)$$

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Specially, in this paper, we assume that

$$h(x, x_\tau) = -bx(t-\tau)^2 + xA\cos(\Omega t),$$
$$g_1(x, x_\theta) = x(t-\theta), \quad g_2(x, x_\tau) = -1, \tag{2}$$

then Eq. (1) characterise an over-damped mono-stable system[11] with time delay driven by multiplicative signal and additive noise, i.e.,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -bx(t-\tau)^2 + x(t-\theta)\xi(t) - \eta(t) + xA\cos(\Omega t), \tag{3}$$

here b is the system parameter, τ denotes the delay time in the deterministic force, θ the delay time in the random force, $\xi(t)$ and $\eta(t)$ are Gaussian white noises with zero mean and correlations

$$\langle \xi(t_1)\xi(t_2)\rangle = 2D\delta(t_1 - t_2),$$
$$\langle \eta(t_1)\eta(t_2)\rangle = 2P\delta(t_1 - t_2), \tag{4}$$
$$\langle \xi(t_1)\eta(t_2)\rangle = \langle \xi(t_2)\eta(t_1)\rangle = 2\lambda\sqrt{DP}\delta(t_1 - t_2), \tag{5}$$

where D and P are the intensities of the multiplicative noise $\xi(t)$ and additive noises $\eta(t)$, respectively, λ denotes the correlation strength between the two noises. The model of system (3) describes the overdamped Brownian motion in a mono-stable potential driven by multiplicative and additive fluctuational forces as well as by multiplicative external force. It is noted that Eq. (3) is different from that in Refs. [11] and [12]: (i) in the present paper, the power number of x is 2, i.e., an even number; while in Ref. [11] it is an positive odd integer, and in Ref. [12] it equals 3, i.e., x^3; (ii) in this paper we introduce delayed-times and multiplicative noise, while there is only an additive noise in Ref. [11]; (iii) the external force in this work is a multiplicative sinusoidal signal, while in Ref. [12], it is made up of a periodic square-wave and an asymmetric dichotomous.

Let $\rho(x,t) = \langle \delta(x - x(t))\rangle$ denote the probability density of the stochastic process defined by Eq. (3). Differentiating $\rho(x,t)$ with respect to time, we have

$$\begin{aligned}
\frac{\partial \rho(x,t)}{\partial t} &= -\left\langle \frac{\partial}{\partial x}\delta(x - x(t))\frac{\mathrm{d}}{\mathrm{d}t}x(t)\right\rangle \\
&= -\frac{\partial}{\partial x}\left\langle h(x(t), x(t-\tau))\delta(x - x(t))\right\rangle - \frac{\partial}{\partial x}\left\langle g_1(x(t), x(t-\theta))\xi(t)\delta(x - x(t))\right\rangle \\
&\quad - \frac{\partial}{\partial x}\left\langle g_2(x(t), x(t-\tau))\eta(t)\delta(x - x(t))\right\rangle \\
&= \frac{\partial}{\partial x}\int h(x, x_\tau)\rho(x, t; x_\tau, t-\tau)\,\mathrm{d}x_\tau - \frac{\partial}{\partial x}\int g_1(x, x_\theta)\left\langle \xi(t)\delta(x - x(t))\delta(x_\theta - x(t-\theta))\right\rangle \mathrm{d}x_\theta \\
&\quad - \frac{\partial}{\partial x}\int g_2(x, x(t-\tau))\left\langle \eta(t)\delta(x - x(t))\delta(x_\tau - x(t-\tau))\right\rangle \mathrm{d}x_\tau,
\end{aligned} \tag{6}$$

here $\rho(x, t; x_\tau, t-\tau)$ is the joint probability density. In order to calculate the average $\langle \xi(t)\delta(x - x(t))\delta(x_\theta - x(t-\theta))\rangle$ and $\langle \eta(t)\delta(x - x(t))\delta(x_\tau - x(t-\tau))\rangle$, one can use the Novikov's theorem[13,14]

$$\langle \xi_k \phi[\xi_1, \xi_2]\rangle = \int_0^t \gamma_{kl}(t, t')\left\langle \frac{\delta\phi[\xi_1, \xi_2]}{\delta\xi_1(t')}\right\rangle \mathrm{d}t', \tag{7}$$

where $\phi[\xi_1, \xi_2]$ is a function of ξ_1 and ξ_2, γ_{kl} are their correlation functions. Thus

$$\begin{aligned}
&\langle \xi(t)\delta(x - x(t))\delta(x_\theta - x(t-\theta))\rangle \\
&= 2D\int_0^t \delta(t-t')\left\langle \frac{\delta(\delta(x - x(t)))\delta(x_\theta - x(t-\theta)}{\delta x(t)}\frac{\delta x(t)}{\delta\xi(t)}\right\rangle \mathrm{d}t' \\
&\quad + 2D\int_0^t \delta(t-t')\left\langle \frac{\delta(\delta(x - x(t)))\delta(x_\theta - x(t-\theta)}{\delta x(t-\theta)}\frac{\delta x(t-\theta)}{\delta\xi(t)}\right\rangle \mathrm{d}t' \\
&\quad + 2\lambda\sqrt{DP}\int_0^t \delta(t-t')\left\langle \frac{\delta(\delta(x - x(t)))\delta(x_\theta - x(t-\theta)}{\delta x(t-\theta)}\frac{\delta x(t)}{\delta\eta(t)}\right\rangle \mathrm{d}t' \\
&\quad + 2\lambda\sqrt{DP}\int_0^t \delta(t-t')\left\langle \frac{\delta(\delta(x - x(t)))\delta(x_\theta - x(t-\theta)}{\delta x(t-\theta)}\frac{\delta x(t-\theta)}{\delta\eta(t)}\right\rangle \mathrm{d}t' \\
&= -D\frac{\partial}{\partial}g_1(x, x_\theta)\rho(x, t; x_\theta, t-\theta) - \lambda\sqrt{DP}\frac{\partial}{\partial x}g_2(x, x_\tau)\rho(x, t; x_\tau, t-\tau).
\end{aligned} \tag{8}$$

Similarly,

$$\langle \eta(t)\delta(x-x(t))\delta(x_\tau - x(t-\tau))\rangle = -P\frac{\partial}{\partial}g_2(x,x_\tau)\rho(x,t;x_\tau,t-\tau) - \lambda\sqrt{DP}\frac{\partial}{\partial x}g_1(x,x_\theta)\rho(x,t;x_\theta,t-\theta). \quad (9)$$

Substituting Eqs. (8) and (9) into Eq. (6), we can obtain the delay Fokker–Planck equation corresponding to Eq. (1). Simplifying the delay Fokker–Planck equation by expanding those functions containing small delayed-time τ and θ in series and applying the first-order approximation[3,15,16] and the Fox approach,[17] we derive the approximate Fokker–Plank equation of the probability density $\rho(x,t)$:

$$\frac{\partial\rho(x,t)}{\partial t} = -\frac{\partial}{\partial x}F(x,\tau)\rho(x,t) + \frac{\partial^2}{\partial x^2}G(x,\tau)\rho(x,t), \quad (10)$$

with

$$F(x,\tau) = [-bx^2 + D(1+\theta)^2 x - \lambda\sqrt{DP}(1+\theta) + xA\cos(\Omega t)]C(x,\tau), \quad (11)$$

$$G(x,\tau) = [D(1+\theta)^2x^2 - 2\lambda\sqrt{DP}(1+\theta)x + P]C(x,\tau)^2, \quad C(x,\tau) = 1 + 2b\tau x. \quad (12)$$

The term $(-\lambda\sqrt{DP}(1+\theta))$ in Eq. (11) is regarded as a constant force of the system.[18,19] From Eq. (11), one can see that, for the case of $D \neq 0$, i.e., with the presence of the multiplicative noise, there is one unstable state $x_\mathrm{u} = 0$ and one stable state $x_\mathrm{s} = D(1+\theta)^2/b$ of the delayed mono-stable system; while in Ref. [11], without the presence of multiplicative noise, the corresponding potential has only one stable state, i.e., $x_\mathrm{s} = 0$. Assume that the frequency $\Omega = 2\pi/T$ of the driving sine-wave signal is very small, i.e., frequency smaller than the rate of the intrawell relaxation, thus there is enough time for the system to reach the equilibrium during the period of T, i.e., we make the assumption that the system satisfies the adiabatic approximation condition.[20] The solution of the corresponding Fokker–Planck equation may then be approximated as a steady-state distribution with a slow time variation, which can be written as

$$\rho_\mathrm{st}(x,t) = \frac{N}{\sqrt{G(x,\tau)}}\exp\left[-\frac{\Phi(x,t)}{D}\right], \quad (13)$$

where N is the normalisation constant, $\Phi(x,t)$ is the rectified potential function

$$\begin{aligned}\Phi(x) &= -D\int\frac{F(x,\tau)}{G(x,\tau)}\mathrm{d}x \\ &= -\frac{c}{2(1+\theta)^2}\ln\left[x^2 - \frac{2\lambda\sqrt{P/D}}{D(1+\theta)}x + \frac{P}{D(1+\theta)^2}\right] - \frac{n}{2Db\tau}\ln(1+2b\tau x) \\ &\quad - \frac{c\lambda\sqrt{P/D} + mD(1+\theta)}{(1+\theta)^2\sqrt{P(1-\lambda^2)/D}}\arctan\left[\frac{(1+\theta)\left(x - \lambda\sqrt{P/D}/(1+\theta)\right)}{\sqrt{P(1-\lambda^2)/D}}\right], \end{aligned} \quad (14)$$

with

$$\begin{aligned} c &= A\cos(\Omega t) + D(1+\theta)^2 + 2b\tau\lambda\sqrt{DP}(1+\theta) - \frac{2b\tau P + 2\lambda\sqrt{DP}(1+\theta)}{2b\tau[2b\tau P + 2\lambda\sqrt{DP}(1+\theta)] + D(1+\theta)^2} \\ &\quad \times \left\{b + 2b\tau[D(1+\theta)^2 + 2b\tau\lambda\sqrt{DP}(1+\theta)] + 2b\tau A\cos(\Omega t)\right\}, \end{aligned} \quad (15)$$

$$m = \frac{P\left\{b + 2b\tau[D(1+\theta)^2 + 2b\tau\lambda\sqrt{DP}(1+\theta)] + 2b\tau A\cos(\Omega t)\right\}}{2b\tau[2b\tau P + 2\lambda\sqrt{DP}(1+\theta)] + D(1+\theta)^2} - \lambda\sqrt{DP}(1+\theta), \quad (16)$$

$$n = -\frac{b + 2b\tau[D(1+\theta)^2 + 2b\tau\lambda\sqrt{DP}(1+\theta)] + 2b\tau A\cos(\Omega t)}{2b\tau[2b\tau P + 2\lambda\sqrt{DP}(1+\theta)] + D(1+\theta)^2}. \quad (17)$$

The expression of transition rate from $x_\mathrm{s} = D(1+\theta)^2/b$ to $x_\mathrm{u} = 0$ for small noise intensity D and P can be written as[20,21]

$$W = \frac{\sqrt{|V''(x_\mathrm{s})V''(x_\mathrm{u})|}}{2\pi}\exp\left[\frac{\Phi(x_\mathrm{s}) - \Phi(x_\mathrm{u})}{D}\right] = \frac{D(1+\theta)^2}{2\pi}\exp\left[\frac{\Phi(x_\mathrm{s}) - \Phi(x_\mathrm{u})}{D}\right], \quad (18)$$

in which $V''(x) = \mathrm{d}^2V(x)/\mathrm{d}x^2 = -D(1+\theta)^2 + 2bx$. The output SNR is the ratio between the power density of the signal and the noise at the signal frequency, which has the form

$$SNR = \frac{\pi W_1 A^2}{4W_0}\left[1 - \frac{W_1^2 A^2}{2(W_0^2 + \Omega^2)}\right]^{-1}, \qquad (19)$$

with

$$W_0 = 2W|_{A\cos(\Omega t)=0},$$
$$W_1 = 2\frac{\mathrm{d}W}{\mathrm{d}(A\cos(\Omega t))}|_{A\cos(\Omega t)=0}. \qquad (20)$$

3. Discussion and conclusion

In this section, we discuss the dependence of SNR on the noises, on the delay time, on the input signal, as well as on the system parameters. By virtue of Eq. (19), the effects of these parameters on the SNR are discussed through Figs. 1–4.

Multiplicative fluctuations emerge naturally in a variety of systems with ensuring applications in different areas ranging from physics to biology.[22−24] In fact, in a realistic model one must always deal with various sources for fluctuations acting upon collective variables. For example, the fluctuation of values of a resistor, a capacitor or an inductor of an electric circuit can lead to multiplicative noise.[23] Multiplicative processes always have a lot of different influence from additive ones. The most probable values in a multiplicative process depend strongly on the strength of the fluctuations, while in additive process the dependence is very weak. We plot the curve of the SNR versus the multiplicative noise intensity D in Fig. 1. We can conclude that the SR phenomenon occurs in the SNR curve as a function of the strength of the multiplicative noise D. In addition, by the analyses of the curves in Figs. 1 and 2, we find that, with the increase of the delayed-time θ in the random force and τ in the deterministic force, the SNR varies non-monotonically, too. In Fig. 1, for small multiplicative strength $D(D < 0.12)$, the SNR increases with the increase of the delayed-time θ, while for relatively larger strength $(D > 0.15)$, the SNR decreases with the increase of θ. Similarly, one can see that from Fig. 2, for small amount of additive noise intensity P $(P < 0.05)$, the SNR raises with the growth of the delayed-time τ, while for relatively larger amount of noise intensity P $(P > 0.08)$, the SNR decreases with the growth of τ. Thus, there is an optimum value of the delay time θ and τ, for which the output of the system reaches its maximum. The non-monotonous behaviour of the multiplicative and the delayed-time is not studied in the mono-stable system of Ref. [11].

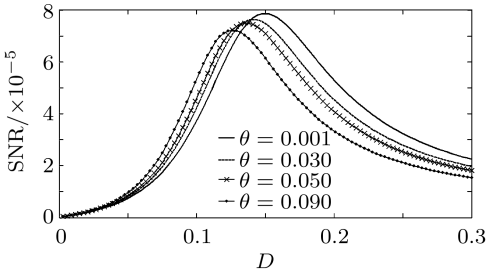


Fig. 1. The SNR as a function of the multiplicative noise intensity D for $b = 5$, $P = 0.3$, $\tau = 0.07$, $A = 0.002$, $\Omega = 0.01$, $\lambda = 0.98$ with different values of the delay time θ.

We investigate the effect of the additive noise strength P on the SNR through Figs. 2 and 3. From the two figures, one can conclude that the SNR is a non-monotonous function of the intensity of the additive noise. Therefore, we observe the SR phenomenon in the delayed mono-stable system. Moreover, the study of Fig. 3 shows that, the SNR can reach a minimum value with the growth of the correlation strength λ ($\lambda = -0.10$) between the multiplicative and additive noise. The values of the SNR for $\lambda = -0.17, 0.12, 0.15$ are greater than that for $\lambda = -0.10$. Therefore, the SNR also varies non-monotonically with the correlation strength between the two noises.

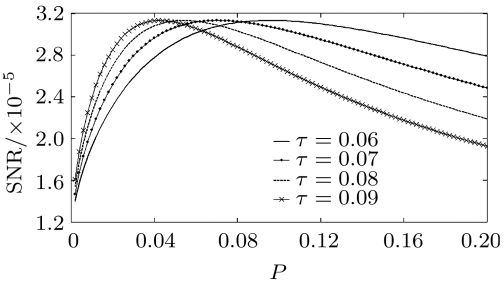


Fig. 2. The SNR as a function of the additive noise intensity P for $b = 5$, $D = 0.05$, $\theta = 0.01$, $A = 0.01$, $\Omega = 0.01$, $\lambda = 0.5$ with different values of the delay time τ.

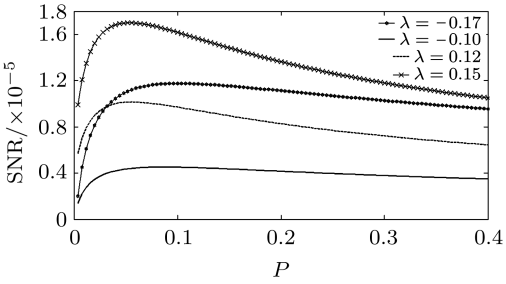


Fig. 3. The SNR as a function of the additive noise intensity P for $b = 10$, $D = 0.01$, $\tau = 0.02$, $\theta = 0.05$, $A = 0.04$, $\Omega = 0.01$ with different values of the correlation noise strength λ.

We analyse the effect of the system parameter on the SNR from Fig. 4. One can see that, the SNR shows non-monotonic behaviour with the increase of the system parameter b, i.e., the SR in a broad sense occurs. This can be explained by using the potential function $\Phi(x)$.[25] From Eq. (8), one can see that the potential function $\Phi(x)$ is a function of the system parameters a and b, so the height $\Delta V = V(x_{\rm u}) - V(x_{\rm s})$ of the potential barrier is also a function of the parameters. Therefore, the system SNR can be maximised at some mediate system parameter. In addition, the non-monotonous dependence of the system parameter on the system SNR is also not analysed in Ref. [11].

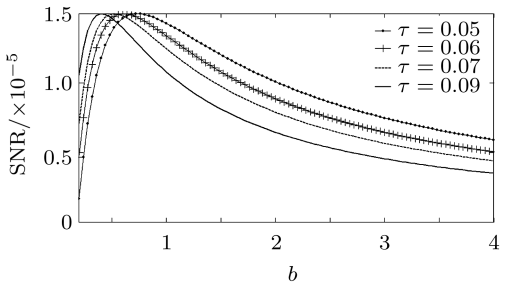


Fig. 4. The SNR as a function of the system parameter b for $D = 0.001$, $P = 0.5$, $\theta = 0.01$, $A = 0.02$, $\Omega = 0.01$, $\lambda = -0.4$ with different values of the delay time τ.

The numerical simulations are performed by directly integrating Eq. (3), simulation results are shown in Figs. 5 and 6. The numerical data of time series are obtained by using the forward Euler procedure with a time step $\Delta t = 2 \times 10^{-5}$ and ensemble size $N = 10^7$. The data for each run are saved at 100 different times and 200 independent realisations. Then the power spectra are calculated by virtue of a fast Fourier transform on auto-correlation function. The output SNR is defined as the height of the peak in the power spectrum at the input frequency derived by the height of the noisy background in the power spectrum around the input frequency. The comparison between Fig. 5, Fig. 6 and Fig. 1, Fig. 2 shows that, the analytical results derived in adiabatic approximation and small delay assumption are consistent with the simulations.

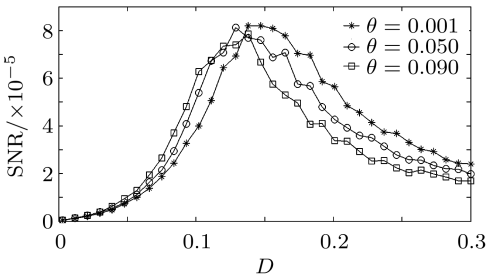


Fig. 5. The SNR as a function of the multiplicative noise intensity D for $b = 5$, $P = 0.3$, $\tau = 0.07$, $A = 0.002$, $\Omega = 0.01$, $\lambda = 0.98$ with different values of the delay time θ.

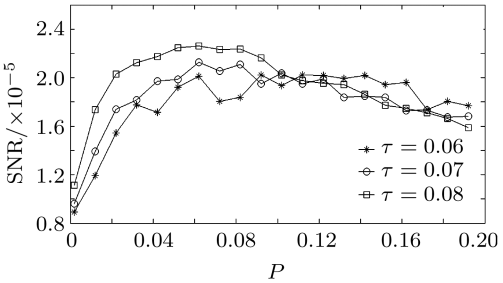


Fig. 6. The SNR as a function of the additive noise intensity P for $b = 5$, $D = 0.05$, $\theta = 0.01$, $A = 0.01$, $\Omega = 0.01$, $\lambda = 0.5$ with different values of the delay time τ.

To summarise, in the present paper, we investigate the SR phenomenon in a time-delayed mono-stable system driven by a multiplicative periodic signal as well as by multiplicative and additive white noise. With the addition of multiplicative noise and delayed-time in Ref. [11], the mono-stable system thus has two states, one is unstable and the other stable. Based on the adiabatic approximation theory, we obtain the expression of the output SNR. We find that the SNR exhibits SR phenomenon as a function of the intensities of the multiplicative and additive noise, as a function of the delayed times, as well as of the system parameter.

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