Solving ground eigenvalue and eigenfunction of spheroidal wave equation at low frequency by supersymmetric quantum mechanics method*

Tang Wen-Lin(唐文林)† and Tian Gui-Hua(田贵花)‡

School of Science, Beijing University of Posts And Telecommunications, Beijing 100876, *China*

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The spheroidal wave functions are found to have extensive applications in many branches of physics and mathematics. We use the perturbation method in supersymmetric quantum mechanics to obtain the analytic ground eigenvalue and the ground eigenfunction of the angular spheroidal wave equation at low frequency in a series form. Using this approach, the numerical determinations of the ground eigenvalue and the ground eigenfunction for small complex frequencies are also obtained.

Keywords: spheroidal wave equation, the perturbation method in supersymmetric quantum mechanics, super-potential, eigenvalue and eigenfunction

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1. Introduction

The spheroidal wave functions are found to have extensive applications in many branches of physics and mathematics, ranging from electromagnetic wave propagation,[1] black hole perturbation theory,[2] random matrices,[3] etc. We can refer to Refs. [4]–[8] for more detailed information about their applications. In general relativity, they were introduced by Teukolsky[2] into the study of the perturbation of a rotating black hole. These functions, which are denoted by $S_n(m,c;\theta), n = 0,1,2,\ldots$, satisfy the differential equations

$$\left(\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}+c^2\cos^2\theta-\frac{m^2}{\sin^2\theta}\right)S_n(m,c;\theta)$$
$$=-E_n(m,c)S_n(m,c;\theta),\quad n=0,1,2,\ldots, \tag{1}$$

where the independent variable θ lies in the interval $(0,\pi)$, m and c are the constants. These eigenfunctions $S_n(m,c;\theta), n = 0,1,2,\ldots$, correspond to eigenvalues $E_n(m,c)$ that need be finite at $\theta = 0,\pi$. In the following, we assume that m is zero or a positive integer since the eigenvalues and eigenfunctions for $-m$ can be obtained from the eigenvalues and eigenfunctions for m (see Ref. [5]). The parameter c is a complex number, if c is real, the eigenfunctions of Eq. (1) are called the oblate spheroidal angle functions, and if c is pure imaginary, the eigenfunctions of Eq. (1) are referred to as the prolate spheroidal angle functions.[5]

The small c expansion was studied by a large number of authors. The classical book by Flammer[5] was largely devoted to tabulating eigenvalues and eigenfunctions of spheroidal wave equation, but these tables only for the prolate and oblate cases. Press and Teukolsky[6] used the ordinary perturbation theory to obtain an expansion in small real c for the eigenvalues and eigenfunctions, and used a continuation technique for lager values of real parameter c. Frackerell and Crossman[7] expressed the spheroidal wave functions as a series involving Jacobi polynomials. The general complex frequencies were considered in their work, and later corrected by Seidel.[8] Here we also present the ground eigenvalue and eigenfunction of the angular spheroidal wave equation at low frequency in terms of a series in c. However, our results are much better than all the results mentioned above, in particular, the ground eigenfunction is presented by a product, which makes our results useful in applications.

When c vanishes, Eq. (1) becomes the one satisfied by the associated Legendre functions P_l^m. The associated Legendre functions have many properties, such as the recurrence relations between them, which give the P_l^m with the high values of l from the low ones. Although the spheroidal wave equations are ex-

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†Corresponding author. E-mail: jiabin1986@gmail.com

‡E-mail: tgh-2000@263.net

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tensions of the spherical wave functions, their corresponding recurrence relations were not found until the advent of Refs. [9]–[13]. In Refs. [9]–[13], the perturbation method in the supersymmetric quantum mechanics (SUSYQM) was used to obtain the recurrence relations up to the first order of c^2 (in these two references, c^2 was represented by the parameter α). In the present paper, we also use the perturbation method in SUSYQM to solve Eq. (1) in the case of small c, though we give only the ground eigenvalue and eigenfunction of Eq. (1). Our results are better than those in Refs. [9]–[13]. Our ground eigenvalue and eigenfunction are given in a complete series of c^2, they can be regarded as extensions to those of Refs. [9]–[13].

Now, our results about ground eigenvalue and eigenfunction of Eq. (1) are shown as follows:

$$E_0(m,c) = m(m+1) - \sum_{n=1}^{\infty} \frac{a_{n,0}(m)}{(2m+3)^n} c^{2n}, \quad (2)$$

and

$$S_0(m,c;\theta) = \sin^m\theta \prod_{n=1}^{\infty} \exp\left[\frac{c^{2n}}{(2m+3)^n} \times \sum_{\ell=0}^{n-1} \frac{a_{n,\ell}(m)}{2\ell+2} \cos^{2\ell+2}\theta\right], \quad (3)$$

where the coefficients $a_{n,\ell}(m)$ satisfy simple recurrence relations. To present our main results in a clear way, we first give the ground state eigenvalue and the ground eigenfunction in Section 2, whose detailed proofs are shown in the Appendix.

2. Ground state eigenvalue and eigenfunction

In order to apply the SUSYQM approach, we should transform Eq. (1) into a Schrödinger equation.[14,15] Applying the transformation[9,10]

$$S_n(m,c;\theta) = \frac{\psi_n(m,c;\theta)}{\sin^{1/2}\theta}, \quad (0<\theta<\pi), \quad n=0,1,\ldots \quad (4)$$

to Eq. (1) yields

$$-\frac{\mathrm{d}^2\psi_n(m,c;\theta)}{\mathrm{d}\theta^2} + V(m,c;\theta)\psi_n(m,c;\theta) = E_n(m,c)\psi_n(m,c;\theta), \quad n=0,1,\ldots, \quad (5)$$

where $V(m,c;\theta)$ is

$$V(m,c;\theta) = -\frac{1}{4}\left(1+\frac{1}{\sin^2\theta}\right) + \frac{m^2}{\sin^2\theta} - c^2\cos^2\theta. \quad (6)$$

The boundary condition of $S_n(m,c;\theta)$ leads the term

$$\lim_{\theta\to 0,\pi} \frac{\psi_n(m,c;\theta)}{\sin^{1/2}\theta}, \quad n=0,1,\ldots, \quad (7)$$

to be finite. When c is small, the ground state eigenvalue $E_0(m,c)$ is expanded into

$$E_0(m,c) = \sum_{n=0}^{\infty} E_{0n}(m) c^{2n}. \quad (8)$$

By introducing the superpotential

$$W(m,c;\theta) = -\frac{\mathrm{d}\ln\psi_0}{\mathrm{d}\theta}, \quad (9)$$

we can obtain[14]

$$W(m,c;\theta)^2 - W'(m,c;\theta) = V(m,c;\theta) - E_0(m,c). \quad (10)$$

In Eq. (10) and in what follows, the symbol prime denotes the derivative of a function with respect to θ. Equation (5) becomes

$$-\frac{\mathrm{d}^2\psi_n(m,c;\theta)}{\mathrm{d}\theta^2} + [V(m,c;\theta) - E_0(m,c)]\psi_n(m,c;\theta) = [E_n(m,c) - E_0(m,c)]\psi_n(m,c;\theta), \quad n=0,1,\ldots. \quad (11)$$

According to the theory of supersymmetric quantum mechanics, the ground state eigenfunction is

$$\psi_0(m,c;\theta) = \exp\left[-\int^{\theta} W(m,c;\phi)\,\mathrm{d}\phi\right]. \quad (12)$$

Expanding the super-potential W by

$$W(m,c;\theta) = \sum_{n=0}^{\infty} W_{on}(m;\theta) c^{2n}, \quad (13)$$

correspondingly, the boundary condition (7) now turns into the term

$$\lim_{\theta\to 0,\pi} \frac{\exp\left[-\int^{\theta} W_{00}(m;\phi)\,\mathrm{d}\phi\right]}{\sin^{1/2}\theta} \quad (14)$$

that exists and is finite, and for any $n \geq 1$, the term

$$\lim_{\theta\to 0,\pi} \exp\left[-c^{2n}\int^{\theta} W_{0n}(m;\phi)\,\mathrm{d}\phi\right] \quad (15)$$

that also exists and is finite. Substituting Eq. (13) into Eq. (10) and equating the coefficients of c^{2n} on both sides of Eq. (10), it is not difficult to obtain

$$W'_{00}(m;\theta) - W^2_{00}(m;\theta) = E_{00}(m) + \frac{1}{4}\left(1+\frac{1}{\sin^2(\theta)}\right) - \frac{m^2}{\sin^2\theta}, \quad (16)$$

$$W'_{01}(m;\theta) - 2W_{00}(m;\theta)W_{01}(m;\theta) = E_{01}(m) + \cos^2\theta, \tag{17}$$

and for $n \geq 2$,

$$W'_{0n}(m;\theta) - 2W_{00}(m;\theta)W_{0n}(m;\theta) = E_{0n}(m) + \sum_{k=1}^{n-1} W_{0k}(m;\theta)W_{0,n-k}(m;\theta). \tag{18}$$

Obviously, the values of $E_{0n}(m)$ and $W_{0n}(m;\theta)$, $n = 0, 1, \ldots$, can be given by the above equations (16)–(18) and the boundary conditions (14) and (15). Here the evaluations of these $E_{0n}(m)$ and $W_{0n}(m;\theta)$ are presented whose detailed proofs are shown in the Appendix. The results are

$$E_{00} = m(m+1) \tag{19}$$

and

$$W_{00}(m;\theta) = -(m+1/2)\cot\theta, \tag{20}$$

respectively. Substitute Eq. (20) into Eqs. (17) and (18), then we will obtain

$$E_{0n}(m) = -\frac{a_{n,0}(m)}{(2m+3)^n} \tag{21}$$

and

$$W_{0n}(m;\theta) = \frac{1}{(2m+3)^n}\sin\theta\cos\theta \times \sum_{\ell=0}^{n-1} a_{n,\ell}(m)\cos^{2\ell}\theta \tag{22}$$

for any $n \geq 1$, where the coefficients $a_{n,\ell}(m)$ in the above equation satisfy the following simple recurrence relations: for $n \geq 1$,

$$a_{n+1,0}(m) = \sum_{k=1}^{n}\sum_{\ell=0}^{n-1}\left[\frac{(1/2)_{\ell+1}}{(m+3/2)_{\ell+1}} - \frac{(1/2)_{\ell+2}}{(m+3/2)_{\ell+2}}\right] \sum_{i=0,\ell-i\leq n-k}^{\min(\ell,k-1)} a_{k,i}(m)a_{n+1-k,\ell-i}(m), \tag{23}$$

$$a_{n+1,p}(m) = \frac{(m+3/2)_p}{(3/2)_p}\sum_{k=1}^{n}\sum_{\ell=p}^{n-1}\frac{(1/2)_{\ell+1}}{(m+3/2)_{\ell+1}} \sum_{i=0,\ell-i\leq n-k}^{\min(\ell,k-1)} a_{k,i}(m)a_{n+1-k,\ell-i}(m) - \frac{(m+3/2)_p}{(3/2)_p}\sum_{k=1}^{n}\sum_{\ell=p-1}^{n-1}\frac{(1/2)_{\ell+2}}{(m+3/2)_{\ell+2}} \sum_{i=0,\ell-i\leq n-k}^{\min(\ell,k-1)} a_{k,i}(m)a_{n+1-k,\ell-i}(m), \tag{24}$$

and

$$a_{n+1,n}(m) = -\frac{1}{2m+2n+3}\sum_{k=1}^{n} a_{k,k-1}(m)a_{n+1-k,n-k}(m) \tag{25}$$

with the initial condition

$$a_{1,0}(m) = 1, \tag{26}$$

where $(a)_n = a \times (a+1) \times \ldots \times (a+n-1)$ for any constant a. Substituting the above evaluations (19)–(22) of $E_{0n}(m)$ and $W_{0n}(m;\theta)$ into Eqs. (8), (12) and (13), one directly obtains the ground state eigenvalue

$$E_0(m,c) = m(m+1) - \sum_{n=1}^{\infty}\frac{a_{n,0}(m)}{(2m+3)^n}c^{2n}, \tag{27}$$

and the ground state eigenfunction

$$\psi_0(m,c;\theta) = \sin^{m+\frac{1}{2}}\theta\prod_{n=1}^{\infty}\exp\left[\frac{c^{2n}}{(2m+3)^n}\sum_{\ell=0}^{n-1}\frac{a_{n,\ell}(m)}{2\ell+2}\cos^{2\ell+2}\theta\right], \tag{28}$$

or

$$S_0(m,c;\theta) = \sin^{m}\theta\prod_{n=1}^{\infty}\exp\left[\frac{c^{2n}}{(2m+3)^n}\sum_{\ell=0}^{n-1}\frac{a_{n,\ell}(m)}{2\ell+2}\cos^{2\ell+2}\theta\right]. \tag{29}$$

The above expressions of the ground state eigenvalue and the ground eigenfunction are in good forms. In particular, we can evaluate them up to any order of c^2 with the help of computer since the coefficients $a_{n,\ell}(m)$ satisfy simple recurrence relations (23), (24), (25) and (26), that is to say, we can obtain the values of $a_{n+1,p}(m)(0 \le p \le n)$ if we know all values of $a_{k,p}(m)$ $(0 \le k \le n$ and $1 \le p \le k-1$ for any $k)$. Therefore, we also give the numerical determinations of the ground eigenvalue and the ground eigenfunction for small complex frequencies.

To show the correctness of our result, we consider the ground eigenvalue and the ground eigenfunction of the prolate spheroidal wave equation

$$\begin{aligned}&(1-t^2)\frac{\mathrm{d}^2 y_n(t)}{\mathrm{d}t^2} - 2t\frac{\mathrm{d}y_n(t)}{\mathrm{d}t}\\&+(\chi_n(c) - c^2t^2)y_n(t) = 0,\\&n = 0, 1, 2, \ldots,\end{aligned} \tag{30}$$

which is equivalent to the Eq. (1) with special integer $m = 0$, the parameter c being a pure imaginary number, and $\chi_n(c) = E_n(0, \mathrm{i}\,c)$. Because it is easy to obtain from Eqs. (23)–(26) that

$$\begin{aligned}&a_{1,0}(0) = 1, \quad a_{2,0}(0) = \frac{2}{15}, \quad a_{2,1}(0) = -\frac{1}{5},\\&a_{3,0}(0) = \frac{4}{315}, \ldots.\end{aligned} \tag{31}$$

Substitute them into Eqs. (27) and (29), then we will have the ground eigenvalue

$$\begin{aligned}\chi_0(c) &= E_n(0, \mathrm{i}\,c)\\&= -\sum_{n=1}^{\infty}\frac{(-1)^n a_{n,0}(0)}{3^n}c^{2n}\\&= \frac{c^2}{3} - \frac{2c^4}{135} + \frac{4c^6}{8505} - \ldots,\end{aligned} \tag{32}$$

and the ground eigenfunction

$$\begin{aligned}y_0(t) &= S_0(0, \mathrm{i}\,c; \theta)\\&= \exp\left(\frac{c^2t^2}{6}\right)\exp\left[\frac{c^4t^2}{45}\left(\frac{1}{3} - \frac{t^2}{4}\right)\right] \times \ldots.\end{aligned} \tag{33}$$

The expression of $\chi_0(c)$ is the same as that given by Flammer,[5] therefore, our results are meaningful.

Appendix: Proofs of formulae for eigen-value and superpotential

In this appendix, the detailed evaluations of $E_{0n}(m)$ and $W_{0n}(m;\theta)$, $n = 0, 1, 2, \ldots$ are given. They satisfy Eqs. (16)–(18), and their determinations are also dependent on boundary conditions (14) and (15). In order to solve Eq. (16), it is convenient to assume $W_{00}(m;\theta) = \dfrac{a + b\cos\theta}{\sin\theta}$ with a and b being some undetermined constants. Hence, one obtains

$$E_{00} = m(m+1), \tag{A1}$$

and

$$W_{00}(m;\theta) = -(m + 1/2)\cot\theta. \tag{A2}$$

Calculating the term

$$\exp\left[-\int^{\theta} W_{00}(m;\phi)\,\mathrm{d}\phi\right] = \sin^{(m+1)/2}\theta, \tag{A3}$$

gives

$$\begin{aligned}&\exp\left[-\int^{\theta} W_{00}(m;\phi)\,\mathrm{d}\phi\right]\\&\to\begin{cases}\theta^{m+1/2} & \text{if } \theta \to 0;\\(\pi-\theta)^{m+1/2} & \text{else if } \theta \to \pi\ ,\end{cases}\end{aligned} \tag{A4}$$

and condition (14) is satisfied since we have assumed that $m \ge 0$. For simplicity, it is useful to introduce functions $f_n(m;\theta), n = 1, 2, \ldots$, as follows:

$$f_1(m,\theta) \equiv E_{01}(m) + \cos^2\theta, \tag{A5}$$

and for $n \ge 2$,

$$f_n(m;\theta) \equiv E_{0n}(m) + \sum_{k=1}^{n-1} W_{0k}(m;\theta)W_{0,n-k}(m;\theta). \tag{A6}$$

Hence Eqs. (17) and (18) can be concisely written as

$$\begin{aligned}&W'_{0n}(m;\theta) + (2m+1)\cot\theta W_{0n}(m;\theta) = f_n(m;\theta),\\&n = 1, 2, \ldots.\end{aligned} \tag{A7}$$

It is easy to give the solutions of the above equation as

$$\begin{aligned}&W_{0n}(m;\theta) = \frac{1}{\sin^{2m+1}\theta}\int^{\theta} f_n(m;\varphi)\sin^{2m+1}\varphi\,\mathrm{d}\varphi,\\&n = 1, 2, \ldots.\end{aligned} \tag{A8}$$

Before evaluating $W_{0n}(m;\theta)$, some helpful formulas which are given for later use. By partial integration, it is easy to obtain

$$\int^{\theta} \cos^{2\ell}\phi \sin^{2m+1}\phi \,\mathrm{d}\phi = \frac{(\frac{1}{2})_\ell}{(m+\frac{3}{2})_\ell}\left[\int^{\theta} \sin^{2m+1}\phi \,\mathrm{d}\phi + \sum_{k=1}^{\ell} \frac{(m+\frac{3}{2})_{\ell-k}}{(\frac{1}{2})_{\ell-k}} \cos^{2\ell-2k+1}\theta \sin^{2m+2}\theta\right], \quad \ell = 1, 2, \ldots \tag{A9}$$

and

$$h(\theta) := \frac{1}{\sin^{2m+1}\theta}\int^{\theta} \sin^{2m+1}\phi \,\mathrm{d}\phi = -\sum_{\ell=0}^{m}(-1)^\ell \binom{m}{\ell}\frac{1}{2\ell+1}\frac{\cos^{2\ell+1}\theta}{\sin^{2m+1}\theta}, \tag{A10}$$

$$\begin{aligned} f(\theta) :=& \int^{\theta} h(\phi)\,\mathrm{d}\phi = -\sum_{\ell=0}^{m}(-1)^\ell \binom{m}{\ell}\frac{1}{2\ell+1}\int^{\theta}\frac{\cos^{2\ell+1}\phi}{\sin^{2m+1}\phi}\,\mathrm{d}\phi \\ =& -\frac{1}{2^{2m}}\sum_{\ell=0}^{m}(-1)^\ell \binom{m}{\ell}\frac{1}{2\ell+1}\sum_{n=0,n\neq m}^{2m}\frac{1}{2(n-m)} \\ & \times \binom{2m-2\ell-1}{n}{}_2F_1(-n,-2\ell-1;2m-2\ell-n;-1)\tan^{2(n-m)}(\theta/2), \end{aligned} \tag{A11}$$

where ${}_2F_1(a,b;c;x)$ is the hypergeometric function.[16,17] One should notice the fact that the functions $h(\theta)$ and $f(\theta)$ blow up whenever $\theta \to 0, \pi$. Form the above discussion, it is easy to obtain

$$\begin{aligned} W_{01}(m;\theta) &= \frac{1}{\sin^{2m+1}\theta}\int^{\theta}(E_{01}(m)+\cos^2\varphi)\sin^{2m+1}\varphi\,\mathrm{d}\varphi \\ &= \left[E_{01}(m)+\frac{1}{2m+3}\right]h(\theta)+\frac{1}{2m+3}\sin\theta\cos\theta, \end{aligned} \tag{A12}$$

and

$$\exp\left[-c^2\int^{\theta} W_{01}(m;\phi)\,\mathrm{d}\phi\right] = \exp\left[-c^2\left(E_{01}(m)+\frac{1}{2m+3}\right)f(\theta)\right]\times\exp\left[-\frac{c^2}{2(2m+3)}\sin^2\theta\right]. \tag{A13}$$

In order to satisfy condition (15), one must have

$$E_{01}(m) = -\frac{1}{2m+3}, \tag{A14}$$

and W_{01} is simplified into

$$W_{01}(m,\theta) = \frac{1}{2m+3}\sin\theta\cos\theta. \tag{A15}$$

Once again, we re-stress our viewpoint, that is, whatever the non-zero coefficients of the summations on the right-hand side of Eq. (A11) are, the function $\exp\left(a\times\tan^{2(n-m)}(\theta/2)\right)$ $(a>0)$ tends to infinity as θ tends to zero or π, and c can be real or pure imaginary, then we need the coefficient of $h(\theta)$ in $W_{01}(m;\theta)$ that must be zero-valued, which is Eq. (A14). In the following, we will use this fact and condition (15) to determine $E_{0n}(m)$ and $W_{0n}(m;\theta), n = 2, 3, \ldots$. Now the evaluation is stated in the following conclusion and proved by mathematical induction.

From Eq. (A7) and the boundary condition (15), we obtain the following conclusions:

(i) for any $n \geq 1$, $W_{0n}(m;\theta)$ has a from

$$W_{0n}(m;\theta) = \frac{1}{(2m+3)^n}\sin\theta\cos\theta g_n(m;\theta), \tag{A16}$$

where $g_n(m;\theta)$ is a polynomial of $\cos^2\theta$, and its highest degree is $n-1$. In particular, if

$$g_n(m;\theta) = \sum_{\ell=0}^{n-1} a_{n,\ell}(m)\cos^{2\ell}\theta, \tag{A17}$$

then there are relations: for $n \geq 1$,

$$a_{n+1,0}(m) = \sum_{k=1}^{n}\sum_{\ell=0}^{n-1}\left[\frac{(1/2)_{\ell+1}}{(m+3/2)_{\ell+1}} - \frac{(1/2)_{\ell+2}}{(m+3/2)_{\ell+2}}\right]\sum_{i=0,\ell-i\leq n-k}^{\min(\ell,k-1)} a_{k,i}(m)a_{n+1-k,\ell-i}(m), \tag{A18}$$

and for $1 \leq p \leq n-1$,

$$a_{n+1,p}(m) = \frac{(m+3/2)_p}{(3/2)_p}\sum_{k=1}^{n}\sum_{\ell=p}^{n-1}\frac{(1/2)_{\ell+1}}{(m+3/2)_{\ell+1}}\sum_{i=0,\ell-i\leq n-k}^{\min(\ell,k-1)} a_{k,i}(m)a_{n+1-k,\ell-i}(m)$$

$$-\frac{(m+3/2)_p}{(3/2)_p}\sum_{k=1}^{n}\sum_{\ell=p-1}^{n-1}\frac{(1/2)_{\ell+2}}{(m+3/2)_{\ell+2}}\sum_{i=0,\ell-i\le n-k}^{\min(\ell,k-1)}a_{k,i}(m)a_{n+1-k,\ell-i}(m), \tag{A19}$$

and

$$a_{n+1,n}(m)=-\frac{1}{2m+2n+3}\sum_{k=1}^{n}a_{k,k-1}(m)a_{n+1-k,n-k}(m) \tag{A20}$$

with the initial condition

$$a_{1,0}(m)=1, \tag{A21}$$

where $(a)_n=a\times(a+1)\times\ldots\times(a+n-1)$ for any constant a. Hence the super-potential is

$$W(m,c;\theta)=-\left(m+\frac{1}{2}\right)\cot\theta+\sin\theta\cos\theta\sum_{n=1}^{\infty}\frac{1}{(2m+3)^n}\left(\sum_{\ell=0}^{n-1}a_{n,\ell}(m)\cos^{2\ell}\theta\right)c^{2n}; \tag{A22}$$

(ii) for any $n\ge 1$,

$$E_{0n}(m)=-\frac{a_{n,0}(m)}{(2m+3)^n}. \tag{A23}$$

The conclusion is drawn in the following. From the preceding evaluation, this conclusion holds for $n=1$. Assume that it holds for $1\le k\le n$ for $n>1$, then from Eq. (A6), one must have

$$\begin{aligned}f_{n+1}(m;\theta)=&E_{0,n+1}(m)+\sum_{k=1}^{n}W_{0k}(m;\theta)W_{0,n+1-k}(m;\theta)\\ =&E_{0,n+1}(m)+\frac{1}{(2m+3)^{n+1}}(1-\cos^2\theta)\cos^2\theta\sum_{k=1}^{n}\sum_{i=0}^{k-1}\sum_{j=0}^{n-k}a_{k,i}(m)a_{n+1-k,j}(m)\cos^{2(i+j)}\theta,\end{aligned} \tag{A24}$$

which is a polynomial in $\cos^2\theta$ with degree $n+1$, therefore, from Eq. (A8), there is

$$\begin{aligned}W_{0,n+1}(m;\theta)=&\frac{1}{\sin^{2m+1}\theta}\int^{\theta}f_{n+1}(m;\varphi)\sin^{2m+1}\varphi\,\mathrm{d}\varphi\\ =&\frac{1}{\sin^{2m+1}\theta}\int^{\theta}\Bigg[E_{0,n+1}(m)+\frac{1}{(2m+3)^{n+1}}\sum_{k=1}^{n}\sum_{i=0}^{k-1}\sum_{j=0}^{n-k}a_{k,i}(m)a_{n+1-k,j}(m)\\ &\times\left(\cos^{2(i+j+1)}\phi-\cos^{2(i+j+2)}\phi\right)\Bigg]\sin^{2m+1}\varphi\,\mathrm{d}\varphi\\ =&\Bigg[E_{0,n+1}(m)+\frac{1}{(2m+3)^{n+1}}\sum_{k=1}^{n}\sum_{i=0}^{k-1}\sum_{j=0}^{n-k}a_{k,i}(m)a_{n+1-k,j}(m)\\ &\times\left(\frac{(\frac{1}{2})_{i+j+1}}{(m+\frac{3}{2})_{i+j+1}}-\frac{(\frac{1}{2})_{i+j+2}}{(m+\frac{3}{2})_{i+j+2}}\right)\Bigg]h(\theta)\\ &+\frac{\sin\theta\cos\theta}{(2m+3)^{n+1}}\Bigg[\sum_{k=1}^{n}\sum_{i=0}^{k-1}\sum_{j=0}^{n-k}a_{k,i}(m)a_{n+1-k,j}(m)\\ &\times\left(\frac{(\frac{1}{2})_{i+j+1}}{(m+\frac{3}{2})_{i+j+1}}\sum_{p=1}^{i+j+1}\frac{(m+\frac{3}{2})_{i+j+1-p}}{(\frac{3}{2})_{i+j+1-p}}\cos^{2(i+j+1-p)}\theta\right.\\ &\left.-\frac{(\frac{1}{2})_{i+j+2}}{(m+\frac{3}{2})_{i+j+2}}\sum_{p=1}^{i+j+2}\frac{(m+\frac{3}{2})_{i+j+2-p}}{(\frac{3}{2})_{i+j+2-p}}\cos^{2(i+j+2-p)}\theta\right)\Bigg].\end{aligned} \tag{A25}$$

The coefficient of $h(\theta)$ in $W_{0,n+1}(m;\theta)$ needs to be zero-valued in order to satisfy condition (15). Thus, for the same reason as that in the process of determining E_{01}, one has the following results:

$$E_{0,n+1}(m)=-\frac{1}{(2m+3)^{n+1}}\sum_{k=1}^{n}\sum_{i=0}^{k-1}\sum_{j=0}^{n-k}a_{k,i}(m)a_{n+1-k,j}(m)\times\left(\frac{(\frac{1}{2})_{i+j+1}}{(m+\frac{3}{2})_{i+j+1}}-\frac{(\frac{1}{2})_{i+j+2}}{(m+\frac{3}{2})_{i+j+2}}\right), \tag{A26}$$

and

$$W_{0,n+1}(m;\theta) = \frac{\sin\theta\cos\theta}{(2m+3)^{n+1}}\Bigg[\sum_{k=1}^{n}\sum_{i=0}^{k-1}\sum_{j=0}^{n-k} a_{k,i}(m)a_{n+1-k,j}(m) \times \Bigg(\frac{(\frac{1}{2})_{i+j+1}}{(m+\frac{3}{2})_{i+j+1}}\sum_{p=1}^{i+j+1}\frac{(m+\frac{3}{2})_{i+j+1-p}}{(\frac{3}{2})_{i+j+1-p}}\cos^{2(i+j+1-p)}\theta - \frac{(\frac{1}{2})_{i+j+2}}{(m+\frac{3}{2})_{i+j+2}}\sum_{p=1}^{i+j+2}\frac{(m+\frac{3}{2})_{i+j+2-p}}{(\frac{3}{2})_{i+j+2-p}}\cos^{2(i+j+2-p)}\theta\Bigg)\Bigg] \equiv \frac{\sin\theta\cos\theta}{(2m+3)^{n+1}}\sum_{p=0}^{n} a_{n+1,p}(m)\cos^{2p}\theta. \quad \text{(A27)}$$

From Eqs. (A25) and (A27), we obtain Eqs. (A18)–(A23) in the above conclusion.

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