Drinfeld twist and the domain wall partition function of the eight-vertex model*

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With the help of the $\boldsymbol{F}$-basis provided by the Drinfeld twist or factorising $\boldsymbol{F}$-matrix for the spatial optical soliton model associated with the eight-vertex model, we calculate the partition function for the eight-vertex model on an $N \times N$ square lattice with domain wall boundary condition.

Keywords: eight-vertex model, domain wall boundary condition, Drinfeld twist

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1. Introduction

The domain wall (DW) boundary condition of statistical model on a finite square lattice was introduced in Ref. [1] for the six-vertex model. The partition function of the model (or the DW partition function) is the inner product of the pseudo-vacuum and some Bethe state. It is well known that in the framework of quantum inverse scattering method (QISM)[2] Bethe states are obtained by applying pseudo-particle creation operators to pseudo-vacuum. However, the apparently simple action of creation operators is plagued with non-local effects arising from polarisation clouds or compensating exchange terms on the level of local operators. This makes the direct calculation of the partition function (also norms of Bethe states) challenging. To avoid the direct computation the author[1] derived some recursive relation of the DW partition function instead, which fully determines the partition function. It was then found in Refs. [3] and [4] that the partition function can be represented as a determinant. Such an explicit expression of the partition function has played an important role in constructing norms of Bethe states, correction functions[2,5,6] and thermodynamical properties of the six-vertex model,[7] and also in the Toda theories.[8] Moreover, it has been proven to be very uesful in solving some pure mathematical problems, such as the problem of alternating sign matrices.[9,10] Recently, the approach in Ref. [1] has been used to obtain the DW partition functions for various models.[11−14]

On the other hand, Maillet and Sanchez de Santos[15] showed how the creation and annihilation operators of the six-vertex model can be simplified by using the Drinfeld twists or factorising $\boldsymbol{F}$-matrices.[16] This leads to the so-called $\boldsymbol{F}$-basis provided by the $\boldsymbol{F}$-matrix for the analysis of the model. In the $\boldsymbol{F}$-basis, the creation and annihilation operators take completely symmetric forms and contain no compensating exchange terms on the level of local operators (i.e. polarisation free). As a result, the computation of the DW partition function is simplified dramatically and can be directly calculated.[6] Such a magic $\boldsymbol{F}$-matrix has been obtained for other models.[17−25] In this paper, we shall use the the $\boldsymbol{F}$-basis provided by the $\boldsymbol{F}$-matrix for the spatial optical soliton (SOS) model associated with the eight-vertex model to compute the partition function for the eight-vertex model on an $N \times N$ square lattice with domain wall boundary condition.

This paper is organised as follows. In Section 2, we introduce our notation and some basic ingredients. In Section 3, using the vertex-face correspondence relation we express the DW partition function of the eight-vertex model in terms of the matrix element of the face type mondromy matrix. In Section 4, with help of the $\boldsymbol{F}$-basis provided by the $\boldsymbol{F}$-matrix for the SOS model associated with the eight-vertex model, we obtain the explicit expression of the DW partition function. In Section 5, we summarise our results and give some discussions.

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2. The eight-vertex model

In this section, we briefly review the DW boundary condition for the eight-vertex model on an $N \times N$ square lattice.[14]

2.1. The eight-vertex $\boldsymbol{R}$-matrix

Let us fix τ such that $\mathrm{Im}(\tau) > 0$ and a generic complex number η. Introduce the following elliptic functions

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (u, \tau) = \sum_{n=-\infty}^{\infty} \exp\left\{ \mathrm{i}\pi \left[(n+a)^2 \tau + 2(n+a)(u+b) \right] \right\}, \quad (1)$$

$$\boldsymbol{\theta}^{(j)}(u) = \theta \begin{bmatrix} \frac{1}{2} - \frac{j}{2} \\ \frac{1}{2} \end{bmatrix} (u, 2\tau), j = 1, 2;$$

$$\boldsymbol{\sigma}(u) = \theta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (u, \tau). \quad (2)$$

The σ-function[1)] satisfies the so-called Riemann identity:

$$\begin{aligned} &\sigma(u+x)\sigma(u-x)\sigma(v+y)\sigma(v-y) \\ &- \sigma(u+y)\sigma(u-y)\sigma(v+x)\sigma(v-x) \\ &= \sigma(u+v)\sigma(u-v)\sigma(x+y)\sigma(x-y), \end{aligned} \quad (3)$$

which will be useful in the following.

Let $\boldsymbol{V}$ be a two-dimensional vector space $\mathbb{C}^2$ and $\{\epsilon_i | i = 1, 2\}$ be the orthonormal basis of $\boldsymbol{V}$ such that $\langle \epsilon_i, \epsilon_j \rangle = \delta_{ij}$. The well-known eight-vertex model $\boldsymbol{R}$-matrix $\boldsymbol{R}(u) \in \mathrm{End}(\boldsymbol{V} \otimes \boldsymbol{V})$ is given by

$$\bar{\boldsymbol{R}}(u) = \begin{pmatrix} a(u) & & & d(u) \\ & b(u) & c(u) & \\ & c(u) & b(u) & \\ d(u) & & & a(u) \end{pmatrix}. \quad (4)$$

The non-vanishing matrix elements are[27]

$$\begin{aligned} a(u) &= \frac{\theta^{(1)}(u)\theta^{(0)}(u+\eta)\,\sigma(\eta)}{\theta^{(1)}(0)\theta^{(0)}(\eta)\sigma(u+\eta)}, \\ b(u) &= \frac{\theta^{(0)}(u)\,\theta^{(1)}(u+\eta)\,\sigma(\eta)}{\theta^{(1)}(0)\,\theta^{(0)}(\eta)\,\sigma(u+\eta)}, \\ c(u) &= \frac{\theta^{(1)}(u)\,\theta^{(1)}(u+\eta)\,\sigma(\eta)}{\theta^{(1)}(0)\,\theta^{(1)}(\eta)\,\sigma(u+\eta)}, \\ d(u) &= \frac{\theta^{(0)}(u)\,\theta^{(0)}(u+\eta)\,\sigma(\eta)}{\theta^{(1)}(0)\theta^{(1)}(\eta)\,\sigma(u+\eta)}. \end{aligned} \quad (5)$$

Here u is the spectral parameter and η is the so-called crossing parameter. The $\boldsymbol{R}$-matrix satisfies the quantum Yang–Baxter equation (QYBE)

$$\begin{aligned} &\bar{R}_{1,2}(u_1 - u_2)\bar{R}_{1,3}(u_1 - u_3)\bar{R}_{2,3}(u_2 - u_3) \\ &= \bar{R}_{2,3}(u_2 - u_3)\bar{R}_{1,3}(u_1 - u_3)\bar{R}_{1,2}(u_1 - u_2). \end{aligned} \quad (6)$$

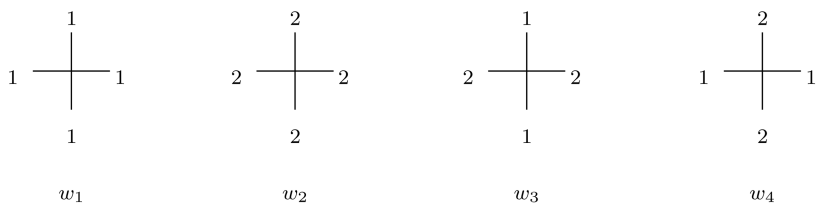
Throughout we adopt the standard notation: for any matrix $\boldsymbol{A} \in \mathrm{End}(\boldsymbol{V})$, A_j (or A^j)is an embedding operator in the tensor space $\boldsymbol{V} \otimes \boldsymbol{V} \otimes \cdots$, which acts as $\boldsymbol{A}$ on the j-th space and as identity on the other factor spaces; $R_{i,j}(u)$ is an embedding operator of $\boldsymbol{R}$-matrix in the tensor space, which acts as identity on the factor spaces except for the i-th and j-th ones.

2.2. The model

The partition function of a statistical model on a two-dimensional lattice is defined by the following

$$Z = \sum \exp\left\{ -\frac{E}{kT} \right\},$$

where E is the energy of the system, k is the Bolzmann constant, T is the temperature of the system, and the summation is taken over all possible configurations under the particular boundary condition such as the DW boundary condition. The model we consider here has eight allowed local vertex configurations



1) Our σ-function is the ϑ-function $\vartheta_1(u)$.[26] It has the following relation with the Weierstrassian σ-function $\sigma_w(u)$: $\sigma_w(u) \propto \mathrm{e}^{\eta_1 u^2}\sigma(u)$ with $\eta_1 = \pi^2\left(1/6 - 4\sum_{n=1}^{\infty}(nq^{2n}0/(1-q^{2n}))\right)$ and $q = \mathrm{e}^{\mathrm{i}\tau}$.

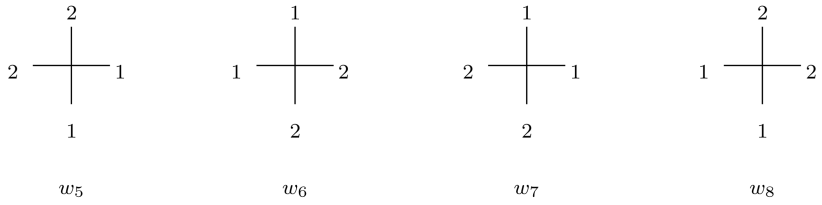


Fig. 1. Vertex configurations and their associated Boltzmann weights.

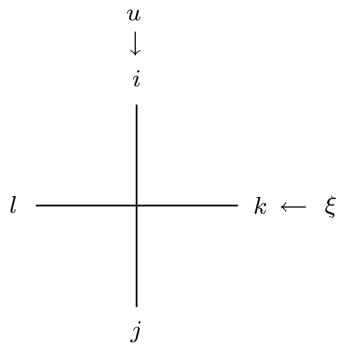
where 1 and 2 respectively denote the spin up and spin down states. Each of these eight configurations is assigned a statistical weight (or Boltzmann weight) w_i. Then the partition function can be rewritten as

$$Z = \sum {w_1}^{n_1} {w_2}^{n_2} {w_3}^{n_3} {w_4}^{n_4} {w_5}^{n_5} {w_6}^{n_6} {w_7}^{n_7} {w_8}^{n_8},$$

where the summation is over all possible vertex configurations with n_i being the number of the vertices of type i. If the local Boltzmann weights have Z_2-symmetry, i.e.,

$$\begin{aligned} &a \equiv w_1 = w_2, b \equiv w_3 = w_4, \\ &c \equiv w_5 = w_6, d \equiv w_7 = w_8, \end{aligned} \tag{7}$$

and the variables a, b, c, d satisfy a function relation, or equivalently, the local Boltzmann weights $\{w_i\}$ can be parameterised by the matrix elements of the eight-vertex $\boldsymbol{R}$-matrix in expressions (4) and (5) as in Fig. 2,



$$\xi \;=\; \bar{R}^{j\,l}_{i\,k}(u-\xi), \quad i,j,k,l = 1,2.$$

Fig. 2. The Bolzmann weights and elements of the eight-vertex $\boldsymbol{R}$-matrix.

then the corresponding model is called the eight-vertex model which can be exactly solved.[27] Therefore the partition function of the eight-vertex model is give by

$$Z = \sum a^{n_1+n_2}\, b^{n_3+n_4}\, c^{n_5+n_6}\, d^{n_7+n_8}.$$

In order to parameterise the local Boltzmann weights in terms of the elements of the $\boldsymbol{R}$-matrix, one needs to assign spectral parameters u and ξ respectively to the vertical line and horizontal line of each vertex of the lattice, as shown in Fig. 2. In an inhomogeneous model, the statistical weights are site-dependent. Hence two sets of spectral parameters $\{u_\alpha\}$ and $\{\xi_i\}$ are needed (see Fig. 3). The horizontal lines are enumerated by indices $1, \ldots, N$ with spectral parameters $\{\xi_i\}$, while the vertical lines are enumerated by indices $\bar{1}, \ldots, \bar{N}$ with spectral parameters $\{u_\alpha\}$. The DW boundary condition is specified by four boundary states $|\Omega^{(1)}(\lambda)\rangle$, $|\bar{\Omega}^{(1)}(\lambda)\rangle$, $\langle\Omega^{(2)}(\lambda+\eta N\hat{2})|$ and $\langle\bar{\Omega}^{(2)}(\lambda+\eta N\hat{2})|$ (the definitions of the boundary states will be given later, see expressions (26)–(29) below). These four states correspond to the particular choices of spin states on the four boundaries of the lattice. In contrast to the six-vertex case,[5] our boundary states not only depend on the spectral parameters ($|\Omega^{(1)}(\lambda)\rangle$ and $\langle\Omega^{(2)}(\lambda+\eta N\hat{2})|$ depend on $\{\xi_i\}$, while $|\bar{\Omega}^{(1)}(\lambda)\rangle$ and $\langle\bar{\Omega}^{(2)}(\lambda+\eta N\hat{2})|$ depend on $\{u_\alpha\}$), but also on two continuous parameters λ_1 and λ_2 (it is convenient to introduce a vector $\boldsymbol{\lambda} \in \boldsymbol{V}$ associated with these two parameters $\{\lambda_i\}$: $\lambda = \sum_{i=1}^{2} \lambda_i \epsilon_i$). However,

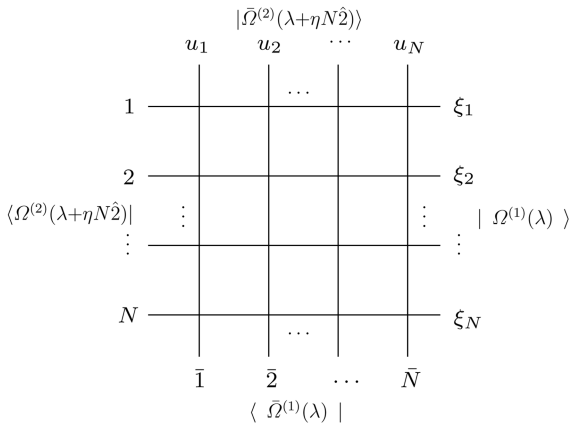


Fig. 3. The eight-vertex model with DW boundary condition.

in the trigonometric limit (i.e., setting $\lambda_2 = \tau/2$ and then taking $\tau \to +\mathrm{i}\infty$), the corresponding boundary states $|\Omega^{(1)}(\lambda)\rangle$ and $\langle\bar{\Omega}^{(1)}(\lambda)|$ (or $|\bar{\Omega}^{(2)}(\lambda+\eta N\hat{2})\rangle$ and $\langle\Omega^{(2)}(\lambda+\eta N\hat{2})|$) become the state of all spin up and

its dual (or the state of all spin down and its dual) up to some over-all scalar factors. Hence the partition function in the limit reduces to that of the six-vertex model.[1,3,4] Then, the partition function corresponding to the boundary condition given in Fig. 3 has been called the DW partition function of the eight-vertex model.[14]

The partition function of the eight-vertex model with DW boundary condition is a function of $2N+2$ variables $\{u_\alpha\}$, $\{\xi_i\}$, λ_1, and λ_2, which is denoted by $Z_N(\{u_\alpha\};\{\xi_i\};\lambda)$. Due to the fact that the local Boltzmann weights of each vertex of the lattice are given by the matrix elements of the eight-vertex $\boldsymbol{R}$-matrix (see Fig. 2), the partition function can be expressed in terms of the product of the $\boldsymbol{R}$-matrices and the four boundary states

$$\begin{aligned} & Z_N(\{u_\alpha\};\{\xi_i\};\lambda) \\ = {} & \langle \Omega^{(2)}(\lambda+\eta N\hat{2})|\langle \bar{\Omega}^{(1)}(\lambda)|\, \bar{R}_{\bar{1},N}(u_1-\xi_N)\dots \bar{R}_{\bar{1},1}(u_1-\xi_1)\dots \\ & \times \bar{R}_{\bar{N},N}(u_N-\xi_N)\dots \bar{R}_{\bar{N},1}(u_N-\xi_1)|\bar{\Omega}^{(2)}(\lambda+\eta N\hat{2})\rangle|\Omega^{(1)}(\lambda)\rangle. \end{aligned} \tag{8}$$

One can rearrange the product of the $\boldsymbol{R}$-matrices in Eq. (8) in terms of a product of the row-to-row monodromy matrices, namely,

$$Z_N(\{u_\alpha\};\{\xi_i\};\lambda) = \langle \Omega^{(2)}(\lambda+\eta N\hat{2})|\langle \bar{\Omega}^{(1)}(\lambda)|\, T_{\bar{1}}(u_1)\dots T_{\bar{N}}(u_N)\,|\bar{\Omega}^{(2)}(\lambda\eta N\hat{2})\rangle|\Omega^{(1)}(\lambda)\rangle, \tag{9}$$

where the monodromy matrix $\boldsymbol{T}_{\bar{i}}(u)$ is given by

$$\boldsymbol{T}_{\bar{i}}(u) \equiv T_{\bar{i}}(u;\xi_1,\dots,\xi_N) = \bar{R}_{\bar{i},N}(u-\xi_N)\ \dots \bar{R}_{\bar{i},1}(u-\xi_1). \tag{10}$$

The QYBE (6) of the $\boldsymbol{R}$-matrix gives rise to the so-called 'RLL' relation satisfied by the monodromy matrix $\boldsymbol{T}_{\bar{i}}(u)$,

$$\bar{R}_{\bar{i},\bar{j}}(u_i-u_j)\, T_{\bar{i}}(u_i)\, T_{\bar{j}}(u_j) = T_{\bar{j}}(u_j)\, T_{\bar{i}}(u_i)\, \bar{R}_{\bar{i},\bar{j}}(u_i-u_j). \tag{11}$$

2.3. The boundary states

From the orthonormal basis $\{\epsilon_i\}$ of $\boldsymbol{V}$, we define

$$\begin{aligned} & \hat{i} = \epsilon_i - \bar{\epsilon}, \quad \bar{\epsilon} = \frac{1}{2}\sum_{k=1}^{2}\epsilon_k, \\ & i=1,2, \quad \text{then} \quad \sum_{k=1}^{2}\hat{k} = 0. \end{aligned} \tag{12}$$

Let $\mathfrak{h}$ be the Cartan subalgebra of A_1 and $\mathfrak{h}^*$ be its dual. A finite dimensional diagonalisable $\mathfrak{h}$-module is a complex finite dimensional vector space $\boldsymbol{W}$ with a weight decomposition $\boldsymbol{W} = \oplus_{\mu\in\mathfrak{h}^*}\boldsymbol{W}[\mu]$, so that $\mathfrak{h}$ acts on $\boldsymbol{W}[\mu]$ by $x\,v = \mu(x)\,v$, ($x\in\mathfrak{h}$, $v\in\boldsymbol{W}[\mu]$). For example, the non-zero weight spaces of the fundamental representation $V_{\Lambda_1} = \mathbb{C}^2 = \boldsymbol{V}$ are

$$W[\hat{i}] = \mathbb{C}\epsilon_i, \quad i=1,2. \tag{13}$$

For a generic $m\in\boldsymbol{V}$, define

$$\begin{aligned} & m_i = \langle m, \epsilon_i\rangle, \\ & m_{ij} = m_i - m_j = \langle m, \epsilon_i-\epsilon_j\rangle, \quad i,j=1,2. \end{aligned} \tag{14}$$

Let $\boldsymbol{R}(u;m)\in \mathrm{End}(\boldsymbol{V}\otimes\boldsymbol{V})$ be the $\boldsymbol{R}$-matrix of the SOS model associated with the eight-vertex model[27] given by

$$\begin{aligned} & R(u;m) \\ = {} & \sum_{i=1}^{2} R(u;m)_{ii}^{ii} E_{ii}\otimes E_{ii} + \sum_{i\neq j}^{2}\Big\{ R(u;m)_{ij}^{ij} E_{ii}\otimes E_{jj} \\ & + R(u;m)_{ij}^{ji} E_{ji}\otimes E_{ij}\Big\}, \end{aligned} \tag{15}$$

where E_{ij} is the matrix with elements $(E_{ij})_k^l = \delta_{jk}\delta_{il}$. The coefficient functions are

$$R(u;m)_{ii}^{ii} = 1, \tag{16}$$

$$R(u;m)_{ij}^{ij} = \frac{\sigma(u)\sigma(m_{ij}-\eta)}{\sigma(u+\eta)\sigma(m_{ij})}, \quad i\neq j, \tag{17}$$

$$R(u;m)_{ij}^{ji} = \frac{\sigma(\eta)\sigma(u+m_{ij})}{\sigma(u+\eta)\sigma(m_{ij})}, \quad i\neq j, \tag{18}$$

and m_{ij} is defined in Eq. (14). The $\boldsymbol{R}$-matrix satisfies the dynamical (modified) quantum Yang–Baxter equation (or the star-triangle relation)[27]

$$R_{1,2}(u_1-u_2;m-\eta h^{(3)})R_{1,3}(u_1-u_3;m)$$

$$\times R_{2,3}(u_2-u_3;m-\eta h^{(1)})$$
$$= R_{2,3}(u_2-u_3;m)R_{1,3}(u_1-u_3;m-\eta h^{(2)})$$
$$\times R_{1,2}(u_1-u_2;m). \quad (18)$$

Here we have adopted

$$R_{1,2}(u,m-\eta h^{(3)})\, v_1\otimes v_2\otimes v_3$$
$$=(R(u,m-\eta\mu)\otimes \mathrm{id})\, v_1\otimes v_2\otimes v_3, \text{ if } v_3\in W[\mu]. \quad (19)$$

Moreover, one may check that the $\boldsymbol{R}$-matrix satisfies weight conservation condition,

$$\left[h^{(1)}+h^{(2)},\, R_{1,2}(u;m)\right]=0 \quad (20)$$

and unitary condition

$$R_{1,2}(u;m)\, R_{2,1}(-u;m)=\mathrm{id}\otimes\mathrm{id}. \quad (21)$$

Let us introduce two intertwiners which are 2-component column-vectors $\phi_{m,m-\eta\hat{j}}(u)$ labelled by $\hat{j}=\hat{1},\,\hat{2}$. The k-th element of $\phi_{m,m-\eta\hat{j}}(u)$ is given by

$$\phi^{(k)}_{m,m-\eta\hat{j}}(u)=\theta^{(k)}(u+2m_j). \quad (22)$$

Explicitly,

$$\phi_{m,m-\eta\hat{1}}(u)=\begin{pmatrix}\theta^{(1)}(u+2m_1)\\ \theta^{(2)}(u+2m_1)\end{pmatrix},$$
$$\phi_{m,m-\eta\hat{2}}(u)=\begin{pmatrix}\theta^{(1)}(u+2m_2)\\ \theta^{(2)}(u+2m_2)\end{pmatrix}.$$

It is easy to check that these two intertwiner vectors $\boldsymbol{\phi}_{m,m-\eta\hat{i}}(u)$ are linearly independent for a generic $m\in\boldsymbol{V}$.

Using the intertwiner vectors, one can derive the following vertex-face correspondence relation[27–29]

$$\bar{R}_{1,2}(u_1-u_2)\phi^1_{m,m-\eta\hat{i}}(u_1)\,\phi^2_{m-\eta\hat{i},m-\eta(\hat{i}+\hat{j})}(u_2)$$
$$=\sum_{k,l}R(u_1-u_2;m)^{kl}_{ij}\phi^1_{m-\eta\hat{l},m-\eta(\hat{l}+\hat{k})}$$
$$\times(u_1)\,\phi^2_{m,m-\eta\hat{l}}(u_2). \quad (23)$$

Hereafter we adopt the convention: $\phi^1=\phi\otimes\mathrm{id}\otimes\ldots$, $\phi^2=\mathrm{id}\otimes\phi\otimes\mathrm{id}\otimes\ldots$, etc. The QYBE (6) of the vertex-type $\boldsymbol{R}$-matrix $\bar{\boldsymbol{R}}(u)$ is equivalent to the dynamical Yang–Baxter equation (18) of the SOS $\boldsymbol{R}$-matrix $\boldsymbol{R}(u;m)$. For a generic m, we can introduce two row-vector intertwiners $\tilde{\phi}$ satisfying the conditions,

$$\tilde{\phi}_{m+\eta\hat{\mu},m}(u)\phi_{m+\eta\hat{\nu},m}(u)=\delta_{\mu\nu},\ \mu,\nu=1,2, \quad (24)$$

from which one derives the relation

$$\sum_{\mu=1}^{2}\phi_{m+\eta\hat{\mu},m}(u)\,\tilde{\phi}_{m+\eta\hat{\mu},m}(u)=\mathrm{id}. \quad (25)$$

Now we are in the position to construct the boundary states to specify the DW boundary condition of the eight-vertex model (see Fig. 3). For any vector $m\in\boldsymbol{V}$, we introduce four states[2)] which live in the two N-tensor spaces of $\boldsymbol{V}$ (one is indexed by $1,\ldots,N$, and the other is indexed by $\bar{1},\ldots,\bar{N}$) or their dual spaces as follows:

$$|\Omega^{(i)}(m)\rangle = \phi^1_{m,m-\eta\hat{i}}(\xi_1)\phi^2_{m-\eta\hat{i},m-2\eta\hat{i}}(\xi_2)\ldots\phi^N_{m-\eta(N-1)\hat{i},m-\eta N\hat{i}}(\xi_N), i=1,2, \quad (26)$$

$$|\bar{\Omega}^{(i)}(m)\rangle = \phi^{\bar{1}}_{m,m-\eta\hat{i}}(u_1)\phi^{\bar{2}}_{m-\eta\hat{i},m-2\eta\hat{i}}(u_2)\ldots\phi^{\bar{N}}_{m-\eta(N-1)\hat{i},m-\eta N\hat{i}}(u_N), i=1,2, \quad (27)$$

$$\langle\Omega^{(i)}(m)| = \tilde{\phi}^1_{m,m-\eta\hat{i}}(\xi_1)\tilde{\phi}^2_{m-\eta\hat{i},m-2\eta\hat{i}}(\xi_2)\ldots\tilde{\phi}^N_{m-\eta(N-1)\hat{i},m-\eta N\hat{i}}(\xi_N), i=1,2, \quad (28)$$

$$\langle\bar{\Omega}^{(i)}(m)| = \tilde{\phi}^{\bar{1}}_{m,m-\eta\hat{i}}(u_1)\tilde{\phi}^{\bar{2}}_{m-\eta\hat{i},m-2\eta\hat{i}}(u_2)\ldots\tilde{\phi}^{\bar{N}}_{m-\eta(N-1)\hat{i},m-\eta N\hat{i}}(u_N), i=1,2. \quad (29)$$

The boundary states which have been used to define the DW boundary condition can be obtained through the above states by special choices of m and i (for example, m is specified to λ or $\lambda+\eta N\hat{2}$). Then the DW partition function $Z_N(\{u_\alpha\};\{\xi_i\};\lambda)$ given by Eq. (8) becomes

$$Z_N(\{u_\alpha\};\{\xi_i\};\lambda)$$
$$=\tilde{\phi}^1_{\lambda+\eta N\hat{2},\lambda+\eta(N-1)\hat{2}}(\xi_1)\ldots\tilde{\phi}^N_{\lambda+\eta\hat{2},\lambda}(\xi_N)\ \tilde{\phi}^{\bar{1}}_{\lambda,\lambda-\eta\hat{1}}(u_1)\ldots\tilde{\phi}^{\bar{N}}_{\lambda-\eta(N-1)\hat{1},\lambda-\eta N\hat{1}}(u_N)$$
$$\times R_{\bar{1},N}(u_1-\xi_N)\ldots R_{\bar{1},1}(u_1-\xi_1)\ldots\ R_{\bar{N},N}(u_N-\xi_N)\ldots R_{\bar{N},1}(u_N-\xi_1)$$

[2)] Among them, $|\Omega^{(i)}(m)\rangle$, with special choice of m, are the complete reference states of the open XYZ spin chain,[30,31] and have played an important role in constructing the extra centre elements of the elliptic algebra at roots of unity.[32,33]

$$\times\ \phi^{\bar{1}}_{\lambda+\eta N\hat{2},\lambda+\eta(N-1)\hat{2}}(u_1)\dots\phi^{\bar{N}}_{\lambda+\eta\hat{2},\lambda}(u_N)\ \phi^{1}_{\lambda,\lambda-\eta\hat{1}}(\xi_1)\dots\phi^{N}_{\lambda-\eta(N-1)\hat{1},\lambda-\eta N\hat{1}}(\xi_N). \tag{30}$$

3. The partition function in terms of the face type monodromy matrix

Similarly to the vertex type of monodromy matrix (10), let us introduce the face type monodromy matrix by using the $\boldsymbol{R}$-matrix (15)

$$\begin{aligned}\boldsymbol{T}_F(l|u) &\equiv T^F_{0,1\dots N}(l|u)\\ &= R_{0,N}(u-z_N;l-\eta\sum_{i=1}^{N-1}h^{(i)})\dots R_{0,2}(u-z_2;l-\eta h^{(1)})R_{0,1}(u-z_1;l),\\ &= \begin{pmatrix} T_F(l|u)^1_1 & T_F(l|u)^1_2\\ T_F(l|u)^2_1 & T_F(l|u)^2_2\end{pmatrix}\end{aligned} \tag{31}$$

where l is a generic vector in $\boldsymbol{V}$. The monodromy matrix satisfies the face type quadratic exchange relation.[34–37] Applying $\boldsymbol{T}_F(l|u)^i_j$ to an arbitrary vector $|i_1,\dots,i_N\rangle$ in the N-tensor product space $\boldsymbol{V}^{\otimes N}$ given by

$$|i_1,\dots,i_N\rangle = \epsilon^1_{i_1}\dots\epsilon^N_{i_N}, \tag{32}$$

we have

$$\begin{aligned}&\boldsymbol{T}_F(l|u)^i_j|i_1,\dots,i_N\rangle\\ \equiv\ &\boldsymbol{T}_F(m;l|u)^i_j|i_1,\dots,i_N\rangle\\ =\ &\sum_{\alpha_{N-1}\dots\alpha_1}\sum_{i'_N\dots i'_1}R\left(u-z_N;l-\eta\sum_{k=1}^{N-1}\hat{i}'_k\right)^{i\qquad i'_N}_{\alpha_{N-1}\ i_N}\dots R(u-z_2;l-\eta\hat{i}'_1)^{\alpha_2\ i'_2}_{\alpha_1\ i_2}R(u-z_1;l)^{\alpha_1\ i'_1}_{j\ i_1}\ |i'_1,\dots,i'_N\rangle,\end{aligned} \tag{33}$$

where $m = l-\eta\sum_{k=1}^N\hat{i}_k$. Associated with the vertex type monodromy matrix $\boldsymbol{T}(u)$ (10), we introduce the following operator

$$T(m,l|u)^j_\mu = \tilde{\phi}^0_{m+\eta\hat{\jmath},m}(u)\,T_0(u)\,\phi^0_{l+\eta\hat{\mu},l}(u). \tag{34}$$

Moreover, for the case of $m = l-\eta\sum_{k=1}^N\hat{i}_k$, we introduce a generic state in the quantum space from the intertwiner vector (22)

$$|i_1,\dots,i_N\rangle^m_l = \phi^1_{l,l-\eta\hat{i}_1}(z_1)\phi^2_{l-\eta\hat{i}_1,l-\eta(\hat{i}_1+\hat{i}_2)}(z_2)\dots\phi^N_{l-\eta\sum_{k=1}^{N-1}\hat{i}_k,l-\eta\sum_{k=1}^N\hat{i}_k}(z_N). \tag{35}$$

We can evaluate the action of the operator $T(m,l|u)$ on the state $|i_1,\dots,i_N\rangle^m_l$ from the face-vertex correspondence relation (23)

$$\begin{aligned}T(m,l|u)^j_\mu|i_1,\dots,i_N\rangle^m_l &= \tilde{\phi}^0_{m+\eta\hat{\jmath},m}(u)\,T_0(u)\,\phi^0_{l+\eta\hat{\mu},l}(u)|i_1,\dots,i_N\rangle^m_l\\ &= \tilde{\phi}^0_{m+\eta\hat{\jmath},m}(u)\bar{R}_{0,N}(u-z_N)\dots\bar{R}_{0,1}(u-z_1)\phi^0_{l+\eta\hat{\mu},l}(u)\phi^1_{l,l-\eta\hat{i}_1}(z_1)\dots\\ &= \sum_{\alpha_1,i'_1}R(u-z_1;l+\eta\hat{\mu})^{\alpha_1\,i'_1}_{\mu\ i_1}\phi^1_{l+\eta\hat{\mu},l+\eta\hat{\mu}-\eta\hat{i}'_1}(z_1)\tilde{\phi}^0_{m+\eta\hat{\jmath},m}(u)\bar{R}_{0,N}(u-z_N)\dots\\ &\quad\times\bar{R}_{0,2}(u-z_2)\phi^0_{l+\eta\hat{\mu}-\eta\hat{i}'_1,l-\eta\hat{i}_1}(u)\phi^2_{l-\eta\hat{i}_1,l-\eta(\hat{i}_1+\hat{i}_2)}(z_2)\dots\\ &\quad\vdots\\ &= \sum_{\alpha_1\dots\alpha_{N-1}}\sum_{i'_1\dots i'_N}R(u-z_N;l+\eta\hat{\mu}-\eta\sum_{k=1}^{N-1}\hat{i}'_k)^{j\qquad i'_N}_{\alpha_{N-1}\qquad i_N}\dots\\ &\quad\times R(u-z_1;l+\eta\hat{\mu})^{\alpha_1\,i'_1}_{\mu\ i_1}|i'_1,\dots,i'_N\rangle^{l+\eta\hat{\mu}-\eta\sum_{k=1}^N\hat{i}'_k}_{l+\eta\hat{\mu}}.\end{aligned} \tag{36}$$

Here we have used the following property of the $\boldsymbol{R}$-matrix

$$R(u;m)^{i'\,j'}_{ij} = R(u;m\pm\eta(\hat{\imath}+\hat{\jmath}))^{i'\,j'}_{ij} = R(u;m\pm\eta(\hat{\imath}'+\hat{\jmath}'))^{i'\,j'}_{ij}, \tag{37}$$

and the weight conservation condition (20).

Now we compute the partition function $Z_N(\{u_\alpha\};\{\xi_i\};\lambda)$

$$\begin{aligned} Z_N(\{u_\alpha\};\{\xi_i\};\lambda) &= \langle\Omega^{(2)}(\lambda+\eta N\hat{2})|\,\tilde{\phi}^{\bar{1}}_{\lambda,\lambda-\eta\hat{1}}(u_1)\,T_{\bar{1}}(u_1)\,\phi^{\bar{1}}_{\lambda+\eta N\hat{2},\lambda+\eta(N-1)\hat{2}}(u_1)\dots \\ &\quad\times\tilde{\phi}^{\bar{N}}_{\lambda-\eta(N-1)\hat{1},\lambda-\eta N\hat{1}}(u_N)\,T_{\bar{N}}(u_N)\,\phi^{\bar{N}}_{\lambda+\eta\hat{2},\lambda}(u_N)|\Omega^{(1)}(\lambda)\rangle \\ &= \langle\Omega^{(2)}(\lambda+\eta N\hat{2})|\,T(\lambda-\eta\hat{1};\lambda+\eta(N-1)\hat{2}|u_1)^1_2\dots\times\ T(\lambda-\eta N\hat{1};\lambda|u_N)^1_2\,|\Omega^{(1)}(\lambda)\rangle. \end{aligned}$$

Then Eq. (36) allows us to express the partition function in terms of the matrix element of the face-type monodromy matrix given by Eq. (33) as follows

$$\begin{aligned} Z_N(\{u_\alpha\};\{\xi_i\};\lambda) &= \langle 2,\dots,2|\,T_F(\lambda-\eta\hat{1}+\eta\hat{2};\lambda+\eta(N-1)\hat{2}+\eta\hat{2}|u_1)^1_2\dots \\ &\quad\times T_F(\lambda-\eta N\hat{1}+\eta\hat{2};\lambda+\eta\hat{2}|u_N)^1_2\,|1,\dots,1\rangle. \end{aligned} \tag{38}$$

4. The Drinfeld twist and the DW partition function

In this section, after briefly reviewing the result in Ref. [19] about the Drinfeld twist[16] (factorising $\boldsymbol{F}$-matrix) of the SOS model associated with the eight-vertex model, we directly calculate the partition function $Z_N(\{u_\alpha\};\{\xi_i\};\lambda)$ for the eight-vertex model on an $N\times N$ square lattice with domain wall boundary condition.

4.1. Factorising drinfeld twist $\boldsymbol{F}$

Let $\mathcal{S}_N$ be the permutation group over indices $1,\dots,N$, and $\{\sigma_i|i=1,\dots,N-1\}$ be the set of elementary permutations in $\mathcal{S}_N$. For each elementary permutation σ_i, we introduce the associated operator $R^{\sigma_i}_{1\dots N}$ on the quantum space

$$\begin{aligned} &R^{\sigma_i}_{1\dots N}(l) \\ &\equiv R^{\sigma_i}(l) \\ &= R_{i,i+1}\left(\xi_i-\xi_{i+1};l-\eta\sum_{k=1}^{i-1}h^{(k)}\right), \end{aligned} \tag{39}$$

where l is a generic vector in V. For any σ, $\sigma'\in\mathcal{S}_N$, operator $R^{\sigma\sigma'}_{1\dots N}$ associated with $\sigma\sigma'$ satisfies the following composition law[15,18,19,24,25]

$$R^{\sigma\sigma'}_{1\dots N}(l)=R^{\sigma'}_{\sigma(1\dots N)}(l)\,R^{\sigma}_{1\dots N}(l). \tag{40}$$

Let σ be decomposed in a minimal way in terms of elementary permutations,

$$\sigma=\sigma_{\beta_1}\dots\sigma_{\beta_p}, \tag{41}$$

where $\beta_i=1,\dots,N-1$, and the positive integer p is the length of σ. The composition law (40) enables one to obtain operator $R^{\sigma}_{1\dots N}$ associated with each $\sigma\in\mathcal{S}_N$. The dynamical quantum Yang–Baxter equation (18), weight conservation condition (20) and unitary condition (21) guarantee the uniqueness of $R^{\sigma}_{1\dots N}$. Moreover, one may check that $R^{\sigma}_{1\dots N}$ satisfies the following exchange relation with the face-type one-row monodromy matrix (31)

$$\begin{aligned} &R^{\sigma}_{1\dots N}(l)T^F_{0,1\dots N}(l|u) \\ &=T^F_{0,\sigma(1\dots N)}(l|u)R^{\sigma}_{1\dots N}(l-\eta h^{(0)}),\ \ \forall\sigma\in\mathcal{S}_N. \end{aligned} \tag{42}$$

Now, we construct the face-type Drinfeld twist $F_{1\dots N}(l)\equiv F_{1\dots N}(l;\xi_1,\dots,\xi_N)$[3] on the N-fold tensor product space $V^{\otimes N}$, which satisfies the following three properties[18,24,25]

I. lower-triangularity; (43)

II. non-degeneracy; (44)

III. factorising property : $R^{\sigma}_{1\dots N}(l)=F^{-1}_{\sigma(1\dots N)}(l)F_{1\dots N}(l),\ \forall\sigma\in\mathcal{S}_N.$ (45)

Substituting Eq. (45) into the exchange relation (42), we have

$$F^{-1}_{\sigma(1\dots N)}(l)F_{1\dots N}(l)T^F_{0,1\dots N}(l|u)=T^F_{0,\sigma(1\dots N)}(l|u)F^{-1}_{\sigma(1\dots N)}(l-\eta h^{(0)})F_{1\dots N}(l-\eta h^{(0)}). \tag{46}$$

[3] In this paper, we adopt the convention: $F_{\sigma(1\dots N)}(l)\equiv F_{\sigma(1\dots N)}(l;\xi_{\sigma(1)},\dots,\xi_{\sigma(N)})$.

Equivalently,

$$F_{1\ldots N}(l)T^F_{0,1\ldots N}(l|u)F^{-1}_{1\ldots N}(l-\eta h^{(0)}) = F_{\sigma(1\ldots N)}(l)T^F_{0,\sigma(1\ldots N)}(l|u)F^{-1}_{\sigma(1\ldots N)}(l-\eta h^{(0)}). \tag{47}$$

Let us introduce the twisted monodromy matrix $\tilde{\boldsymbol{T}}^F_{0,1\ldots N}(l|u)$ by using

$$\tilde{\boldsymbol{T}}^F_{0,1\ldots N}(l|u) = F_{1\ldots N}(l)T^F_{0,1\ldots N}(l|u)F^{-1}_{1\ldots N}(l-\eta h^{(0)}) = \begin{pmatrix} \tilde{T}_F(l|u)^1_1 & \tilde{T}_F(l|u)^1_2 \\ \tilde{T}_F(l|u)^2_1 & \tilde{T}_F(l|u)^2_2 \end{pmatrix}. \tag{48}$$

Then Eq. (47) implies that the twisted monodromy matrix is symmetric under $\mathcal{S}_N$, namely,

$$\tilde{T}^F_{0,1\ldots N}(l|u) = \tilde{T}^F_{0,\sigma(1\ldots N)}(l|u), \forall\sigma\in\mathcal{S}_N. \tag{49}$$

Define the $\boldsymbol{F}$-matrix

$$\boldsymbol{F}_{1\ldots N}(l) = \sum_{\sigma\in\mathcal{S}_N}\sum_{\{\alpha_j\}=1}^{2\,*}\prod_{j=1}^{N} P^{\sigma(i)}_{\alpha_{\sigma(i)}}\, R^{\sigma}_{1\ldots N}(l), \tag{50}$$

where P^i_α is the embedding of the project operator P_α in the ith space with matric elements $(P_\alpha)_{kl} = \delta_{kl}\delta_{k\alpha}$. The sum $\sum^*$ in Eq. (50) is taken over all non-decreasing sequences of the labels $\alpha_{\sigma(i)}$

$$\begin{aligned} &\alpha_{\sigma(i+1)} \geq \alpha_{\sigma(i)} \text{ if } \sigma(i+1) > \sigma(i), \\ &\alpha_{\sigma(i+1)} > \alpha_{\sigma(i)} \text{ if } \sigma(i+1) < \sigma(i). \end{aligned} \tag{51}$$

From Eq. (51), $\boldsymbol{F}_{1\ldots N}(l)$ obviously is a lower-triangular matrix. Moreover, the $\boldsymbol{F}$-matrix is non-degenerate because all its diagonal elements are non-zero. One may check that the $\boldsymbol{F}$-matrix given by Eq. (50) indeed satisfies the factorising property Eq. (45).

In the $\boldsymbol{F}$-basis, the twisted operators $\tilde{T}_F(l|u)^j_i$ defined by Eq. (48) become[19,38]

$$\tilde{\boldsymbol{T}}_F(l|u)^2_2 = \frac{\sigma(l_{21}-\eta)}{\sigma\left(l_{21}-\eta+\eta\langle H,\epsilon_1\rangle\right)} \otimes_i \begin{pmatrix} \dfrac{\sigma(u-\xi_i)}{\sigma(u-\xi_i+\eta)} & \\ & 1 \end{pmatrix}_{(i)}, \tag{52}$$

$$\tilde{\boldsymbol{T}}_F(l|u)^2_1 = \sum_{i=1}^{N} \frac{\sigma(\eta)\sigma(u-\xi_i+l_{12})}{\sigma(u-\xi_i+\eta)\sigma(l_{12})} E^i_{12} \otimes_{j\neq i} \begin{pmatrix} \dfrac{\sigma(u-\xi_j)\sigma(\xi_i-\xi_j+\eta)}{\sigma(u-\xi_j+\eta)\sigma(\xi_i-\xi_j)} & \\ & 1 \end{pmatrix}_{(j)}, \tag{53}$$

$$\begin{aligned} \tilde{\boldsymbol{T}}_F(l|u)^1_2 = {} & \frac{\sigma(l_{21}-\eta)}{\sigma(l_{21}+\eta\langle H,\epsilon_1-\epsilon_2\rangle)} \sum_{i=1}^{N} \frac{\sigma(\eta)\sigma(u-\xi_i+l_{21}+\eta+\eta\langle H,\epsilon_1-\epsilon_2\rangle)}{\sigma(u-\xi_i+\eta)\sigma(l_{21}+\eta+\eta\langle H,\epsilon_1-\epsilon_2\rangle)} \\ & \times E^i_{21} \otimes_{j\neq i} \begin{pmatrix} \dfrac{\sigma(u-\xi_j)}{\sigma(u-\xi_j+\eta)} & \\ & \dfrac{\sigma(\xi_j-\xi_i+\eta)}{\sigma(\xi_j-\xi_i)} \end{pmatrix}_{(j)}, \end{aligned} \tag{54}$$

where $H = \sum_{k=1}^N h^{(k)}$. Applying the above operators to the arbitrary state $|i_1,\ldots,i_N\rangle$ given by Eq. (32), we find that the pseudo-particle creation/annihilation operators in the $\boldsymbol{F}$-basis have the following completely symmetric polarisation free forms:

$$\tilde{\boldsymbol{T}}_F(m,l|u)^2_2 = \frac{\sigma(l_{21}-\eta)}{\sigma\left(l_2-m_1-\eta\right)} \otimes_i \begin{pmatrix} \dfrac{\sigma(u-\xi_i)}{\sigma(u-\xi_i+\eta)} & \\ & 1 \end{pmatrix}_{(i)}, \tag{55}$$

$$\tilde{\boldsymbol{T}}_F(m,l|u)^2_1 = \sum_{i=1}^{N} \frac{\sigma(\eta)\sigma(u-\xi_i+l_{12})}{\sigma(u-\xi_i+\eta)\sigma(l_{12})} \times E^i_{12} \otimes_{j\neq i} \begin{pmatrix} \dfrac{\sigma(u-\xi_j)\sigma(\xi_i-\xi_j+\eta)}{\sigma(u-\xi_j+\eta)\sigma(\xi_i-\xi_j)} & \\ & 1 \end{pmatrix}_{(j)}, \tag{56}$$

$$\tilde{\boldsymbol{T}}_F(m,l|u)_2^1 = \frac{\sigma(l_{21}-\eta)}{\sigma(m_{21}-2\eta)}\sum_{i=1}^{N}\frac{\sigma(\eta)\sigma(u-\xi_i+m_{21}-\eta)}{\sigma(u-\xi_i+\eta)\sigma(m_{21}-\eta)} \times E_{21}^i \otimes_{j\neq i}\begin{pmatrix} \dfrac{\sigma(u-\xi_j)}{\sigma(u-\xi_j+\eta)} & \\ & \dfrac{\sigma(\xi_j-\xi_i+\eta)}{\sigma(\xi_j-\xi_i)} \end{pmatrix}_{(j)}. \tag{57}$$

4.2. The DW partition function

Now we compute the DW partition function $Z_N(\{u_\alpha\};\{\xi_i\};\lambda)$ from Eq. (38) and the expansion (57) of the twisted operator $\tilde{\boldsymbol{T}}_F(m,l|u)_2^1$. It is easy to check that the states $|i,\ldots,i\rangle$ and their dual $\langle i,\ldots,i|$ $(i=1,2)$ are invariant under the action of the $\boldsymbol{F}$-matrix $F_{1\ldots N}(l)$ given by Eq. (50), namely,

$$F_{1\ldots N}(l)|i,\ldots,i\rangle = |i,\ldots,i\rangle, \quad i=1,2, \tag{58}$$

$$\langle i,\ldots,i|F_{1\ldots N}(l) = \langle i,\ldots,i|, \quad i=1,2. \tag{59}$$

Hence the DW partition function $Z_N(\{u_\alpha\};\{\xi_j\};\lambda)$ can be expressed in terms of the twisted operator $\tilde{\boldsymbol{T}}_F(m,l|u)_2^1$ as follows:

$$\begin{aligned}
&\boldsymbol{Z}_N(\{u_\alpha\};\{\xi_i\};\lambda)\\
&= \langle 2,\ldots,2|\, T_F(\lambda-\eta\hat{1}+\eta\hat{2};\lambda+\eta(N-1)\hat{2}+\eta\hat{2}|u_1)_2^1\ldots\times T_F(\lambda-\eta N\hat{1}+\eta\hat{2};\lambda+\eta\hat{2}|u_N)_2^1\,|1,\ldots,1\rangle\\
&= \langle 2,\ldots,2|\, F_{1\ldots N}(\lambda+\eta N\hat{2}+\eta\hat{2})\times T_F(\lambda-\eta\hat{1}+\eta\hat{2};\lambda+\eta(N-1)\hat{2}+\eta\hat{2}|u_1)_2^1\ldots\\
&\quad\times T_F(\lambda-\eta N\hat{1}+\eta\hat{2};\lambda+\eta\hat{2}|u_N)_2^1 F_{1\ldots N}^{-1}(\lambda+\eta\hat{2})\,|1,\ldots,1\rangle\\
&= \langle 2,\ldots,2|\,\tilde{T}_F(\lambda-\eta\hat{1}+\eta\hat{2};\lambda+\eta(N-1)\hat{2}+\eta\hat{2}|u_1)_2^1\ldots\times\tilde{T}_F(\lambda-\eta N\hat{1}+\eta\hat{2};\lambda+\eta\hat{2}|u_N)_2^1\,|1,\ldots,1\rangle.
\end{aligned} \tag{60}$$

Substituting Eq. (57) of the twisted $\tilde{\boldsymbol{T}}_F(m,l|u)_2^1$ into the above equation, we have

$$\begin{aligned}
&\boldsymbol{Z}_N(\{u_\alpha\};\{\xi_i\};\lambda)\\
&= \langle 2,\ldots,2|\times\sum_{i=1}^{N}\frac{\sigma(\eta)\sigma(u_1-\xi_i+\lambda_{21}+\eta)}{\sigma(u_1-\xi_i+\eta)\sigma(\lambda_{21}+\eta)}E_{21}^i\otimes_{j\neq i}\begin{pmatrix} \dfrac{\sigma(u_1-\xi_j)}{\sigma(u_1-\xi_j+\eta)} & \\ & \dfrac{\sigma(\xi_j-\xi_i+\eta)}{\sigma(\xi_j-\xi_i)}\end{pmatrix}_{(j)}\\
&\quad\vdots\\
&\quad\times\sum_{i=1}^{N}\frac{\sigma(\eta)\sigma(u_l-\xi_i+\lambda_{21}+l\eta)}{\sigma(u_l-\xi_i+\eta)\sigma(\lambda_{21}+l\eta)}E_{21}^i\otimes_{j\neq i}\begin{pmatrix} \dfrac{\sigma(u_l-\xi_j)}{\sigma(u_l-\xi_j+\eta)} & \\ & \dfrac{\sigma(\xi_j-\xi_i+\eta)}{\sigma(\xi_j-\xi_i)}\end{pmatrix}_{(j)}\\
&\quad\vdots\\
&\quad\times\sum_{i=1}^{N}\frac{\sigma(\eta)\sigma(u_N-\xi_i+\lambda_{21}+N\eta)}{\sigma(u_N-\xi_i+\eta)\sigma(\lambda_{21}+N\eta)}E_{21}^i\otimes_{j\neq i}\begin{pmatrix} \dfrac{\sigma(u_N-\xi_j)}{\sigma(u_N-\xi_j+\eta)} & \\ & \dfrac{\sigma(\xi_j-\xi_i+\eta)}{\sigma(\xi_j-\xi_i)}\end{pmatrix}_{(j)}\times|1,\ldots 1\rangle.
\end{aligned} \tag{61}$$

Expanding the last sum term of the above equation corresponding to the spectral parameter u_N yields

$$\begin{aligned}
&\boldsymbol{Z}_N(\{u_\alpha\};\{\xi_i\};\lambda)\\
&= \sum_{i=1}^{N}\frac{\sigma(\eta)\sigma(u_N-\xi_i+\lambda_{21}+N\eta)}{\sigma(u_N-\xi_i+\eta)\sigma(\lambda_{21}+N\eta)}\prod_{j\neq i}\frac{\sigma(u_N-\xi_j)\sigma(\xi_i-\xi_j+\eta)}{\sigma(u_N-\xi_j+\eta)\sigma(\xi_i-\xi_j)}\langle 2,\ldots,2|\\
&\quad\times\sum_{l\neq i}^{N}\frac{\sigma(\eta)\sigma(u_1-\xi_l+\lambda_{21}+\eta)}{\sigma(u_1-\xi_l+\eta)\sigma(\lambda_{21}+\eta)}E_{21}^l\otimes_{j\neq l,i}\begin{pmatrix} \dfrac{\sigma(u_1-\xi_j)}{\sigma(u_1-\xi_j+\eta)} & \\ & \dfrac{\sigma(\xi_j-\xi_l+\eta)}{\sigma(\xi_j-\xi_l)}\end{pmatrix}_{(j)}
\end{aligned}$$

$$\vdots$$

$$\times\sum_{l\neq i}^{N}\frac{\sigma(\eta)\sigma(u_{N-1}-\xi_l+\lambda_{21}+(N-1)\eta)}{\sigma(u_{N-1}-\xi_l+\eta)\sigma(\lambda_{21}+(N-1)\eta)}E_{21}^{l}\otimes_{j\neq l,i}\begin{pmatrix}\dfrac{\sigma(u_{N-1}-\xi_j)}{\sigma(u_{N-1}-\xi_j+\eta)} & \\ & \dfrac{\sigma(\xi_j-\xi_l+\eta)}{\sigma(\xi_j-\xi_l)}\end{pmatrix}_{(j)}$$

$$\times E_{21}^{i}\,|1,\ldots 1\rangle. \tag{62}$$

Iterating the above procedure, we have

$$\begin{aligned}\boldsymbol{Z}_N(\{u_\alpha\};\{\xi_i\};\lambda) = &\sum_{i_1\neq i_2\ldots\neq i_N}\prod_{l=1}^{N}\left\{\frac{\sigma(\eta)\sigma(u_l-\xi_{i_l}+\lambda_{21}+l\eta)}{\sigma(u_l-\xi_{i_l}+\eta)\sigma(\lambda_{21}+l\eta)}\prod_{k=1}^{l-1}\frac{\sigma(u_l-\xi_{i_k})\sigma(\xi_{i_l}-\xi_{i_k}+\eta)}{\sigma(u_l-\xi_{i_k}+\eta)\sigma(\xi_{i_l}-\xi_{i_k})}\right\}\\ &\times\langle 2,\ldots,2|\,E_{21}^{i_1}\ldots E_{21}^{i_N}\,|1,\ldots,1\rangle.\end{aligned} \tag{63}$$

This leads to the complete symmetric expression of the DW partition function $Z_N(\{u_\alpha\};\{\xi_j\};\lambda)$

$$\boldsymbol{Z}_N(\{u_\alpha\};\{\xi_j\};\lambda) = \sum_{s\in S_N}\prod_{l=1}^{N}\left\{\frac{\sigma(\eta)\sigma(u_l-\xi_{s(l)}+\lambda_{21}+l\eta)}{\sigma(u_l-\xi_{s(l)}+\eta)\sigma(\lambda_{21}+l\eta)}\times\prod_{k=1}^{l-1}\frac{\sigma(u_l-\xi_{s(k)})\sigma(\xi_{s(l)}-\xi_{s(k)}+\eta)}{\sigma(u_l-\xi_{s(k)}+\eta)\sigma(\xi_{s(l)}-\xi_{s(k)})}\right\}, \tag{64}$$

where S_N is the permutation group of N indices.

5. Conclusions and discussions

We have obtained the explicit expression (64) of the partition function $Z_N(\{u_\alpha\};\{\xi_j\};\lambda)$ of the eight-vertex model on an $N\times N$ square lattice with the domain wall boundary condition by directly computing the inner product Eq. (38). The completely symmetric polarisation free form (57) of the creation/annihilation operator in the very $\boldsymbol{F}$-basis provided by the Drinfeld twist (50) has played an important role in the calculation. The approach in this paper is complementary to the one used recently in Refs. [37], [39], [46], [47] and [14] to obtain the partition function via the recursive relation. Moreover, the advantage of our approach over the previous ones is that it can be used to obtain partition functions of the eight-vertex model with other boundary conditions such as open boundary condition.

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