Rapid Communication

Nonlinear dynamical symmetries of Smorodinsky–Winternitz and and Fokas–Lagerstorm systems*

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General solutions of the Smorodinsky–Winternitz system and the Fokas–Lagerstorm system, which are superintegrable in two-dimensional Euclidean space, are obtained using the algebraic method (structure function). Their dynamical symmetries, which are governed by deformed angular momentum algebras, are revealed.

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1. Introduction

The concepts of integrability and superintegrability play a fundamental role in the description of physical systems due to their many interesting properties, from both mathematical and physical viewpoints.[1] In classical Hamiltonian mechanics, an n-freedom Hamiltonian system is said to be Arnold–Liouville integrable if there are n functionally independent integrals of motion in involution (including the Hamiltonian itself),[2] and called superintegrable if it admits more than n integrals of motion.[3] Note that not all the integrals of motion in the superintegrable system can be in involution. In quantum mechanics, the definitions of integrability and superintegrability are quite similar to those in classical mechanics,[4] that is, an n-dimensional quantum system is called integrable if it has $n-1$ algebraically independent conservational quantities X_i ($i = 1, 2, \ldots, n-1$), which, together with the Hamiltonian H of the system, satisfy

$$[H,\ X_i] = 0, \quad [X_i,\ X_j] = 0,$$

where $[,]$ denotes the commutator. Such a system is said to be superintegrable if there exist further m ($1 \leq m \leq n-1$) conservational quantities Y_a ($a = 1, 2, \ldots, m$) such that they satisfy

$$[H,\ X_i] = [H,\ Y_a] = 0, \quad [X_i,\ X_j] = 0.$$

Similarly, the operators Y_a need commute neither among each other, nor with the operators X_i.

The Kepler (Coulomb) problem ($\sim r^{-1}$) and the isotropic harmonic oscillator ($\sim r^2$) may be the most well-known cases of maximally superintegrable systems.[5] In N-dimensional Euclidean space, their symmetry algebras are ordinary Lie algebras so($N+1$) and su(N) respectively, higher than those of the other central potentials. For the Kepler problem and the isotropic harmonic oscillator, we may determine their eigenenergies and eigenstates by purely algebraic methods because their algebras are relatively clear and simple. However, identification of Lie algebraic structures for general superintegrable systems is not easy. Following the introduction of quantum algebras[6] and later the other deformations of Lie algebras,[7−11] the applications of the algebraic method based upon various deformed Lie algebras to quantum systems are increased.[12−17] In this paper, we study two superintegrable systems in two-dimensional Euclidean space, the Smorodinsky–Winternitz system[18] and the Fokas-Lagerstorm system,[19] by using algebraic methods (structure function and dynamical symmetry).

The rest the paper is organized as follows. In Section 2 we first construct the nonlinear Lie algebra that describes the Smorodinsky–Winternitz system, then use it to study the general solution of this system. In Section 3 we reveal the nonlinear dynamical symmetry of the system. In Section 4 the general solution of the Fokas–Lagerstorm system is studied from its nonlinear dynamical symmetry based on the corresponding nonlinear Lie algebra. Some simple discussion is given

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in the final section. In this paper, $\hbar$ and m are set to be 1.

2. General solution of Smorodinsky–Winternitz system

Friš *et al.*[18] have proven that only four independent two-dimensional superintegrable systems exist if the condition that conservational operators are quadratic in momenta is required, whose potentials are respectively as follows: $V_1 = \alpha r^2 + \beta_1 x_1^{-2} + \beta_2 x_2^{-2}$, $V_2 = \alpha(4x_1^2 + x_2^2) + \beta_1 x_1 + \beta_2 x_2^{-2}$, $V_3 = \alpha r^{-1} + r^{-2}[\beta_1 \cos^{-2}(\phi/2) + \beta_2 \sin^{-2}(\phi/2)]$, and $V_4 = \alpha r^{-1} + r^{-1/2}[\beta_1 \cos(\phi/2) + \beta_2 \sin(\phi/2)]$, where $r = \sqrt{x_1^2 + x_2^2}$, $x_1 = r\cos\phi$, and $x_2 = r\sin\phi$. Then Tempesta *et al.*[20] gave the solution of V_1 by solving directly the corresponding Schrödinger equation and found that the solutions of V_3 and V_4 could easily be obtained from the solution of V_1. In this section, we study the general solution of V_1 by using the algebraic method.

For the system of V_1, the corresponding Hamiltonian is

$$H_1 = \frac{1}{2}(p_1^2 + p_2^2) + \alpha(x_1^2 + x_2^2) + \frac{\beta_1}{x_1^2} + \frac{\beta_2}{x_2^2}, \quad (1)$$

where α is a positive real number, and β_1 and β_2 are real numbers. This Hamiltonian has two conservational operators:[18]

$$T_1 = \frac{1}{2}(p_2^2 - p_1^2) + \alpha(x_2^2 - x_1^2) + \frac{\beta_2}{x_2^2} - \frac{\beta_1}{x_1^2}, \quad (2)$$

$$T_2 = \frac{2\beta_2 x_1^2}{x_2^2} + \frac{2\beta_1 x_2^2}{x_1^2} + x_1^2 p_2^2 + x_2^2 p_1^2 - \{x_1 x_2, p_1 p_2\}, \quad (3)$$

where $\{\,,\,\}$ is the anticommutator. Furthermore, the commutation relation between T_1 and T_2 may give a new conservational operator T_3, i.e.,

$$[T_1, T_2] = 4\mathrm{i}\left\{\left(\frac{1}{2}p_2^2 - \alpha x_2^2 + \frac{\beta_2^2}{x_2^2}\right)\{x_1, p_1\} - \left(\frac{1}{2}p_1^2 - \alpha x_1^2 + \frac{\beta_1^2}{x_1^2}\right)\{x_2, p_2\}\right\} \equiv T_3. \quad (4)$$

It is obvious from Eqs. (2)–(4) that the expressions of T_1 and T_3 are antisymmetric by exchanging subscripts 1 and 2, while T_2 is symmetric. They satisfy the following commutation relations:

$$\begin{aligned}
[T_1, T_2] &= T_3, \\
[T_1, T_3] &= 16\alpha(2T_2 - 1) + 8(T_1^2 - H_1^2), \\
[T_3, T_2] &= 16T_2T_1 + 32T_1(\beta_1 + \beta_2 - 1) \\
&\quad + 32H_1(\beta_1 - \beta_2) + 8T_3. \qquad (5)
\end{aligned}$$

These commutation relations are closed. However, we should note two significant features: the Hamiltonian H_1 of the system and nonlinear terms T_1^2 and T_2T_1 appear on the right-hand sides of the second and third commutation relations of Eqs. (5) respectively.

It is more useful to express the three operators T_i $(i = 1, 2, 3)$ in raising/lowering form correspondingly:

$$\begin{aligned}
L_0 &= \frac{1}{\sqrt{32\alpha}}T_1, \\
L_+ &= 4\alpha T_2 + \sqrt{\frac{\alpha}{2}}T_3 + T_1^2 - H_1^2 - 2\alpha, \\
L_- &= 4\alpha T_2 - \sqrt{\frac{\alpha}{2}}T_3 + T_1^2 - H_1^2 - 2\alpha, \qquad (6)
\end{aligned}$$

which are nonlinear expansions with respect to T_i and satisfy

$$\begin{aligned}
&[L_0,\, L_\pm] = \pm L_\pm, \\
&L_+L_- = \Phi(H_1, L_0), \\
&L_-L_+ = \Phi(H_1, L_0 + 1), \qquad (7)
\end{aligned}$$

where $\Phi(H_1, L_0)$ is called the structure operator function,[21] whose explicit expression is given by

$$\Phi(H_1, L_0) = 1024\alpha^2 \times \prod_{\mu=\pm}\prod_{i=1}^{2}\left[L_0 + (-)^i\frac{H_1}{\sqrt{32\alpha}} - B_\mu^{(i)}\right], \quad (8)$$

in which symbols $B_\pm^{(i)}$ are defined as

$$B_\pm^{(i)} \equiv \frac{1}{2} \pm \frac{\sqrt{1 + 8\beta_i}}{4}. \quad (9)$$

In the framework of the algebraic method, we need to assume that the vector space on which these equations (7) are allowed to act is the energy eigenspace, that is, the common eigenspaces of H_1 and L_0. After replacing H_1 and L_0 in Eqs. (7) by their respective eigenvalues E and l, the operators L_i $(i = 0, +, -)$ thus generate a closed nonlinear Lie algebra. It can be seen as the polynomial deformation of the ordinary Lie algebra so(3), which is used to deal with the problem of angular momentum in quantum mechanics,[22] so we call it a deformed angular momentum algebra, denoted by $\mathcal{R}_3$. Correspondingly, $\Phi(H_1, L_0)$ becomes the structure function $\Phi(E, l)$ with double variables E and l. It follows from the second equation in Eq. (7) that $\Phi(E, l)$ should satisfy the constraint condition:

$$\Phi(E, l) \geq 0. \quad (10)$$

Since we are only interested in the finite spectra of H_1, we can immediately write down the following equations from Eq. (10):

$$\begin{aligned} &\Phi(E_N, l_{\min}) = 0, \\ &\Phi(E_N, l_{\max}+1) = 0, \\ &l_{\max} - l_{\min} = N. \end{aligned} \tag{11}$$

For a given E_N, l may take $N+1$ different values, so the degeneracy of the system is $N+1$.

There should be sixteen different solutions satisfying Eqs. (11). However, among them, eight solutions will lead to expressions like $B_+^{(2)} - B_-^{(2)} + N + 1 = 0$, in which no eigenenergy E_N appears, therefore, they cannot produce any physical meaningful results. Four of the other solutions will give negative energies, except for the special case $\beta_1 = \beta_2 = 0$, H_1 is an ordinary two-dimensional oscillator, whose eigenenergies are positive. After neglecting the above twelve solutions, we may obtain all possible eigenvalues of H_1 as follows:

$$\begin{aligned} l_{\min} &= B_+^{(2)} - \frac{E_N}{\sqrt{32\alpha}}, \\ E_N &= \sqrt{8\alpha}(N + B_+^{(1)} + B_+^{(2)}) \end{aligned} \tag{12}$$

for $\beta_1, \beta_2 \geq -1/8$;

$$\begin{aligned} l_{\min} &= B_+^{(2)} - \frac{E_N}{\sqrt{32\alpha}}, \\ E_N &= \sqrt{8\alpha}(N + B_-^{(1)} + B_+^{(2)}) \end{aligned} \tag{13}$$

for $3/8 \geq \beta_1 \geq -1/8$, $\beta_2 \geq -1/8$;

$$\begin{aligned} l_{\min} &= B_-^{(2)} - \frac{E_N}{\sqrt{32\alpha}}, \\ E_N &= \sqrt{8\alpha}(N + B_+^{(1)} + B_-^{(2)}) \end{aligned} \tag{14}$$

for $\beta_1 \geq -1/8$, $3/8 \geq \beta_2 \geq -1/8$;

$$\begin{aligned} l_{\min} &= B_-^{(2)} - \frac{E_N}{\sqrt{32\alpha}}, \\ E_N &= \sqrt{8\alpha}(N + B_-^{(1)} + B_-^{(2)}) \end{aligned} \tag{15}$$

for $3/8 \geq \beta_1 \geq -1/8$, $3/8 \geq \beta_2 \geq -1/8$.

We end this section by giving unitary irreducible representations (UIRs) of $\mathcal{R}_3$. Owing to the fact that H_1 commutes with the generators L_i $(i = 0, +, -)$ of $\mathcal{R}_3$, the eigenenergies E_N of H_1 may be used to label the UIRs of $\mathcal{R}_3$, the eigenstates with respect to E_N may act as basis vectors carrying the corresponding UIRs, and the degeneracy of E_N is the dimension of the corresponding UIRs. Let $\{|E_N, l\rangle\}$ be a set of common eigenstates of H_1 and L_0, with E_N and l being the eigenvalues of H_1 and L_0 respectively, and then choose the proper phase factors between the eigenstates, we then have

$$\begin{aligned} H|E_N, l\rangle &= E_N|E_N, l\rangle, \quad L_0|E_N, l\rangle = l|E_N, l\rangle, \\ L_+|E_N, l\rangle &= 32\alpha\sqrt{\prod_{\mu=\pm}\prod_{i=1}^{2}\left[(-)^i\frac{E_N}{\sqrt{32\alpha}} + l + B_\mu^{(i)}\right]}|E_N, l+1\rangle, \\ L_-|E_N, l\rangle &= 32\alpha\sqrt{\prod_{\mu=\pm}\prod_{i=1}^{2}\left[(-)^i\frac{E_N}{\sqrt{32\alpha}} + l - B_\mu^{(i)}\right]}|E_N, l-1\rangle, \end{aligned} \tag{16}$$

where l, corresponding to the given E_N (see Eqs. (12)–(15)), may take one of the following four values, i.e.,

$$\begin{aligned} &l = \frac{1}{2}(B_\pm^{(2)} - B_\pm^{(1)} - N) + n, \\ &n = 0, 1, \ldots, N, \end{aligned} \tag{17}$$

and the subscripts + and − in the expressions $B_\pm^{(2)} - B_\pm^{(1)}$ may occur independently. In order to obtain UIRs (16), the constraint conditions (10)–(11) need imposing.

3. Dynamical symmetry of Smorodinsky–Winternitz system

Dynamical symmetries have received intensive attention over the past few years, and have important applications in various branches of physics[23,24] and in solving differential equations.[25] In this section, we shall generalize the concept of dynamical symmetry based upon the Lie algebra to the nonlinear Lie algebra:[26] a quantum system possesses a nonlinear dynamical symmetry, if its Hamiltonian is a function of a Casimir invariant C_g of some nonlin-

ear Lie algebra g, or of the Casimir invariants C_g, C_{g1}, C_{g2}, ..., of a chain of nested nonlinear Lie algebras $g \supset g_1 \supset g_2 \supset \ldots$. As we will show, the Smorodinsky–Winternitz system discussed above has $\mathcal{R}_3$ dynamical symmetry.

Using Eqs. (7)–(9), the explicit commutation relations of $\mathcal{R}_3$ are

$$\begin{aligned}&[L_0,\ L_\pm] = \pm L_\pm,\\ &[L_+, L_-] = c_0 + c_1 L_0 + c_2 L_0^2 + c_3 L_0^3,\end{aligned} \tag{18}$$

where

$$\begin{aligned}&c_0 = \sqrt{32\alpha}(\beta_1 - \beta_2)H_1,\\ &c_1 = 1024\alpha^2\left(\beta_1 + \beta_2 + \frac{H_1^2}{8\alpha} - \frac{3}{4}\right),\\ &c_2 = 0, \quad c_3 = -4096\alpha^2,\end{aligned} \tag{19}$$

and the Casimir invariant of $\mathcal{R}_3$ reads[9]

$$\begin{aligned}\mathcal{C} =\ & L_+L_- + L_-L_+ + \left(2c_0 + \frac{1}{3}c_2\right)L_0\\ &+ \left(c_1 + \frac{1}{2}c_3\right)L_0^2 + \frac{2}{3}c_2L_0^3 + \frac{1}{2}c_3L_0^4\\ =\ & 2\prod_{\mu=\pm}\prod_{i=1}^{2}\left[H_1 + (-)^{i-1}\sqrt{32\alpha}B_\mu^{(i)}\right].\end{aligned} \tag{20}$$

It is clear from Eq. (20) that there is a relationship between the Hamiltonian H_1 and the Casimir invariant $\mathcal{C}$ of $\mathcal{R}_3$, that is, H_1 is a root of the quartic equation, whose explicit form involves $\mathcal{C}$ and two parameters, β_1 and β_2, only. Therefore, we conclude that the Smorodinsky–Winternitz system possesses $\mathcal{R}_3$ dynamical symmetry.

Due to the fact that the Casimir invariant $\mathcal{C}$ of $\mathcal{R}_3$ commutes with all generators L_i $(i = 0, +, -)$ of $\mathcal{R}_3$, we may simultaneously diagonalize $\mathcal{C}$ and the Cartan operator L_0. Since $\mathcal{C}^\dagger = \mathcal{C}$ and $L_0^\dagger = L_0$, the eigenvalues of $\mathcal{C}$ and L_0 are real. Let $|\lambda, m\rangle$ be a common eigenvector of $\mathcal{C}$ and L_0 with eigenvalues λ and m (the weight) respectively, i.e.,

$$\mathcal{C}|\lambda, m\rangle = \lambda|\lambda, m\rangle, \tag{21}$$

$$L_0|\lambda, m\rangle = m|\lambda, m\rangle. \tag{22}$$

From the commutation relations $[L_0, L_\pm] = \pm L_\pm$, we see that for an arbitrary nonnegative integer k, $(L_+)^k|\lambda, m\rangle$ (or $(L_-)^k|\lambda, m\rangle$) is also an eigenvector of L_0 with an eigenvalue $m + k$ (or $m - k$). It follows that L_-L_+ is a diagonal operator, and in particular

$$\langle\lambda, m|L_-L_+|\lambda, m\rangle = \|L_+|\lambda, m\rangle\|^2 \geq 0. \tag{23}$$

This equation requires that the set of eigenvalues of L_0 should be bounded above, and also below. Therefore, there is a highest weight and a lowest weight state, such that the UIR of $\mathcal{R}_3$ in the representation space $\{|\lambda, m\rangle\}$ is finite-dimensional. Let $\overline{m}$ and $\underline{m}$ be the highest weight and the lowest one respectively, then we have

$$\begin{aligned}&L_0|\lambda, \overline{m}\rangle = \overline{m}|\lambda, \overline{m}\rangle,\\ &L_+|\lambda, \overline{m}\rangle = 0,\\ &L_0|\lambda, \underline{m}\rangle = \underline{m}|\lambda, \underline{m}\rangle,\\ &L_-|\lambda, \underline{m}\rangle = 0,\\ &\underline{m} + N = \overline{m},\end{aligned} \tag{24}$$

where $N + 1$ is the dimension of the corresponding UIR. Substituting Eqs. (20) and (24) into Eq. (21), we obtain the following two equations:

$$\left(\underline{m} + \frac{E_N}{\sqrt{32\alpha}} - B_+^{(2)}\right)\left(\underline{m} + \frac{E_N}{\sqrt{32\alpha}} - B_-^{(2)}\right)\left(\underline{m} - \frac{E_N}{\sqrt{32\alpha}} - B_+^{(1)}\right)\left(\underline{m} - \frac{E_N}{\sqrt{32\alpha}} - B_-^{(1)}\right) = 0, \tag{25}$$

$$\begin{aligned}&\left(\underline{m} + N + \frac{E_N}{\sqrt{32\alpha}} + B_+^{(2)}\right)\left(\underline{m} + N + \frac{E_N}{\sqrt{32\alpha}} + B_-^{(2)}\right)\\ &\times\left(\underline{m} + N - \frac{E_N}{\sqrt{32\alpha}} + B_+^{(1)}\right)\left(\underline{m} + N - \frac{E_N}{\sqrt{32\alpha}} + B_-^{(1)}\right) = 0.\end{aligned} \tag{26}$$

The solutions of Eqs. (25) and (26) recover the results obtained in Section 2. The value ranges of β_1 and β_2 may easily be obtained by solving Eq. (23).

Finally, we discuss another origin of the Smorodinsky–Winternitz system and its relation to the radial Schrödinger equation of the two-dimensional isotropic harmonic oscillator. The Hamiltonian of the four-dimensional isotropic harmonic oscillator is

$$\bar{H} = \frac{1}{2}\boldsymbol{p}^2 + \alpha\boldsymbol{r}^2, \tag{27}$$

where α is a positive real number and the Cartesian coordinates are $\boldsymbol{r}$ are (x_1, x_2, x_3, x_4). Introducing new coordinates $(x, \theta_x, y, \theta_y)$,[27] which have the following relations with (x_1, x_2, x_3, x_4):

$$x_1 = x\cos\theta_x, \quad x_2 = x\sin\theta_x,$$
$$x_3 = y\cos\theta_y, \quad x_4 = y\sin\theta_y, \tag{28}$$

Hamiltonian (27) becomes

$$\bar{H} = \frac{1}{2}\left(p_x^2 + p_y^2 + \frac{p_{\theta_x}^2}{x^2} + \frac{p_{\theta_y}^2}{y^2}\right) + \alpha(x^2 + y^2). \tag{29}$$

After separating coordinates (θ_x, θ_y) and replacing p_{θ_x} and p_{θ_y} by two real numbers k_1 and k_2 respectively, Eq. (29) reduces to

$$\hat{H} = \frac{1}{2}(p_x^2 + p_x^2) + \alpha(x^2 + y^2) + \frac{k_1}{x^2} + \frac{k_2}{y^2}, \tag{30}$$

which in fact is H_1 (see Eq. (1)).

On the other hand, consider the radial Schrödinger equation for the two-dimensional isotropic harmonic oscillator, separated in the polar coordinate system, i.e.,

$$\hat{\Omega}(\alpha, r)\, v(r) = Ev(r), \tag{31}$$

where

$$\hat{\Omega}(\alpha, r) = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \alpha r^2 + \frac{1}{2r^2}\left(m^2 - \frac{1}{4}\right), \tag{32}$$

with m ($= 0, \pm 1, \ldots$) being the magnetic quantum number. It follows by setting $\beta_1 = \frac{1}{2}(m_1^2 - \frac{1}{4})$, $\beta_2 = \frac{1}{2}(m_2^2 - \frac{1}{4})$ in Eq. (1) that we observe that H_1, formally, is a combination of two differential operator functions $\hat{\Omega}(\alpha, x)$ and $\hat{\Omega}(\alpha, y)$ corresponding, respectively, to two isotropic harmonic oscillators with the same frequency, that is, $H_1 = \hat{\Omega}(\alpha, x) + \hat{\Omega}(\alpha, y)$. However, there exist three evident differences. First of all, the range of the radial variable r in $\hat{\Omega}(\alpha, r)$ is $(0, \infty)$, whereas that of the one-dimensional variable x (or y) in $\hat{\Omega}(\alpha, x)$ (or $\hat{\Omega}(\alpha, y)$) in H_1 is $(-\infty, \infty)$. Secondly, the vacuum energy of H_1 may not be an integer for the arbitrary β_i ($i = 1, 2$). Thirdly, if the further restrictions on the parameters (β_1 and β_2) are valid, the other solutions can also be permitted.

4. Dynamical symmetry of Fokas–Lagerstorm system

In this section, we consider another superintegrable system: a special two-dimensional anisotropic harmonic oscillator, whose Hamiltonian is

$$H' = \frac{1}{2}(p_1^2 + p_2^2) + \frac{x_1^2}{2} + \frac{x_2^2}{18}. \tag{33}$$

In classical mechanics, this system was first studied in detail by Fokas and Lagerstorm.[19]

The Hamiltonian H' has two conservational operators:

$$\begin{aligned} U_1 &= \frac{1}{4}\left(p_1^2 - p_2^2\right) + \frac{x_1^2}{4} - \frac{x_2^2}{36}, \\ U_2 &= \frac{1}{2}\{x_1 p_2 - x_2 p_1, p_2^2\} \\ &\quad - \frac{1}{6}\{x_1 x_2^2, p_2\} + \frac{x_2^3 p_1}{27}. \end{aligned} \tag{34}$$

We note two differences between the Smorodinsky–Winternitz system and the Fokas–Lagerstorm system: 1) the Hamiltonian H_1 (see Eq. (1)) of the Smorodinsky–Winternitz system is multiseparable in two different coordinate systems, whereas the Hamiltonian H' of the Fokas–Lagerstorm system is not multiseparable; 2) the powers in momenta of all the conservational operators of the Smorodinsky–Winternitz system are quadratic, whereas the highest powers in momenta of the conservational operators of the Fokas–Lagerstorm system are unequal: that of U_1 is quadratic, that of U_2 is cubic.

Using U_1 and U_2, the closed commutation relations may be obtained as follows:

$$\begin{aligned} [U_1, U_2] &= U_3, \\ [U_1, U_3] &= U_2, \\ [U_3, U_2] &= 2H'(H' - 1)(H' + 1) + \frac{56}{9}U_1 \\ &\quad - 24H'U_1^2 + 32U_1^3. \end{aligned} \tag{35}$$

Similarly, expressing the three operators U_i ($i = 1, 2, 3$) in raising/lowering form

$$J_0 = U_1, \quad J_\pm = U_2 \pm U_3, \tag{36}$$

which are linear expansions of U_i. It is easy to obtain the following commutation relations:

$$\begin{aligned} [J_0,\ J_\pm] &= \pm J_\pm, \\ [J_+, J_-] &= c_0 + c_1 J_0 + c_2 J_0^2 + c_3 J_0^3, \end{aligned} \tag{37}$$

where

$$\begin{aligned} c_0 &= -4H'\left(4H'^2 - \frac{23}{9}\right), \\ c_1 &= 16\left(6H'^2 + \frac{7}{9}\right), \\ c_2 &= -144H', \quad c_3 = 64. \end{aligned} \tag{38}$$

We observe that Eqs. (37) have the same algebraic structure as Eqs. (18) except for different c_i's, so here it is denoted by $\mathcal{R}'_3$.

Using Eqs. (35) and (36), the Casimir invariant $\mathcal{C}'$ of $\mathcal{R}'_3$ reads

$$\begin{aligned}\mathcal{C}' &= J_+J_- + J_-J_+ + \left(2c_0 + \frac{1}{3}c_2\right)J_0 \\ &\quad + \left(c_1 + \frac{1}{2}c_3\right)J_0^2 + \frac{2}{3}c_2J_0^3 + \frac{1}{2}c_3J_0^4 \\ &= \frac{2}{9}(-1+H')(1+H')(5+9H'^2). \end{aligned} \tag{39}$$

The second equation shows that the Hamiltonian H' is a root of the quartic equation, that is, H' is only a function of $\mathcal{C}'$. Therefore, we say the Fokas–Lagerstorm system possesses $\mathcal{R}'_3$ dynamical symmetry.

Now, we apply the same algebraic method used in Section 3 to $\mathcal{R}'_3$ to obtain the solution of the Fokas–Lagerstorm system. Let $\{|\lambda', m'\rangle\}$ be a set of basis vectors carrying the UIRs of $\mathcal{R}'_3$, where $|\lambda', m'\rangle$ is a common eigenvector of $\mathcal{C}'$ and J_0 with eigenvalues λ' and m' respectively, that is,

$$\mathcal{C}'|\lambda', m'\rangle = \lambda'|\lambda', m'\rangle, \tag{40}$$

$$J_0|\lambda', m'\rangle = m'|\lambda', m'\rangle. \tag{41}$$

Denoting the highest weight and the lowest one by $\overline{m}'$ and $\underline{m}'$ respectively, we have

$$\begin{aligned}&J_0|\lambda', \overline{m}'\rangle = \overline{m}'|\lambda', \overline{m}'\rangle, \\ &J_+|\lambda', \overline{m}'\rangle = 0, \\ &J_0|\lambda', \underline{m}'\rangle = \underline{m}'|\lambda,' \underline{m}'\rangle, \\ &J_-|\lambda', \underline{m}'\rangle = 0, \\ &\underline{m}' + N = \overline{m}', \end{aligned} \tag{42}$$

where $N+1$ is the dimension of the corresponding UIR.

Substitution of Eqs. (39), (42) into Eq. (40) gives rise to the following two equations for E'_N and $\underline{m}'$:

$$\begin{aligned}&(3E'_N - 6\underline{m}' + 1)(3E'_N - 6\underline{m}' + 5) \\ &\times (E'_N - 2\underline{m}' + 1)(E'_N + 2\underline{m}' - 1) = 0, \\ &(3E'_N - 6\underline{m}' - 6N - 5)(3E'_N - 6\underline{m}' - 6N - 1) \\ &\times (E'_N - 2\underline{m}' - 2N - 1) \\ &\times (E'_N + 2\underline{m}' + 2N + 1) = 0. \end{aligned} \tag{43}$$

Solving Eq. (43) and neglecting all of the negative energy solutions and any expressions with no physical meaning, we obtain all the possible eigenvalues of H':

$$\begin{aligned}E'_N &= N + \frac{2}{3} \quad \text{with} \quad \underline{m}' = \frac{1}{2} - \frac{1}{2}E'_N, \\ E'_N &= N + 1 \quad \text{with} \quad \underline{m}' = \frac{1}{2} - \frac{1}{2}E'_N, \\ E'_N &= N + \frac{4}{3} \quad \text{with} \quad \underline{m}' = \frac{1}{2} - \frac{1}{2}E'_N, \end{aligned} \tag{44}$$

where $N = 0, 1, 2, \ldots$. The degeneracy of the energy level E'_N is $N+1$. These results (44) coincide with the usual results:

$$E'_{n,m} = n + \frac{1}{2} + \frac{1}{3}\left(m + \frac{1}{2}\right), \tag{45}$$

where m, $n = 0, 1, 2, \ldots$.

5. Discussion

In this paper we have constructed the deformed angular momentum algebra $\mathcal{R}_3$ for two different two-dimensional superintegrable systems, the Smorodinsky–Winternitz system and the Fokas–Lagerstorm system, and have obtained their general solutions by using the algebraic method based upon $\mathcal{R}_3$. Furthermore, we have revealed that both systems possess the same $\mathcal{R}_3$ dynamical symmetries because both Hamiltonians are functions of the Casimir invariant of $\mathcal{R}_3$.

There are some interesting questions raised which merit further investigation. First, it is clear that the Hamiltonian $\tilde{H} = \frac{1}{2}\boldsymbol{p}^2 + V(r)$ for the central potential $V(r)$ in $2N$-dimensional space is also integrable, and its symmetry algebra is $(2N)$. Using the same method of gaining Eq. (29) in four-dimensional space, $\tilde{H}$ may have a similar deformation, that is, $\tilde{H}' = \frac{1}{2}\boldsymbol{p}^2 + V(r) + \sum_i \frac{k_i}{x_i^2}$ (k_i's are arbitrary real numbers). Can a (nonlinear) Lie algebra to describe $\tilde{H}$ be found? In fact, the same question also applies to the $2N$-dimensional anisotropic harmonic oscillator, whose Hamiltonian reads $H^{\mathrm{a}} = \frac{1}{2}\boldsymbol{p}^2 + (\omega^2/2)\sum_i q_i^2 x_i^2$ (q_i's are arbitrary positive real integers). Secondly, is there any superintegrable potential in two-dimensional space which cannot be recognized as a 'projection' of the higher-dimensional Kepler problem or the harmonic oscillator, or a combination of both? These questions are under study.

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