Symplectic eigenvector expansion theorem of a class of operator matrices arising from elasticity theory*

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This paper deals with the completeness of the eigenvector system of a class of operator matrices arising from elasticity theory, i.e., symplectic eigenvector expansion theorem. Under certain conditions, the symplectic orthogonality of eigenvectors of the operator matrix is demonstrated. Based on this, a necessary and sufficient condition for the completeness of the eigenvector system of the operator matrix is given. Furthermore, the obtained results are tested for the free vibration of rectangular thin plates.

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1. Introduction

In 1991, the separation of variables based on Hamiltonian systems was proposed, which provides a new approach to elasticity and related fields.[1−5] This method is an extension of the traditional separation of variables, and its mathematical foundation is the completeness of the eigenvector system of the corresponding Hamiltonian operator.[6−11]

It is well known that many partial differential equations can be written as various differential systems. Like the case of Hamiltonian systems, whether we can analytically solve these differential systems by separation of variables depends on the completeness of the eigenvector system of the corresponding operator matrices. However, in general, the eigenvector system of operator matrices has no (symplectic) orthogonality, which is one of the difficulties we need to overcome.

In this paper, we investigate the completeness of the eigenvector system of a class of operator matrices arising from elasticity theory and obtain a necessary and sufficient condition in symplectic spaces. Moreover, the results are applied to the free vibration of rectangular thin plate.

Throughout this paper, an operator or operator matrix is always linear (not necessarily bounded). In the following, we present some basic notions.

Definition 1.1.[11] Let Z be a Hilbert space over $\mathbb{C}$, and the function $(\cdot\,,\,\cdot)$ be an inner product on the product space $Z \times Z$. Then, the function $(\cdot\,,\,J\,\cdot\,)$ is called a symplectic form generated by the inner product $(\cdot\,,\,\cdot)$, where

$$\boldsymbol{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

with I being the identity operator on Z.

A linear space equipped with a symplectic form is called a symplectic space. In the present paper, the symplectic form on the symplectic space $Z \times Z$ is always defined by the function $(\cdot\,,\,J\,\cdot\,)$.

Definition 1.2. Let X be a Hilbert space. Then, the symplectic orthogonal system $\{\boldsymbol{U}_k, \boldsymbol{U}_{-k}, \boldsymbol{V}_k, \boldsymbol{V}_{-k} | \, k = 1, 2, \dots\}$ is said to be complete in the symplectic space $X \times X \times X \times X$ (written in short as X^4), if for each $\Delta \in X^4$, there exists a unique constant sequence $\{c_{\pm k}, d_{\pm k} | \, k = 1, 2, \dots\}$ such that

$$\Delta = \sum_{k=1}^{+\infty} c_k U_k + c_{-k} U_{-k} + d_k V_k + d_{-k} V_{-k}. \qquad (1)$$

In the following, consider the operator matrix $\boldsymbol{M}$

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in X^4

$$\boldsymbol{M} = \begin{pmatrix} 0 & P & 0 & 0 \\ Q_1 & 0 & 0 & D \\ 0 & 0 & 0 & Q_2 \\ 0 & 0 & -P & 0 \end{pmatrix}. \tag{2}$$

Note that for the eigenvalue λ of an operator T, the set $E(\lambda; T)$ consists of all associated eigenvectors.

2. Theoretical aspects

In this section, we present the main results of this paper and their proofs.

Lemma 2.1.[11] Let X be a Hilbert space and $\boldsymbol{A} : \mathcal{D}(Q) \times \mathcal{D}(P) \to X \times X$,

$$\boldsymbol{A} = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix}.$$

Then, the following statements hold.

(i) $\lambda \in \sigma_p(A) \Leftrightarrow -\lambda \in \sigma_p(A)$, and $(f \ g)^{\mathrm{T}} \in E(\lambda; A)$ if and only if $(f \ -g)^{\mathrm{T}} \in E(-\lambda; A)$.

(ii) $0 \neq \lambda \in \sigma_p(A) \Leftrightarrow \lambda^2 \in \sigma_p(QP)$, and $(f \ g)^{\mathrm{T}} \in E(\lambda; \boldsymbol{A})$ if and only if $g \in E(\lambda^2; QP)$, where $f = Pg/\lambda$.

(iii) Assume that the operator $\boldsymbol{A}$ possesses countable eigenvalues. By (i), we set

$$\sigma_p(A) = \{\lambda_{\pm k} \in \mathbb{C} \,|\, k = 1, 2, \dots\}$$

with $\lambda_{-k} = -\lambda_k$, then, for $(f_{\pm k} \ g_{\pm k})^{\mathrm{T}} \in E(\lambda_{\pm k}; \boldsymbol{A})$ $(\lambda_{\pm k} \neq 0)$, we have that $f_{-k} = f_k$, $g_{-k} = -g_k$.

(iv) If $0 \notin \sigma_p(PQ) \cup \sigma_p(QP)$, then $0 \notin \sigma_p(\boldsymbol{A})$.

Lemma 2.2. For the operator matrix $\boldsymbol{M}$ with $0 \notin \sigma_p(PQ_1) \cup \sigma_p(PQ_2)$, let both the operators Q_1P and Q_2P possess countable non-zero eigenvalues, and g_k and e_k are eigenvectors of Q_1P and Q_2P associated with the eigenvalues α_k and β_k, respectively. If there exists $y_k \in X$ such that $Q_1Py_k - DPe_k = -\beta_k y_k$ for $k \in \Lambda$, then $\sigma_p(\boldsymbol{M}) = \{\lambda_{\pm k}, \mu_{\pm k} \,|\, k \in \Lambda\}$, and

$$\begin{aligned} \boldsymbol{U}_k &= \left(\frac{Pg_k}{\lambda_k} \ g_k \ 0 \ 0\right)^{\mathrm{T}}, \\ \boldsymbol{U}_{-k} &= \left(\frac{Pg_k}{\lambda_k} \ -g_k \ 0 \ 0\right)^{\mathrm{T}}, \end{aligned} \tag{3}$$

$$\begin{aligned} \boldsymbol{V}_k &= \left(\frac{Py_k}{\mu_k} \ y_k \ e_k \ -\frac{Pe_k}{\mu_k}\right)^{\mathrm{T}}, \\ \boldsymbol{V}_{-k} &= \left(\frac{Py_k}{\mu_k} \ -y_k \ -e_k \ -\frac{Pe_k}{\mu_k}\right)^{\mathrm{T}}, \end{aligned} \tag{4}$$

are eigenvectors of the operator $\boldsymbol{M}$ associated with the eigenvalues λ_k, λ_{-k} and μ_k, μ_{-k}, respectively, where $\Lambda = \{1, 2, \dots\}$, $\alpha_k = (\lambda_k)^2$, $\beta_k = -(\mu_k)^2$, $\lambda_{-k} = -\lambda_k$, $\mu_{-k} = -\mu_k$ and $g_{-k} = -g_k$, $y_{-k} = -y_k$, $e_{-k} = -e_k$.

Proof Since the operator Q_1P possesses countable non-zero eigenvalues and $0 \notin \sigma_p(PQ_1)$, by Lemma 2.1. (ii) (iv), the operator

$$\boldsymbol{A} = \begin{pmatrix} 0 & P \\ Q_1 & 0 \end{pmatrix}$$

also possesses accountable non-zero eigenvalues. Without loss of generality, by Lemma 2.1 (iii), it follows that the eigenvalues of $\boldsymbol{A}$ are given by $\lambda_{\pm k}$ $(k \in \Lambda)$, where $\lambda_{-k} = -\lambda_k$, and

$$(\lambda_{\pm k})^2 = \alpha_k.$$

Similarly, the operator

$$\boldsymbol{A} = \begin{pmatrix} 0 & Q_2 \\ -P & 0 \end{pmatrix}$$

also possesses accountable non-zero eigenvalues, written as $\mu_{\pm k}$ $(k \in \Lambda)$, where $\mu_{-k} = -\mu_k$, and

$$(\mu_{\pm k})^2 = -\beta_k.$$

For $\boldsymbol{U}_k$ and $\boldsymbol{U}_{-k}$ defined in Eq. (3), by the assumptions $Q_1Pg_k = \alpha_k g_k = \lambda_k^2 g_k$, we have

$$H\boldsymbol{U}_{\pm k} = \lambda_{\pm k}\boldsymbol{U}_{\pm k}, \ k \in \Lambda,$$

which shows that $\lambda_{\pm k}$ are the eigenvalue of H, and $\boldsymbol{U}_{\pm k}$ is the corresponding eigenvector, and $g_{-k} = g_k$.

For $\boldsymbol{V}_k$ and $\boldsymbol{V}_{-k}$ defined in Eq. (4), by the assumption $Q_2Pe_k = -\mu_k^2 e_k$ and the existence of $y_k \in X$, such that $Q_1Py_k - DPe_k = -\beta_k y_k$ for $k \in \Lambda$, we have

$$H\boldsymbol{V}_{\pm k} = \mu_{\pm k}\boldsymbol{V}_{\pm k}, \ k \in \Lambda,$$

which shows that $\boldsymbol{V}_{\pm k}$ is the eigenvector associated with the eigenvalue $\mu_{\pm k}$, and $e_{-k} = -e_k$. The proof is completed. □

Lemma 2.3. Under the assumptions of Lemma 2.2, if $(Pg_k, e_k) = (g_k, Pe_k) \neq 0$, $(Pg_k, e_j) = (g_k, P_{e_j}) = 0, (k \neq j \in \Lambda)$, and $(Py_k, e_k) = (y_k, P_{e_k}) \neq 0$, $(Py_k, e_j) = (y_k, P_{e_j}) = 0$, $(k \neq j \in \Lambda)$, then the eigenvector system $\{\boldsymbol{U}_k, \boldsymbol{U}_{-k}, \boldsymbol{V}_k, \boldsymbol{V}_{-k} \,|\, k \in \Lambda\}$ of $\boldsymbol{M}$ has symplectic orthogonality, i.e., for $k, j = \pm 1, \pm 2, \dots,$

$$
\begin{aligned}
&(\boldsymbol{U}_k, J\boldsymbol{U}_j) = 0,\\
&(\boldsymbol{U}_k, J\boldsymbol{V}_j) = \begin{cases} \left(\dfrac{1}{\lambda_k} - \dfrac{1}{\overline{\mu_k}}\right)(Pg_k, e_k), & k-j=0,\\ -\left(\dfrac{1}{\lambda_k} + \dfrac{1}{\overline{\mu_k}}\right)(Pg_k, e_k), & k+j=0,\\ 0, & \text{others}, \end{cases}\\
&(\boldsymbol{V}_k, J\boldsymbol{V}_j) = \begin{cases} 2\left(\dfrac{1}{\mu_k} - \dfrac{1}{\overline{\mu_k}}\right)(Py_k, e_k), & k-j=0,\\ -2\left(\dfrac{1}{\mu_k} + \dfrac{1}{\overline{\mu_k}}\right)(Py_k, e_k), & k+j=0,\\ 0, & \text{others}. \end{cases}
\end{aligned} \tag{5}
$$

Proof Through direct computation, the conclusion is obtained. □

Theorem 2.4. If the operator matrix $\boldsymbol{M}$ satisfies the assumptions of Lemmas 2.2 and 2.3, then the eigenvector system of $\boldsymbol{M}$ is complete in the symplectic space X^4 if and only if the vector systems $\{g_k\}_{k=1}^{\infty}$ and $\{e_k\}_{k=1}^{\infty}$ are both Schauder bases in X.

Proof Sufficiency. Lemmas 2.2 and 2.3 show that $\{\boldsymbol{U}_k, \boldsymbol{U}_{-k}, \boldsymbol{V}_k, \boldsymbol{V}_{-k} \,|\, k \in \Lambda\}$ is an eigenvector system of $\boldsymbol{M}$ and has symplectic orthogonality. Now, for $k \in \Lambda$, we take

$$
\begin{aligned}
d_k &= \frac{(\Delta, J\boldsymbol{U}_k)(\boldsymbol{V}_{-k}, J\boldsymbol{U}_{-k}) - (\Delta, J\boldsymbol{U}_{-k})(\boldsymbol{V}_{-k}, J\boldsymbol{U}_k)}{(\boldsymbol{V}_k, J\boldsymbol{U}_k)(\boldsymbol{V}_{-k}, J\boldsymbol{U}_{-k}) - (\boldsymbol{V}_k, J\boldsymbol{U}_{-k})(\boldsymbol{V}_{-k}, J\boldsymbol{U}_k)},\\
d_{-k} &= \frac{(\Delta, J\boldsymbol{U}_{-k})(\boldsymbol{V}_k, J\boldsymbol{U}_k) - (\Delta, J\boldsymbol{U}_k)(\boldsymbol{V}_k, J\boldsymbol{U}_{-k})}{(\boldsymbol{V}_k, J\boldsymbol{U}_k)(\boldsymbol{V}_{-k}, J\boldsymbol{U}_{-k}) - (\boldsymbol{V}_k, J\boldsymbol{U}_{-k})(\boldsymbol{V}_{-k}, J\boldsymbol{U}_k)},\\
c_k &= \frac{\big((\Delta, J\boldsymbol{V}_k) - d_k(\boldsymbol{V}_k, J\boldsymbol{V}_k) - d_{-k}(\boldsymbol{V}_{-k}, J\boldsymbol{V}_k)\big)(\boldsymbol{U}_{-k}, J\boldsymbol{V}_{-k})}{(\boldsymbol{U}_k, J\boldsymbol{V}_k)(\boldsymbol{U}_{-k}, J\boldsymbol{V}_k) - (\boldsymbol{U}_k, J\boldsymbol{V}_{-k})(\boldsymbol{U}_{-k}, J\boldsymbol{V}_k)}\\
&\quad - \frac{\big((\Delta, J\boldsymbol{V}_{-k}) - d_k(\boldsymbol{V}_k, J\boldsymbol{V}_{-k}) - d_{-k}(\boldsymbol{V}_{-k}, J\boldsymbol{V}_{-k})\big)(\boldsymbol{U}_{-k}, J\boldsymbol{V}_k)}{(\boldsymbol{U}_k, J\boldsymbol{V}_k)(\boldsymbol{U}_{-k}, J\boldsymbol{V}_k) - (\boldsymbol{U}_k, J\boldsymbol{V}_{-k})(\boldsymbol{U}_{-k}, J\boldsymbol{V}_k)},\\
c_{-k} &= \frac{-\big((\Delta, J\boldsymbol{V}_k) - d_k(\boldsymbol{V}_k, J\boldsymbol{V}_k) - d_{-k}(\boldsymbol{V}_{-k}, J\boldsymbol{V}_k)\big)(\boldsymbol{U}_k, J\boldsymbol{V}_{-k})}{(\boldsymbol{U}_k, J\boldsymbol{V}_k)(\boldsymbol{U}_{-k}, J\boldsymbol{V}_k) - (\boldsymbol{U}_k, J\boldsymbol{V}_{-k})(\boldsymbol{U}_{-k}, J\boldsymbol{V}_k)}\\
&\quad + \frac{\big((\Delta, J\boldsymbol{V}_{-k}) - d_k(\boldsymbol{V}_k, J\boldsymbol{V}_{-k}) - d_{-k}(\boldsymbol{V}_{-k}, J\boldsymbol{V}_{-k})\big)(\boldsymbol{U}_k, J\boldsymbol{V}_k)}{(\boldsymbol{U}_k, J\boldsymbol{V}_k)(\boldsymbol{U}_{-k}, J\boldsymbol{V}_k) - (\boldsymbol{U}_k, J\boldsymbol{V}_{-k})(\boldsymbol{U}_{-k}, J\boldsymbol{V}_k)}.
\end{aligned} \tag{6}
$$

then,

$$
\begin{aligned}
&\sum_{k=1}^{+\infty} c_k\boldsymbol{U}_k + c_{-k}\boldsymbol{U}_{-k} + d_k\boldsymbol{V}_k + d_{-k}\boldsymbol{V}_{-k}\\
&= \sum_{k=1}^{+\infty} \begin{pmatrix} \dfrac{(h_1, e_k) - (h_4, y_k) + \dfrac{(h_4, g_k)}{(Pe_k, g_k)}\big((Py_k, e_k) + (e_k, Py_k)\big)}{(Pg_k, e_k)} Pg_k - \dfrac{(h_4, g_k)}{(Pe_k, g_k)} Py_k\\ \dfrac{(h_2, Pe_k) + (h_3, Py_k) - \dfrac{(h_3, Pg_k)}{(Pe_k, g_k)}\big((Py_k, e_k) + (e_k, Py_k)\big)}{(Pg_k, e_k)} g_k + \dfrac{(h_3, Pg_k)}{(Pe_k, g_k)} y_k\\ \dfrac{(h_3, Pg_k)}{(Pe_k, g_k)} e_k\\ \dfrac{(h_4, g_k)}{(Pe_k, g_k)} Pe_k \end{pmatrix}.
\end{aligned} \tag{7}
$$

Since the vector system $\{e_k\}_{k=1}^{\infty}$ is a Schauder basis in X, there exists a unique constant sequence $\{d_k(h_3)\}_{k=1}^{\infty}$ such that

$$
h_3 = \sum_{k=1}^{\infty} d_k(h_3) e_k
$$

for each $h_3 \in X$. Taking the inner product of the above relation by Pg_k on the right-hand side, obvioulsy we clearly have $d_k(h_3) = (h_3, Pg_k)/(e_k, Pg_k)$, i.e.,

$$h_3 = \sum_{k=1}^{+\infty} \frac{(h_3, Pg_k)}{(e_k, Pg_k)} e_k. \tag{8}$$

Then, in view of $\{g_k\}_{k=1}^{\infty}$ being a Schauder basis, similarly we have

$$h_2 = \sum_{k=1}^{+\infty} \frac{(h_2, Pe_k)}{(g_k, Pe_k)} g_k. \tag{9}$$

Based on Eqs. (8) and (9), $\{Pe_k\}_{k=1}^{\infty}$ and $\{Pg_k\}_{k=1}^{\infty}$ are both Schauder bases in X. So

$$h_1 = \sum_{k=1}^{+\infty} \frac{(h_1, e_k)}{(Pg_k, e_k)} Pg_k, \quad h_4 = \sum_{k=1}^{+\infty} \frac{(h_4, g_k)}{(Pe_k, g_k)} Pe_k. \tag{10}$$

Using Eqs. (10), (8) and $(Py_k, e_k) = (y_k, Pe_k) \neq 0, (Py_k, e_j) = 0, (k \neq j \in \Lambda)$, we see that for each $k, j \in \Lambda$

$$\left(\frac{-(h_4, y_k) + \dfrac{(h_4, g_k)}{(Pe_k, g_k)} \big((Py_k, e_k) + (e_k, Py_k) \big)}{(Pg_k, e_k)} Pg_k - \frac{(h_4, g_k)}{(Pe_k, g_k)} Py_k, \ e_j \right) = 0,$$

$$\left(\frac{(h_3, Py_k) - \dfrac{(h_3, Pg_k)}{(Pe_k, g_k)} \big((Py_k, e_k) + (e_k, Py_k) \big)}{(Pg_k, e_k)} g_k + \frac{(h_3, Pg_k)}{(Pe_k, g_k)} y_k, \ Pe_j \right) = 0.$$

Note that $\{e_j\}_{j=1}^{\infty}$ and $\{Pe_j\}_{j=1}^{\infty}$ are both Schauder bases in X. Thus, for $k \in \Lambda$,

$$\frac{-(h_4, y_k) + \dfrac{(h_4, g_k)}{(Pe_k, g_k)} \big((Py_k, e_k) + (e_k, Py_k) \big)}{(Pg_k, e_k)} Pg_k - \frac{(h_4, g_k)}{(Pe_k, g_k)} Py_k = 0,$$

$$\frac{(h_3, Py_k) - \dfrac{(h_3, Pg_k)}{(Pe_k, g_k)} \big((Py_k, e_k) + (e_k, Py_k) \big)}{(Pg_k, e_k)} g_k + \frac{(h_3, Pg_k)}{(Pe_k, g_k)} y_k = 0.$$

Therefore, from Eqs. (7)–(10), we obtain

$$\sum_{k=1}^{+\infty} c_k U_k + c_{-k} U_{-k} + d_k V_k + d_{-k} V_{-k} = \sum_{k=1}^{+\infty} \begin{pmatrix} \dfrac{(h_1, e_k)}{(Pg_k, e_k)} Pg_k \\ \dfrac{(h_2, Pe_k)}{(Pg_k, e_k)} g_k \\ \dfrac{(h_3, Pg_k)}{(Pe_k, g_k)} e_k \\ \dfrac{(h_4, g_k)}{(Pe_k, g_k)} Pe_k \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix}. \tag{11}$$

Thus, Eq. (1) is valid. On the other hand, we assume that there is another constant sequence $\{\widetilde{c}_k, \widetilde{c}_{-k}, \widetilde{d}_k, \widetilde{d}_{-k} | \, k \in \Lambda\}$ such that the expansion (11) is valid. Then, we have

$$\sum_{k=1}^{+\infty} (c_k - \widetilde{c}_k) U_k + (c_{-k} - \widetilde{c}_{-k}) U_{-k} + (d_k - \widetilde{d}_k) V_k + (d_{-k} - \widetilde{d}_{-k}) V_{-k} = 0,$$

which obviously implies that $d_k = \widetilde{d}_k$, $d_{-k} = \widetilde{d}_{-k}$ $c_k = \widetilde{c}_k$, and $c_{-k} = \widetilde{c}_{-k}$, by taking the inner product of the relation as JU_{-k}, JU_k, JV_{-k} and JV_k on the right-hand side, respectively. Therefore, the eigenvector system $\{U_k, U_{-k}, V_k, V_{-k} \, | \, k \in \Lambda\}$ of the operator $\boldsymbol{M}$ is complete in X by definition 1.2.

Necessity. Assume that the eigenvector system $\{U_k, U_{-k}, V_k, V_{-k} \,|\, k \in \Lambda\}$ of the operator matrix $\boldsymbol{M}$ is complete in the symplectic space X^4, i.e., for each $\Delta = (h_1\ h_2\ h_3\ h_4)^{\mathrm{T}} \in X^4$, there exists a unique constant sequence $\{c_k, d_k \,|\, k = \pm 1, \pm 2, \dots\}$ such that the equality (1) holds. Then, taking the inner product of the relation as JU_{-k}, JU_k, JV_{-k} and JV_k on the right-hand side, respectively, we deduce that c_k, d_k are determined by Eq. (6). So

$$\begin{aligned} h_3 &= \sum_{k=1}^{+\infty}(d_k - d_{-k})e_k \\ &= \sum_{k=1}^{+\infty}\frac{(h_3, Pg_k)}{(e_k, Pg_k)}e_k \\ h_4 &= \sum_{k=1}^{+\infty} -\frac{(d_k + d_{-k})Q^* e_k}{\overline{\lambda_k}} \\ &= \sum_{k=1}^{+\infty}\frac{(h_4, g_k)}{(Pe_k, g_k)}Pe_k. \end{aligned} \tag{12}$$

Thus, by the arbitrariness of h_3, h_4, the vector systems $\{e_k\}_{k=1}^{\infty}$ and $\{Pe_k\}_{k=1}^{\infty}$ are both bases in X. So $\{g_k\}_{k=1}^{\infty}$ is also a Schauder basis in X by Eq. (12). Therefore, the proof is completed. □

3. Applications

In this section, we present an example illustrating the previous results. We always assume that $X = L^2[0, 1]$.

Example The free vibration of rectangular thin plates with two opposite edges simply supported is given by

$$D\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 W - \rho\omega^2 W = 0, \tag{13}$$

where $W(x, y)$ is vibration mode, ρ is mass density, D is bending stiffness, and ω is natural frequency. The boundary conditions for simply supported edges $y = 0$ and $y = 1$ are

$$W = 0, \frac{\partial^2 W}{\partial y^2} = 0, \quad \text{for} \quad y = 0 \quad \text{and} \quad y = 1. \tag{14}$$

Let

$$\theta = \frac{\partial W}{\partial x}, \quad u = \frac{\partial^3 W}{\partial x^3} + \frac{\partial^3 W}{\partial x \partial y^2} + \sqrt{\rho\omega^2/D}\frac{\partial W}{\partial x},$$
$$v = -\left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \sqrt{\rho\omega^2/D}W\right).$$

Then,

$$\frac{\partial}{\partial x}\begin{bmatrix} W \\ \theta \\ u \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\partial^2/\partial y^2 - \sqrt{\rho\omega^2/D} & 0 & 0 & -1 \\ 0 & 0 & 0 & \partial^2/\partial y^2 - \sqrt{\rho\omega^2/D} \\ 0 & 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} W \\ \theta \\ u \\ v \end{bmatrix}, \tag{15}$$

which is the required differential system from Eq. (13). Write

$$\boldsymbol{M} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\mathrm{d}^2/\mathrm{d}y^2 - \sqrt{\rho\omega^2/D} & 0 & 0 & -1 \\ 0 & 0 & 0 & \mathrm{d}^2/\mathrm{d}y^2 - \sqrt{\rho\omega^2/D} \\ 0 & 0 & -1 & 0 \end{bmatrix}. \tag{16}$$

According to the boundary conditions (14), we have

$$\mathcal{D}(\boldsymbol{M}) = \left\{ \begin{bmatrix} W \\ \theta \\ u \\ v \end{bmatrix} \in X^4 \,\middle|\, \begin{array}{c} W(0) = W(1) = 0,\ v(0) = v(1) = 0,\ W', v' \text{ are} \\ \text{absolutely continuous},\ W'', v'' \in L^2[0, 1] \end{array} \right\}. \tag{17}$$

Direct calculations show that $0 \notin \sigma_p(M)$, eigenvalues and associated eigenfunctions of the operator $Q_1 P$ and $Q_2 P$ are $\alpha_k = (k\pi)^2 - \sqrt{\rho\omega^2/D}$, $\beta_k = -(k\pi)^2 - \sqrt{\rho\omega^2/D}$ and $g_k = \lambda_k \sin(k\pi y), e_k = \mu_k \sin(k\pi y)$, where $(\lambda_k)^2 = \alpha_k, (\mu_k)^2 = -\beta_k (k = 1, 2, \dots)$. Obviously,

$$(Pg_k, e_k) = (g_k, Pe_k) \neq 0, \quad (Pg_k, e_j) = 0, (k \neq j \in \Lambda).$$

Also, there exists $y_k = \mu_k/2\sqrt{\rho\omega^2/D}\sin(k\pi y)$, such that

$$Q_1 P y_k - DPe_k = -\beta_k y_k$$

for $k \in \Lambda$, and

$$(Py_k, e_k) = (y_k, Pe_k) \neq 0, (Py_k, e_j) = 0, (k, j \in \Lambda).$$

Thus, the assumptions of Theorem 2.4 are satisfied. Note that $\{g_k\}_{k=1}^{\infty} = \{\lambda_k \sin(k\pi y)\}_{k=1}^{\infty}$ and $\{e_k\}_{k=1}^{\infty} = \{\mu_k \sin(k\pi y)\}_{k=1}^{\infty}$ are both orthogonal bases in $\boldsymbol{X}$, which are clearly Schauder bases. Therefore, the eigenvector system of the operator matrix $\boldsymbol{M}$ is complete in the symplectic space X^4 by Theorem 2.4.

On the other hand, it can be seen that the eigenvalues and associated eigenvectors of the operator $\boldsymbol{M}$ are $\lambda_k = (\mathrm{sgn}k)\sqrt{(k\pi)^2 - \sqrt{\rho\omega^2/D}}$, $\mu_k = (\mathrm{sgn}k)\sqrt{(k\pi)^2 + \sqrt{\rho\omega^2/D}}$ $(\pm 1, \pm 2, \dots)$, and

$$U_k = (\sin(k\pi y) \quad \lambda_k \sin(k\pi y) \quad 0 \quad 0)^{\mathrm{T}},$$

$$V_k = \left(\frac{1}{2\sqrt{\rho\omega^2/D}} \sin(k\pi y) \quad \frac{\mu_k}{2\sqrt{\rho\omega^2/D}} \sin(k\pi y) \quad \mu_k \sin(k\pi y) \quad -\sin(k\pi y) \right)^{\mathrm{T}}.$$

We can directly check the eigenvector system of $\boldsymbol{M}$ is complete in the symplectic space X^4, which coincides with the conclusion given by using Theorem 2.4.

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