Disturbed solution of the El Niño-southern oscillation sea–air delayed oscillator*

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A class of delayed oscillators of El Niño-southern oscillation (ENSO) models is considered. Using the delayed theory, the perturbed theory and other methods, the asymptotic expansions of the solutions for ENSO models are obtained and the asymptotic behaviour of solution of corresponding problem is studied.

Keywords: nonlinear, perturbation theory, EI Niño-southern oscillation model
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1. Introduction

The El Niño-southern oscillation (ENSO) is an interannual phenomenon involved in the tropical Pacific ocean–atmosphere interactions. The phenomenon of the ENSO is a very attractive object of study in the international academic circles. A class of problems in atmospheric physics, ocean weather, dynamic system and so on have been considered.[1−18] The essence of ENSO oscillation requires both positive and negative ocean–atmosphere feedbacks. This anomaly reduces the zonal sea surface temperature (SST) gradient and hence the strength of the southern oscillation circulation, resulting in weaker trade winds around the equator. The weaker trade winds in turn drive the ocean circulation changes, thereby reinforcing the SST anomaly. This positive ocean-atmosphere feedback or coupled ocean atmosphere instability leads the equatorial Pacific to a never-ending warm state. Recently many scholars have investigated the nonlinear singularly perturbed problem. Approximation methods have been developed and refined, including the method of averaging, the boundary layer method, the matched asymptotic, the expansion method and the multiple scales method.[19−23] Using the perturbed method, Mo *et al.* considered a class of singularly perturbed problems.[24−35] In this paper we consider the some delayed ENSO models using a simple and special method for perturbation theory.

2. Delayed oscillator model of equatorial Pacific SST

In the delayed oscillator model it is assumed that winds in the western Pacific do not affect the SST anomalies in the eastern Pacific and that the atmosphere is in a steady state or the Niño-4 (over 5°S–5°N, 160°E–150°W) zonal wind stress anomalies are linearly proportional to the Niño-3 (over 5°S–5°N, 150°–90°W) SST anomalies. We can obtain the following nonlinear time delay model of equatorial Pacific:[13]

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{ad}{R}T - \frac{bd}{R}T(t-\tau) - \varepsilon T^3, \tag{1}$$

and initial condition

$$T(t) = h(t), \quad -\tau \le t \le 0, \tag{2}$$

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where T is the Niño-3 SST anomaly, τ is the time for waves to travel to the western boundary and return to the eastern Pacific, a is the coefficient representing the positive feedback between T's, b is the coefficient representing the negative feedback due to waves reflection at the western boundary, d is the positive coefficient that relates the Niño-3 SST anomalies to the Niño-4 zonal wind stress anomalies, R is the damping coefficient, ε is the cubic damping coefficient, and $h(t) \in C^{\infty}(-\tau, \infty)$ is an initial function. We also assume that τ is a small delayed parameter.

Let $\tau = 0$ then we will have

$$\frac{\mathrm{d}T(t)}{\mathrm{d}t} = \frac{(a-b)d}{R}T(t) - \varepsilon[T(t)]^3, \tag{3}$$

and introduce

$$\bar{T} = T^{-2}. \tag{4}$$

From Eq. (3), we have

$$\bar{T} = C\exp\left(-\frac{2(a-b)d}{R}t\right) + \frac{R\varepsilon}{(a-b)d}. \tag{5}$$

Then from Eqs. (4) and (5), we obtain

$$T(t) = \left[C\exp\left(-\frac{2(a-b)d}{R}t\right) + \frac{R\varepsilon}{(a-b)d}\right]^{-1/2}, \tag{6}$$

where C is an arbitrary constant.

3. Perturbed solution of SST

We assume that Eq. (1) possesses the following formal solution:

$$T(t,\tau) = U(t,\tau) + \tau\, V(\eta,\tau), \tag{7}$$

where $U(t,\tau)$ is the outer solution

$$U(t,\tau) \sim \sum_{i=0}^{\infty} U_i(t)\tau^i, \tag{8}$$

and $V(\eta,\tau)$ is the initial-shock layer corrective term

$$V(\eta,\tau) \sim \sum_{i=0}^{\infty} v_i(\eta)\tau^i, \tag{9}$$

$\eta = t/\tau$ is a stretched variable, and it tends to zero as $\eta \to +\infty$ for every $v_i(\eta, x)$.

Developing $T(t-\tau)$ in τ, we have

$$T(t-\tau) \sim T(t) - T_t(t)\tau + \frac{1}{2}T_{tt}(t)\tau^2 + \ldots. \tag{10}$$

Substituting Eqs. (7)–(9) into Eq. (1), developing the nonlinear term in τ, comparing the same coefficients of τ^i and for $i = 0$, and noting that the U_0 satisfies Eqs. (6) and (2), i.e.,

$$\begin{aligned} &U_0(t) \\ &= \left[\left(h^{-2}(0) - \frac{R\varepsilon}{(a-b)d}\right)\right. \\ &\quad \left.\times \exp\left(-\frac{2(a-b)d}{R}t\right) + \frac{R\varepsilon}{(a-b)d}\right]^{-1/2}, \end{aligned} \tag{11}$$

for $i = 1$, we have

$$\frac{\mathrm{d}U_1}{\mathrm{d}t} = \left(\frac{(a-b)d}{R} - 3\varepsilon\, U_0^2\right)U_1 + \frac{\mathrm{d}U_0}{\mathrm{d}t}, \tag{12}$$

and

$$U_1(0) = -v_0(0), \tag{13}$$

for $i = 2, 3\ldots$, we have

$$\begin{aligned} &\frac{\mathrm{d}U_i}{\mathrm{d}t} \\ &= \left(\frac{(a-b)d}{R} - 3\varepsilon\, U_0^2\right)U_i + F_i(t),\ i = 2, 3, \ldots, \end{aligned} \tag{14}$$

and

$$U_i(0) = -v_{i-1}(0), \quad i = 2, 3, \ldots, \tag{15}$$

where F_i $(i = 2, 3, \ldots)$ are determined functions, their constructions are omitted. From Eqs. (11)–(15), we have

$$\begin{aligned} U_1(t) &= -v_0(0)\exp\int_0^t \left(\frac{(a-b)d}{R} - 3\varepsilon\, U_0^2(t_1)\right)\mathrm{d}t_1 \\ &\quad + \int_{\tau_1}^t \frac{U_0}{d\tau_1}(\tau_1)\exp\int_{\tau}^t \left(\frac{(a-b)d}{R} - 3\varepsilon\, U_0^2(t_1)\right)\mathrm{d}t_1\,\mathrm{d}\tau_1. \end{aligned} \tag{16}$$

$$\begin{aligned} U_i(t) &= -v_{i-1}(0)\exp\int_0^t \left(\frac{(a-b)d}{R} - 3\varepsilon\, U_0^2(t_1)\right)\mathrm{d}t_1 \\ &\quad + \int_0^t F_i(\tau_1)\exp\int_{\tau_1}^t \left(\frac{(a-b)d}{R} - 3\varepsilon\, U_0^2(t_1)\right)\mathrm{d}t_1\,\mathrm{d}\tau_1, \quad i = 2, 3, \ldots. \end{aligned} \tag{17}$$

Now we construct the initial-shock layer functions $v_i(\eta,\tau)$ $(i = 0, 1, 2, \ldots)$. Using the integration by parts, they are obtained successively in the intervals $p \le \eta \le p+1$, $p \ge 0$. Substituting Eqs. (7)–(9) into Eqs. (1) and

(2), we have

$$\begin{aligned}\frac{\partial V}{\partial \eta}(\eta,\tau) &= \frac{ad}{R}(\tau V(\tau\eta,\tau)) + \frac{bd}{R}(U(\tau\eta,\tau) - h(\tau(\eta-1)))\\ &\quad + [U(\tau\eta,\tau) + \tau V(\tau\eta,\tau)]^3 - [U(\tau\eta,\tau)]^3. \quad 0 \le \eta \le 1, \qquad (18)\\ \frac{\partial V}{\partial \eta}(\eta,\tau) &= \frac{ad}{R}(\tau V(\tau\eta,\tau)) + \frac{bd}{R}(U(\tau\eta,\tau) - U(\tau(\eta-1),\tau) - \tau V(\tau(\eta-1),\tau))\\ &\quad + [U(\tau\eta,\tau) + \tau V(\tau\eta,\tau)]^3 - [U(\tau\eta,\tau)]^3, \ \eta \ge 1. \qquad (19)\end{aligned}$$

From Eqs. (9) and (2) it follows that for $\tau = 0$, $v_0(\eta)$ satisfies

$$\frac{\partial v_0}{\partial \eta}(\eta) = 0, \quad \eta \ge 0, \qquad (20)$$

and

$$v_0(0) = 0. \qquad (21)$$

Thus, from Eqs. (18), (19), and (21), we can obtain $v_0(\eta) = 0$.

We can also obtain $U_1(0) = -v_0(0)$, i.e., $U_1(0) = 0$. Thus from Eq. (16), we have

$$U_1(t) = \int_0^t \frac{\mathrm{d}U_0}{\mathrm{d}\tau_1}(\tau_1) \exp \int_{\tau_1}^t \left(\frac{(a-b)d}{R} - 3\varepsilon\, U_0^2(t_1)\right) \mathrm{d}t_1 \,\mathrm{d}\tau_1. \qquad (22)$$

Noting Eq. (10), comparing the same coefficients of τ^i in Eqs. (18) and (19), we have

$$\frac{\partial v_i}{\partial \eta}(\eta) = G_p^i(\eta), \quad p \le \eta \le p+1, \ \ i = 1, 2, \ldots, \qquad (23)$$

where G_p^i, $p = 0, 1, \ldots$, are determined functions successively. Therefore from Eq. (23), we can obtain

$$v_i(\eta) = v_i(0) + \sum_{l=0}^{p-1} \int_l^{l+1} G_l^i(s)\,\mathrm{d}s + \int_p^\eta G_p^i(s)\,\mathrm{d}s, \quad p \le \eta \le p+1, \qquad (24)$$

Thus we have

$$U_{i+1}(0) = -v_i(0) = \sum_{l=0}^{\infty} \left(\int_l^{l+1} G_l^i(s)\,\mathrm{d}s\right), \ \ (i = 1, 2, \ldots). \qquad (25)$$

From Eqs. (17) and (25), we have

$$\begin{aligned}U_i(t) &= \left[\sum_{l=0}^{\infty} \int_l^{l+1} G_l^{i-1}(s)\,\mathrm{d}s\right] \left[\exp \int_0^t \left(\frac{(a-b)d}{R} - 3\varepsilon\, U_0^2(t_1)\right) \mathrm{d}t_1\right]\\ &\quad + \int_0^t F_i(\tau_1) \exp \int_{\tau_1}^t \left(\frac{(a-b)d}{R} - 3\varepsilon\, U_0^2(t_1)\right) \mathrm{d}t_1 \,\mathrm{d}\tau_1, \quad i = 2, 3, \ldots. \qquad (26)\end{aligned}$$

Using the above iteration method, we can obtain $U_i(t)$, $v_{i-1}(\eta)$ $(i = 1, 2, \ldots)$, successively. Substituting determined functions into Eqs. (7)–(9), from Eqs. (11), (22), (26) and (24), we obtain the uniformly valid asymptotic expansion of solution for the problem (Eqs. (1) and (2)) as follows:[36,37]

$$\begin{aligned}&T(t,\tau)\\ &= \left[\left(h^{-2}(0) - \frac{R\varepsilon}{(a-b)d}\right) \exp\left(-\frac{2(a-b)d}{R}t\right) + \frac{R\varepsilon}{(a-b)d}\right]^{-1/2}\\ &\quad + \int_0^t \frac{\mathrm{d}U_0}{\mathrm{d}\tau_1}(\tau_1) \exp \int_{\tau_1}^t \left(\frac{(a-b)d}{R} - 3\varepsilon\, U_0^2(t_1)\right) \mathrm{d}t_1 \,\mathrm{d}\tau_1\\ &\quad + \sum_{i=2}^{N} \left[\sum_{l=0}^{\infty} \int_l^{l+1} G_l^{i-1}(s)\,\mathrm{d}s\right] \left[\exp \int_0^t \left(\frac{(a-b)d}{R} - 3\varepsilon\, U_0^2(t_1)\right) \mathrm{d}t_1\right] \tau^i\end{aligned}$$

$$+\tau\int_0^t F_i(\tau)\exp\int_{\tau_1}^t\left(\frac{(a-b)d}{R}-3\varepsilon\,U_0^2(t_1)\right)\mathrm{d}t_1\,\mathrm{d}\tau_1-\sum_{i=0}^{N}\left[\sum_{l=p}^{\infty}\left(\int_l^{l+1}G_l^i(s)\,\mathrm{d}s\right)+\int_p^{\eta}G_p^i(s)\,\mathrm{d}s\right]\tau^{i+1}$$
$$+O(\tau^{N+1}),\quad 0\le t\le T_0,\ \ 0\le\tau<<1. \tag{27}$$

where F_{i+1}, G_l^{i-1} $(0\le p\le\eta\le p+1,\ i=1,2,\dots)$ are successively determined functions, U_0 is expressed as Eq. (11) and T_0 is a large enough psitive constant.

4. Curves for SST

In order to evaluate the accuracy of expansion Eq. (27) of the asymptotic disturbed solution to Eqs. (1) and (2), we give a set of parameters as follows:[5] $a=1\times10^2\ ^\circ\mathrm{C}\cdot\mathrm{m}^2\cdot\mathrm{N}^{-1}\cdot\mathrm{yr}^{-1}$, $b=3\times10^2\ ^\circ\mathrm{C}\cdot\mathrm{m}^2\cdot\mathrm{N}^{-1}\cdot\mathrm{yr}^{-1}$, $d=1\times10^{-2}\ ^\circ\mathrm{C}\cdot\mathrm{m}^{-2}\cdot\mathrm{N}\cdot\mathrm{yr}^{-1}$, $R=2.0\ \mathrm{yr}^{-1}$, $\varepsilon=40$, $h(0)=10$. From the above method we have

$$T_{\mathrm{per}}(t)=U_0(t)+\tau\int_0^t\frac{\mathrm{d}U_0}{\mathrm{d}\tau_1}(\tau_1)\exp\int_{\tau_1}^t[[-1-120U_0^2(\tau)]\mathrm{d}t\,\mathrm{d}\tau_1+O(\tau^2),\quad 0<\tau<<1, \tag{28}$$

where

$$U_0(\tau)=\left[\left(\frac{1}{100}+40\right)(\exp(2\tau)-40\right)\right]^{-1/2}.$$

Now we compare the values of T_{num} from the numerical calculation and T_{per} from the asymptotic disturbed expansion Eq. (28) at different values of $T(^\circ\mathrm{C})$ and $t(\mathrm{yr})$. We obtain the curves for $\tau=0.8$ (Fig. 1) and $\tau=0.2$ (Fig. 2).

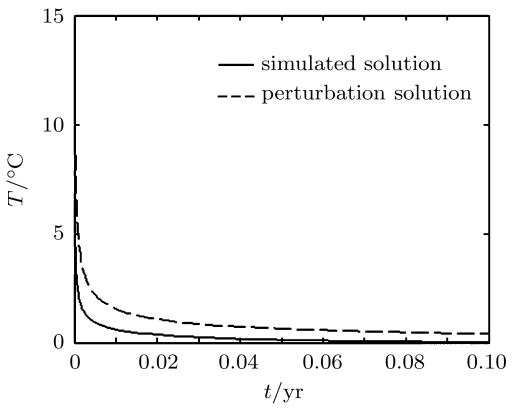


Fig. 1. A comparison between the curves for T_{num} (solid line) and T_{per} (dash line) with $\tau=0.8$.

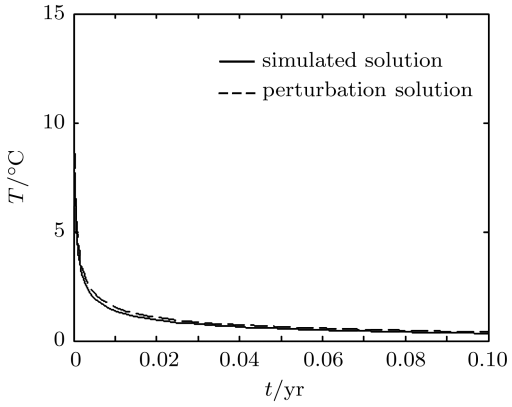


Fig. 2. A comparison between the curves for T_{num} (solid line) and T_{per} (dash line) with $\tau=0.2$.

From Figs. 1 and 2, we can see that the first approximate disturbed solution T_{per} obtained using the perturbation method has a good accuracy and the smaller the τ, the better and the accuracy is.

5. Accuracy of perturbed solution

We give a set of parameters the same as those in Section 4. Then we have the values of expansion Eq. (28) for $\tau=0.2$ (see Table 1).

Table 1. A comparison between the values of T_{num} and T_{per} for $\tau=0.2$.

t	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
T_{num}	15.00	1.36	0.94	0.75	0.63	0.53	0.51	0.44	0.40	0.37	0.34
T_{per}	15.00	1.52	1.07	0.86	0.73	0.64	0.57	0.52	0.47	0.44	0.40

From Table 1, we can also see that the first approximate disturbed solution $T_{\rm per}$ obtained using the perturbation method has a good accuracy.

6. Conclusions

(i) From the comparison of accuracy, we can conclude that for small parameter τ, the asymptotic expansion Eq. (27) is a good approximate expansion of the disturbed solution for the nonlinear time delay model (Eqs. (1) and (2)).

(ii) The ENSO is a complicated natural phenomenon. Hence we need to reduce basic models for the ENSO sea-air oscillator and solve them by using the approximate method. The perturbation method is a valid method.

(iii) The perturbation method is an asymptotic method, which differs from general numerical method. The expansions of disturbed solution obtained through the perturbation method can be used analytically. Thus, from Eq. (27), we can further study that the fixed quality and quantitative behaviours of the temperature anomaly in the equatorial eastern Pacific.

(iv) The approximate solution Eq. (27) can be obtained by using the perturbation method, and the progress of the study on SST and relevant physical quantities is made.

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