The Hirota bilinear method for the coupled Burgers equation and the high-order Boussinesq–Burgers equation*

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This paper studies the coupled Burgers equation and the high-order Boussinesq–Burgers equation. The Hirota bilinear method is applied to show that the two equations are completely integrable. Multiple-kink (soliton) solutions and multiple-singular-kink (soliton) solutions are derived for the two equations.

Keywords: coupled Burgers equation, high-order Boussinesq–Burgers equation, Hirota's bilinear method

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1. Introduction

It is well known that the multi-soliton solutions of completely integrable evolution equations can be obtained using three different methods:[1−8] the inverse scattering method, the Bäcklund transformation method and the Hirota bilinear method. However, the Hirota bilinear method[9−14] is rather heuristic and gives multiple soliton solutions for a wide class of nonlinear evolution equations. Other methods are also used in Refs. [15]–[22] to the handle nonlinear evolution equations for single soliton solutions.

In this paper, we study the following coupled Burgers equation:

$$\begin{aligned} &u_t + \alpha u u_x + \beta v v_x + \gamma u_{xx} = 0, \\ &v_t + \delta (uv)_x + \gamma v_{xx} = 0, \end{aligned} \tag{1}$$

where α, β, γ, δ are arbitrary constants, and the high-order Boussinesq–Burgers equation is given by

$$\begin{aligned} &u_t - 3\sigma u^2 u_x + \frac{3}{2}\sigma (uv)_x - \frac{1}{4}\sigma u_{xxx} = 0, \\ &v_t + \frac{3}{2}\sigma v v_x - 3\sigma (u^2 v)_x + 3\sigma u_x u_{xx} \\ &\quad + \frac{3}{2}\sigma u u_{xxx} - \frac{1}{4}\sigma v_{xxx} = 0, \end{aligned} \tag{2}$$

where σ is an arbitrary constant.

The objectives of this paper are twofold. First, we apply Hirota's bilinear method to two equations. Secondly, we establish multiple-kink (soliton) solutions and multiple- singular-kink (soliton) solutions in order to confirm that the two equations are completely integrable.

2. The Hirota bilinear method

N-soliton solutions can be obtained by following these main steps:

(i) for the dispersion relation, we use

$$u(x,t) = \mathrm{e}^{\theta_i}, \quad \theta_i = k_i x - c_i t; \tag{3}$$

(ii) next we substitute the single soliton solution

$$\begin{aligned} u(x,t) &= R\left[\arctan\left(\frac{f(x,t)}{g(x,t)}\right)\right]_x \\ &= R\frac{f_x g - f g_x}{f^2 + g^2}, \end{aligned} \tag{4}$$

$$\begin{aligned} u(x,t) &= R\big[\ln f(x,t)\big]_x \\ &= R\frac{f_x}{f}, \end{aligned} \tag{5}$$

or

$$u(x,t) = R\big[\ln f(x,t)\big]_{xx} = R\frac{f f_{xx} - f_x^2}{f^2} \tag{6}$$

into the equation under discussion to determine R, where

$$f(x,t) = 1 + f_1(x,t) = 1 + \mathrm{e}^{\theta_1}; \tag{7}$$

(iii) for a single soliton, we use

$$f(x,t) = 1 + \mathrm{e}^{\theta_1}; \tag{8}$$

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(iv) for two-soliton solutions, we use

$$f(x,t) = 1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + a_{12}\,\mathrm{e}^{\theta_1+\theta_2}; \tag{9}$$

(v) for three-soliton solutions, we use

$$\begin{aligned} f(x,t) = {} & 1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + \mathrm{e}^{\theta_3} + a_{12}\,\mathrm{e}^{\theta_1+\theta_2} \\ & + a_{13}\,\mathrm{e}^{\theta_1+\theta_3} + a_{23}\,\mathrm{e}^{\theta_2+\theta_3} \\ & + b_{123}\,\mathrm{e}^{\theta_1+\theta_2+\theta_3}. \end{aligned} \tag{10}$$

If the obtained result shows that $b_{123} = a_{12}a_{13}a_{23}$, then the equation gives N-soliton solutions and the equation is integrable.

However, for multiple singular soliton solutions, we apply the following steps:

(i) for a dispersion relation, we use

$$u(x,t) = \mathrm{e}^{\theta_i},\ \theta_i = k_i x - c_i t; \tag{11}$$

(ii) next we substitute the single soliton solution

$$\begin{aligned} u(x,t) &= R\left[\arctan\left(\frac{f(x,t)}{g(x,t)}\right)\right]_x \\ &= R\frac{f_x g - f g_x}{f^2+g^2}, \end{aligned} \tag{12}$$

$$u(x,t) = R\big[\ln f(x,t)\big]_x = R\frac{f_x}{f}, \tag{13}$$

or

$$u(x,t) = R\big[\ln f(x,t)\big]_{xx} = R\frac{f f_{xx} - f_x^2}{f^2} \tag{14}$$

into the equation under discussion to determine R, where

$$f(x,t) = 1 - f_1(x,t) = 1 - \mathrm{e}^{\theta_1}; \tag{15}$$

(iii) for a single soliton, we use

$$f(x,t) = 1 - \mathrm{e}^{\theta_1}; \tag{16}$$

(iv) for two-soliton solutions, we use

$$f(x,t) = 1 - \mathrm{e}^{\theta_1} - \mathrm{e}^{\theta_2} + a_{12}\,\mathrm{e}^{\theta_1+\theta_2}; \tag{17}$$

(v) for three-soliton solutions, we use

$$\begin{aligned} f(x,t) = {} & 1 - \mathrm{e}^{\theta_1} - \mathrm{e}^{\theta_2} - \mathrm{e}^{\theta_3} + a_{12}\,\mathrm{e}^{\theta_1+\theta_2} \\ & + a_{13}\,\mathrm{e}^{\theta_1+\theta_3} + a_{23}\,\mathrm{e}^{\theta_2+\theta_3} \\ & - b_{123}\,\mathrm{e}^{\theta_1+\theta_2+\theta_3}. \end{aligned} \tag{18}$$

Other approaches will also be used for the multiple singular soliton solutions, as will be examined later.

3. The coupled Burgers equation

3.1. Multiple-kink solutions

Substituting the following equations

$$\begin{aligned} u(x,t) &= A\,\mathrm{e}^{k_i x - c_i t}, \\ v(x,t) &= B\,\mathrm{e}^{k_i x - c_i t} \end{aligned} \tag{19}$$

into Eq. (1) we obtain the dispersion relation

$$c_i = \gamma k_i^2, \tag{20}$$

as a result we have

$$\theta_i = k_i x - \gamma k_i^2 t, \tag{21}$$

where A, B are constants. Using the Cole–Hopf transformation method, we assume that the multiple-kink solutions of Eq. (1) are

$$\begin{aligned} u(x,t) &= R_1(\ln f)_x = R_1\frac{f_x}{f}, \\ v(x,t) &= R_2(\ln f)_x = R_2\frac{f_x}{f}, \end{aligned} \tag{22}$$

where $f(x,t)$, for single-kink solutions, is given by

$$f(x,t) = 1 + \mathrm{e}^{\theta_1} = 1 + \mathrm{e}^{k_1 x - \gamma k_1^2 t}. \tag{23}$$

Substituting Eq. (23) into Eq. (22), using the result from Eq. (1), and solving it for R_1 and R_2 we find

$$\begin{aligned} R_1 &= \frac{\gamma}{\delta}, \\ R_2 &= \pm\frac{\gamma}{\delta}\sqrt{\frac{2\delta-\alpha}{\beta}}. \end{aligned} \tag{24}$$

This means that for the single-kink solution we set

$$f(x,t) = 1 + \mathrm{e}^{\theta_1} = 1 + \mathrm{e}^{k_1 x - \gamma k_1^2 t}. \tag{25}$$

Substituting Eq. (25) into Eq. (22) we obtain the single-kink solutions

$$\begin{aligned} u(x,t) &= \frac{\gamma k_1\,\mathrm{e}^{k_1 x - \gamma k_1^2 t}}{\delta\big(1 + \mathrm{e}^{k_1 x - \gamma k_1^2 t}\big)}, \\ v(x,t) &= \pm\frac{\gamma}{\delta}\sqrt{\frac{2\delta-\alpha}{\beta}}\,\frac{k_1\,\mathrm{e}^{k_1 x - \gamma k_1^2 t}}{\big(1 + \mathrm{e}^{k_1 x - \gamma k_1^2 t}\big)}. \end{aligned} \tag{26}$$

For two-kink solutions we set

$$\begin{aligned} f(x,t) &= 1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + a_{12}\,\mathrm{e}^{\theta_1+\theta_2} \\ &= 1 + \mathrm{e}^{k_1 x - \gamma k_1^2 t} + \mathrm{e}^{k_2 x - \gamma k_2^2 t} \\ &\quad + a_{12}\,\mathrm{e}^{(k_1+k_2)x - \gamma(k_1^2+k_2^2)t}. \end{aligned} \tag{27}$$

Using Eq. (27) in Eq. (22) and substituting the result in Eq. (1), we obtain the phase shift by

$$a_{12} = 0, \tag{28}$$

and hence we set

$$a_{ij} = 0,\ 1 \le i < j \le 3. \tag{29}$$

The two-kink solutions are obtained by substituting Eqs. (28) and (27) into Eq. (22),

$$\begin{aligned} &u(x,t) \\ &= \frac{\gamma}{\delta}\frac{k_1\,\mathrm{e}^{k_1x-\gamma k_1^2t}+k_2\,\mathrm{e}^{k_2x-\gamma k_2^2t}}{\left(1+\mathrm{e}^{k_1x-\gamma k_1^2t}+\mathrm{e}^{k_2x-\gamma k_2^2t}\right)}, \\ &v(x,t) \\ &= \pm\frac{\gamma}{\delta}\sqrt{\frac{2\delta-\alpha}{\beta}}\frac{k_1\,\mathrm{e}^{k_1x-\gamma k_1^2t}+k_2\,\mathrm{e}^{k_2x-\gamma k_2^2t}}{\left(1+\mathrm{e}^{k_1x-\gamma k_1^2t}+\mathrm{e}^{k_2x-\gamma k_2^2t}\right)}. \end{aligned} \tag{30}$$

For three-kink solutions, we set

$$\begin{aligned} f(x,t) &= 1+\mathrm{e}^{\theta_1}+\mathrm{e}^{\theta_2}+\mathrm{e}^{\theta_3} \\ &= 1+\mathrm{e}^{k_1x-\gamma k_1^2t}+\mathrm{e}^{k_2x-\gamma k_2^2t} \\ &\quad +\mathrm{e}^{k_3x-\gamma k_3^2t}. \end{aligned} \tag{31}$$

Proceeding as before, we find that the three-kink solutions are given by

$$\begin{aligned} &u(x,t) \\ &= \frac{\gamma}{\delta}\frac{k_1\,\mathrm{e}^{k_1x-\gamma k_1^2t}+k_2\,\mathrm{e}^{k_2x-\gamma k_2^2t}+k_3\,\mathrm{e}^{k_3x-\gamma k_3^2t}}{\left(1+\mathrm{e}^{k_1x-\gamma k_1^2t}+\mathrm{e}^{k_2x-\gamma k_2^2t}+\mathrm{e}^{k_3x-\gamma k_3^2t}\right)}, \\ &v(x,t) \\ &= \pm\frac{\gamma}{\delta}\sqrt{\frac{2\delta-\alpha}{\beta}} \\ &\quad\times\frac{k_1\,\mathrm{e}^{k_1x-\gamma k_1^2t}+k_2\,\mathrm{e}^{k_2x-\gamma k_2^2t}+k_3\,\mathrm{e}^{k_3x-\gamma k_3^2t}}{\left(1+\mathrm{e}^{k_1x-\gamma k_1^2t}+\mathrm{e}^{k_2x-\gamma k_2^2t}+\mathrm{e}^{k_3x-\gamma k_3^2t}\right)}. \end{aligned} \tag{32}$$

The three-kink solutions are obtained by substituting Eq. (31) into Eq. (22). This shows that Eq. (1) is completely integrable and N-kink solutions can be determined for $u(x,t)$ and $v(x,t)$, for finite N, where $N \ge 1$. Based on Eq. (32), we can set the general kink solutions as

$$\begin{aligned} u(x,t) &= \frac{\gamma}{\delta}\frac{\sum\limits_{i=1}^{N}k_i\,\mathrm{e}^{k_ix-\gamma k_i^2t}}{\left(1+\sum\limits_{i=1}^{N}\mathrm{e}^{k_ix-\gamma k_i^2t}\right)}, \\ v(x,t) &= \pm\frac{\gamma}{\delta}\sqrt{\frac{2\delta-\alpha}{\beta}}\frac{\sum\limits_{i=1}^{N}k_i\,\mathrm{e}^{k_ix-\gamma k_i^2t}}{\left(1+\sum\limits_{i=1}^{N}\mathrm{e}^{k_ix-\gamma k_i^2t}\right)}, \\ N &\ge 1. \end{aligned} \tag{33}$$

3.2. Multiple-singular-kink solutions

Proceeding as before, we find the dispersion relation

$$c_i = \gamma k_i^2, \tag{34}$$

and obtain

$$\theta_i = k_ix - \gamma k_i^2t. \tag{35}$$

Using the Cole–Hopf transformation method, we assume that the single- singular-kink solutions of Eq. (1) are

$$\begin{aligned} u(x,t) &= R_1(\ln f)_x = R_1\frac{f_x}{f}, \\ v(x,t) &= R_2(\ln f)_x = R_2\frac{f_x}{f}, \end{aligned} \tag{36}$$

where $f(x,t)$, for the single-singular-kink solutions, is given by

$$f(x,t) = 1-\mathrm{e}^{\theta_1} = 1-\mathrm{e}^{k_1x-\gamma k_1^2t}. \tag{37}$$

Substituting Eq. (37) into Eq. (36), using the result from Eq. (1), and solving for R_1 and R_2 we find

$$\begin{aligned} R_1 &= \frac{\gamma}{\delta}, \\ R_2 &= \pm\frac{\gamma}{\delta}\sqrt{\frac{2\delta-\alpha}{\beta}}. \end{aligned} \tag{38}$$

Substituting Eq. (37) into Eq. (36) gives the single-singular-kink solutions

$$\begin{aligned} u(x,t) &= -\frac{\gamma k_1\,\mathrm{e}^{k_1x-\gamma k_1^2t}}{\delta\left(1-\mathrm{e}^{k_1x-\gamma k_1^2t}\right)}, \\ v(x,t) &= \pm\frac{\gamma}{\delta}\sqrt{\frac{2\delta-\alpha}{\beta}}\frac{k_1\,\mathrm{e}^{k_1x-\gamma k_1^2t}}{\left(1-\mathrm{e}^{k_1x-\gamma k_1^2t}\right)}. \end{aligned} \tag{39}$$

For two-singular-kink solutions we set

$$\begin{aligned} f(x,t) &= 1-\mathrm{e}^{\theta_1}-\mathrm{e}^{\theta_2}+a_{12}\,\mathrm{e}^{\theta_1+\theta_2} \\ &= 1-\mathrm{e}^{k_1x-\gamma k_1^2t}-\mathrm{e}^{k_2x-\gamma k_2^2t} \\ &\quad +a_{12}\,\mathrm{e}^{(k_1+k_2)x-\gamma(k_1^2+k_2^2)t}. \end{aligned} \tag{40}$$

Using Eq. (40) in Eq. (36) and substituting the result into Eq. (1), we obtain the phase shift

$$a_{12} = 0, \tag{41}$$

and hence we set

$$a_{ij} = 0,\ 1 \le i < j \le 3. \tag{42}$$

The two-singular-kink solutions are obtained by substituting Eqs. (41) and (40) into Eq. (36), where we find

$$u(x,t) = -\frac{\gamma}{\delta}\frac{k_1\,\mathrm{e}^{k_1x-\gamma k_1^2t}+k_2\,\mathrm{e}^{k_2x-\gamma k_2^2}}{\left(1-\mathrm{e}^{k_1x-\gamma k_1^2t}-\mathrm{e}^{k_2x-\gamma k_2^2t}\right)},$$

$$v(x,t) = \pm\frac{\gamma}{\delta}\sqrt{\frac{2\delta-\alpha}{\beta}}\,\frac{k_1\,\mathrm{e}^{k_1x-\gamma k_1^2t}+k_2\,\mathrm{e}^{k_2x-\gamma k_2^2t}}{\left(1-\mathrm{e}^{k_1x-\gamma k_1^2t}-\mathrm{e}^{k_2x-\gamma k_2^2t}\right)}. \tag{43}$$

For three-singular-kink solutions, we set

$$f(x,t) = 1-\mathrm{e}^{\theta_1}-\mathrm{e}^{\theta_2}-\mathrm{e}^{\theta_3}$$

$$= 1-\mathrm{e}^{k_1x-\gamma k_1^2t} - \mathrm{e}^{k_2x-\gamma k_2^2t}-\mathrm{e}^{k_3x-\gamma k_3^2t}. \tag{44}$$

Proceeding as before, we find the following three-singular-kink solutions:

$$u(x,t) = -\frac{\gamma}{\delta}\,\frac{k_1\,\mathrm{e}^{k_1x-\gamma k_1^2t}+k_2\,\mathrm{e}^{k_2x-\gamma k_2^2t}+k_3\,\mathrm{e}^{k_3x-\gamma k_3^2t}}{\left(1-\mathrm{e}^{k_1x-\gamma k_1^2t}-\mathrm{e}^{k_2x-\gamma k_2^2t}-\mathrm{e}^{k_3x-\gamma k_3^2t}\right)},$$

$$v(x,t) = \pm\frac{\gamma}{\delta}\sqrt{\frac{2\delta-\alpha}{\beta}}\,\frac{k_1\,\mathrm{e}^{k_1x-\gamma k_1^2t}+k_2\,\mathrm{e}^{k_2x-\gamma k_2^2t}+k_3\,\mathrm{e}^{k_3x-\gamma k_3^2t}}{\left(1-\mathrm{e}^{k_1x-\gamma k_1^2t}-\mathrm{e}^{k_2x-\gamma k_2^2t}-\mathrm{e}^{k_3x-\gamma k_3^2t}\right)}. \tag{45}$$

Based on the last result, we can set the general singular-kink solutions as

$$u(x,t) = -\frac{\gamma}{\delta}\,\frac{\sum\limits_{i=1}^{N} k_i\,\mathrm{e}^{k_ix-\gamma k_i^2t}}{\left(1-\sum\limits_{i=1}^{N}\mathrm{e}^{k_ix-\gamma k_i^2t}\right)},\quad v(x,t) = \pm\frac{\gamma}{\delta}\sqrt{\frac{2\delta-\alpha}{\beta}}\,\frac{\sum\limits_{i=1}^{N} k_i\,\mathrm{e}^{k_ix-\gamma k_i^2t}}{\left(1-\sum\limits_{i=1}^{N}\mathrm{e}^{k_ix-\gamma k_i^2t}\right)},\quad N\geq 1. \tag{46}$$

4. The high-order Boussinesq–Burgers equation

4.1. Multiple-kink (soliton) solutions

Substituting the following equations

$$u(x,t) = A\,\mathrm{e}^{k_ix-c_it},\quad v(x,t) = B\,\mathrm{e}^{k_ix-c_it} \tag{47}$$

into Eq. (2) we obtain the dispersion relation

$$c_i = -\frac{\sigma k_i^3}{4}, \tag{48}$$

as a result we obtain

$$\theta_i = k_ix+\frac{\sigma k_i^3}{4}t, \tag{49}$$

where A, B are constants. The single-kink (soliton) solutions of Eq. (2) are assumed to be

$$u(x,t) = R_1(\ln f)_x = R_1\frac{f_x}{f},$$

$$v(x,t) = R_2(\ln f)_{xx} = R_2\frac{ff_{xx}-f_x^2}{f^2}, \tag{50}$$

where $f(x,t)$, for the single-kink (soliton) solutions, is given by

$$f(x,t) = 1+\mathrm{e}^{\theta_1} = 1+\mathrm{e}^{k_1x+(\sigma k_1^3/4)t}. \tag{51}$$

Substituting Eq. (51) into Eq. (50), using the result from Eq. (2), and solving for R_1 and R_2 we find

$$R_1 = \pm\frac{1}{2},\quad R_2 = -\frac{1}{2}. \tag{52}$$

This means that for the single-kink (soliton) solutions we set

$$f(x,t) = 1+\mathrm{e}^{\theta_1} = 1+\mathrm{e}^{k_1x+(\sigma k_1^3/4)t}. \tag{53}$$

Substituting Eq. (53) into Eq. (50) gives the single-kink (soliton) solutions

$$u(x,t) = \pm\frac{k_1\,\mathrm{e}^{k_1x+(\sigma k_1^3/4)t}}{2\left(1+\mathrm{e}^{k_1x+(\sigma k_1^3/4)t}\right)},$$

$$v(x,t) = -\frac{k_1^2\,\mathrm{e}^{k_1x+(\sigma k_1^3/4)t}}{2\left(1+\mathrm{e}^{k_1x+(\sigma k_1^3/4)t}\right)^2}. \tag{54}$$

For two-kink (soliton) solutions we set

$$f(x,t) = 1+\mathrm{e}^{\theta_1}+\mathrm{e}^{\theta_2}+a_{12}\,\mathrm{e}^{\theta_1+\theta_2}$$

$$= 1+\mathrm{e}^{k_1x+(\sigma k_1^3/4)t}+\mathrm{e}^{k_2x+(\sigma k_2^3/4)t}$$

$$+ a_{12}\,\mathrm{e}^{(k_1+k_2)x+(\sigma(k_1^3+k_2^3)/4)t}. \tag{55}$$

Using Eq. (55) in Eq. (50) and substituting the result into Eq. (2), we obtain the phase shift

$$a_{12} = 0, \tag{56}$$

and hence we set

$$a_{ij} = 0,\ 1\leq i<j\leq 3. \tag{57}$$

The two-kink (soliton) solutions are obtained by substituting Eqs. (56) and (55) into Eq. (50),

$$u(x,t) = \pm\frac{k_1\,\mathrm{e}^{k_1x+(\sigma k_1^3/4)t} + k_2\,\mathrm{e}^{k_2x+(\sigma k_2^3/4)t}}{2\big(1+\mathrm{e}^{k_1x+(\sigma k_1^3/4)t}+\mathrm{e}^{k_2x+(\sigma k_2^3/4)t}\big)},$$
$$v(x,t) = -\frac{k_1^2\,\mathrm{e}^{k_1x+(\sigma k_1^3/4)t} + k_2^2\,\mathrm{e}^{k_2x+(\sigma k_2^3/4)t} + (k_1-k_2)^2\,\mathrm{e}^{(k_1+k_2)x+(\sigma(k_1^3+k_2^3)/4)t}}{2\big(1+\mathrm{e}^{k_1x+(\sigma k_1^3/4)t}+\mathrm{e}^{k_2x+(\sigma k_2^3/4)t}\big)^2}. \tag{58}$$

For three-kink (soliton) solutions, we set

$$f(x,t) = 1+\mathrm{e}^{\theta_1}+\mathrm{e}^{\theta_2}+\mathrm{e}^{\theta_3} = 1+\mathrm{e}^{k_1x+(\sigma k_1^3/4)t}+\mathrm{e}^{k_2x+(\sigma k_2^3/4)t}+\mathrm{e}^{k_3x+(\sigma k_3^3/4)t}. \tag{59}$$

Proceeding as before, we find that the three-kink (soliton) solutions are

$$u(x,t) = \pm\frac{k_1\,\mathrm{e}^{k_1x+(\sigma k_1^3/4)t} + k_2\,\mathrm{e}^{k_2x+(\sigma k_2^3/4)t} + k_3\,\mathrm{e}^{k_3x+(\sigma k_3^3/4)t}}{2\big(1+\mathrm{e}^{k_1x+(\sigma k_1^3/4)t}+\mathrm{e}^{k_2x+(\sigma k_2^3/4)t}+\mathrm{e}^{k_3x+(\sigma k_3^3/4)t}\big)},$$
$$v(x,t) = -\frac{\sum\limits_{i=1}^{3} k_i^2\,\mathrm{e}^{k_ix+(\sigma k_i^3/4)t} + \sum\limits_{1\le i<j\le 3}(k_i-k_j)^2\,\mathrm{e}^{(k_i+k_j)x+(\sigma(k_i^3+k_j^3)/4)t}}{2\left(1+\sum\limits_{i=1}^{3}\mathrm{e}^{k_ix+(\sigma k_i^3/4)t}\right)^2}. \tag{60}$$

The three-kink (soliton) solutions are obtained by substituting Eq. (59) into Eq. (50). This shows that Eq. (2) is completely integrable and N-kink (soliton) solutions can be determined for $u(x,t)$ and $v(x,t)$, for finite N, where $N \ge 1$. Based on Eq. (60), the general kink (soliton) solutions can be set as

$$u(x,t) = \pm\frac{\sum\limits_{i=1}^{N} k_i\,\mathrm{e}^{k_ix+(\sigma k_i^3/4)t}}{2\left(1+\sum\limits_{i=1}^{N}\mathrm{e}^{k_ix+(\sigma k_i^3/4)t}\right)},$$
$$v(x,t) = -\frac{\sum\limits_{i=1}^{N} k_i^2\,\mathrm{e}^{k_ix+(\sigma k_i^3/4)t} + \sum\limits_{1\le i<j\le N}(k_i-k_j)^2\,\mathrm{e}^{(k_i+k_j)x+(\sigma(k_i^3+k_j^3)/4)t}}{2\left(1+\sum\limits_{i=1}^{N}\mathrm{e}^{k_ix+(\sigma k_i^3/4)t}\right)^2}. \tag{61}$$

4.2. Multiple-singular-kink (soliton) solutions

Proceeding as before, we find the dispersion relation

$$c_i = -\frac{\sigma k_i^3}{4}, \tag{62}$$

and obtain

$$\theta_i = k_ix + \frac{\sigma k_i^3}{4}t. \tag{63}$$

The single-singular-kink (soliton) solutions of Eq. (2) are assumed to be

$$u(x,t) = R_1(\ln f)_x = R_1\frac{f_x}{f},$$
$$v(x,t) = R_2(\ln f)_{xx} = R_2\frac{ff_{xx}-f_x^2}{f^2}, \tag{64}$$

where $f(x,t)$, for the single-singular-kink (soliton) solutions, is given by

$$f(x,t) = 1-\mathrm{e}^{\theta_1} = 1-\mathrm{e}^{k_1x+(\sigma k_1^3/4)t}. \tag{65}$$

Substituting Eq. (65) into Eq. (64), using the result from Eq. (2), and solving for R_1 and R_2 we find

$$R_1 = \pm\frac{1}{2},$$
$$R_2 = -\frac{1}{2}. \tag{66}$$

Substituting Eq. (65) into Eq. (64) gives the single-singular-kink (soliton) solutions

$$u(x,t) = \pm\frac{k_1\,\mathrm{e}^{k_1x+(\sigma k_1^3/4)t}}{2\big(1-\mathrm{e}^{k_1x+(\sigma k_1^3/4)t}\big)},$$
$$v(x,t) = \frac{k_1^2\,\mathrm{e}^{k_1x+(\sigma k_1^3/4)t}}{2\big(1-\mathrm{e}^{k_1x+(\sigma k_1^3/4)t}\big)^2}. \tag{67}$$

For two-singular-kink (soliton) solutions we set

$$f(x,t) = 1-\mathrm{e}^{\theta_1}-\mathrm{e}^{\theta_2}+a_{12}\,\mathrm{e}^{\theta_1+\theta_2}$$
$$= 1-\mathrm{e}^{k_1x+(\sigma k_1^3/4)t}-\mathrm{e}^{k_2x+(\sigma k_2^3/4)t}$$

$$+ a_{12}\,\mathrm{e}^{(k_1+k_2)x+(\sigma(k_1^3+k_2^3)/4)t}. \tag{68}$$

Using Eq. (68) in Eq. (64) and substituting the result from Eq. (2), we obtain the phase shift

$$a_{12} = 0, \tag{69}$$

and hence we set

$$a_{ij} = 0,\ 1 \le i < j \le 3. \tag{70}$$

The two-singular-kink (soliton) solutions are obtained by substituting Eqs. (68) and (69) into Eq. (64), where we find

$$\begin{aligned} u(x,t) &= \pm\frac{k_1\,\mathrm{e}^{k_1x+(\sigma k_1^3/4)t} + k_2\,\mathrm{e}^{k_2x+(\sigma k_2^3/4)t}}{2\left(1-\mathrm{e}^{k_1x+(\sigma k_1^3/4)t}-\mathrm{e}^{k_2x+(\sigma k_2^3/4)t}\right)},\\ v(x,t) &= \frac{k_1^2\,\mathrm{e}^{k_1x+(\sigma k_1^3/4)t} + k_2^2\,\mathrm{e}^{k_2x+(\sigma k_2^3/4)t} - (k_1-k_2)^2\,\mathrm{e}^{(k_1+k_2)x+(\sigma(k_1^3+k_2^3)/4)t}}{2\left(1-\mathrm{e}^{k_1x+(\sigma k_1^3/4)t}-\mathrm{e}^{k_2x+(\sigma k_2^3/4)t}\right)^2}. \end{aligned} \tag{71}$$

For the three-singular-kink (soliton) solutions, we set

$$f(x,t) = 1-\mathrm{e}^{\theta_1}-\mathrm{e}^{\theta_2}-\mathrm{e}^{\theta_3} = 1-\mathrm{e}^{k_1x+(\sigma k_1^3/4)t}-\mathrm{e}^{k_2x+(\sigma k_2^3/4)t}-\mathrm{e}^{k_3x+(\sigma k_3^3/4)t}. \tag{72}$$

Proceeding as before, we find that the three- singular-kink (soliton) solutions are

$$\begin{aligned} u(x,t) &= \pm\frac{\sum\limits_{i=1}^{3} k_i\,\mathrm{e}^{k_ix+(\sigma k_i^3/4)t}}{2\left(1-\sum\limits_{i=1}^{3}\mathrm{e}^{k_ix+(\sigma k_i^3/4)t},\right)},\\ v(x,t) &= \frac{\sum\limits_{i=1}^{3} k_i^2\,\mathrm{e}^{k_ix+(\sigma k_i^3/4)t} - \sum\limits_{1\le i<j\le 3}(k_i-k_j)^2\,\mathrm{e}^{(k_i+k_j)x+(\sigma(k_i^3+k_j^3)/4)t}}{2\left(1-\sum\limits_{i=1}^{3}\mathrm{e}^{k_ix+(\sigma k_i^3/4)t}\right)^2}. \end{aligned} \tag{73}$$

Based on the last result, general singular-kink (soliton) solutions can be set as

$$\begin{aligned} u(x,t) &= \pm\frac{\sum\limits_{i=1}^{N} k_i\,\mathrm{e}^{k_ix+(\sigma k_i^3/4)t}}{2\left(1-\sum\limits_{i=1}^{N}\mathrm{e}^{k_ix+(\sigma k_i^3/4)t}\right)},\\ v(x,t) &= \frac{\sum\limits_{i=1}^{N} k_i^2\,\mathrm{e}^{k_ix+(\sigma k_i^3/4)t} - \sum\limits_{1\le i<j\le N}(k_i-k_j)^2\,\mathrm{e}^{(k_i+k_j)x+(\sigma(k_i^3+k_j^3)/4)t}}{2\left(1-\sum\limits_{i=1}^{N}\mathrm{e}^{k_ix+(\sigma k_i^3/4)t}\right)^2}, \quad N \ge 1. \end{aligned} \tag{74}$$

5. Discussion

The Hirota bilinear method is applied to emphasise the integrability of the coupled Burgers equation and the Boussinesq–Burgers equation. Multiple-kink (soliton) solutions and multiple-singular-kink (soliton) solutions are formally derived. The analysis confirms the fact that certain equations that have N-kink (soliton) solutions simultaneously have N-singular kink (soliton) solutions. The results obtained for the phase shift a_{ij} show that the system is resonance free.

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