Noether's theory of generalized linear nonholonomic mechanical systems*

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(Received 7 December 2009; revised manuscript received 28 June 2010)

By introducing the quasi-symmetry of the infinitesimal transformation of the transformation group G_r, the Noether's theorem and the Noether's inverse theorem for generalized linear nonholonomic mechanical systems are obtained in a generalized compound derivative space. An example is given to illustrate the application of the result.

Keywords: generalized linear nonholonomic mechanics, symmetry, conserved quantity, Noether's theory

PACS: 02.20.Sv, 11.30.−j, 45.05.+x **DOI:** 10.1088/1674-1056/20/1/010204

1. Introduction

The conserved quantity of a dynamic system (or the first integral) is not only important in mathematics, but also reflects the profound nature of physics. In 1918, Noether revealed the potential relationship between conserved quantities and symmetry:[1] each type of symmetry corresponds to a conservation law and there always is a conserved quantity. Over the past two decades, Noether's theorem has been extensively studied by mechanics researchers and brought about many important research results.[2−11]

In 2002, Luo[12] combined generalized classical mechanics and nonholonomic mechanics, which was developed independently more than 100 years ago, to create a generalized nonholonomic mechanical system. The variational principles and differential equations of motion of the system are given. The integral variants and the Poincaré–Cartan-type and Poincaré-type integral invariants are also obtained. Li *et al.*[13,14] studied the canonical symmetry and the Poincaré–Cartan integral invariant of a constrained singular generalized mechanical system. In the present paper, we study Noether's theory of generalized linear nonholonomic mechanical systems. By introducing the quasi-symmetry of an infinitesimal transformation of the r–parameter transformation group, we establish the Noether theorem and the Noether inverse theorem for the system. Finally, an example is presented to illustrate the use of the result. Because the generalized nonholonomic mechanical systems hold for both generalized classical mechanics and for first-order to higher-order nonholonomic mechanics, the solution given in this paper has more universal significance than those given in previous studies.

2. Motion differential equation

Consider that a generalized linear nonholonomic mechanical system consists of n degrees of freedom, and suppose that its configuration is specified by the n generalized coordinates q_s $(s = 1, \cdots, n)$, where its motion is subjected to k-order linear nonholonomic constraints:

$$f_\beta\left(t, q_s, \dot{q}_s, \cdots, \overset{(k)}{q_s}\right) = 0, \quad (\beta = 1, 2, \cdots, g)\,. \tag{1}$$

The virtual displacement of the generalized compound derivative space satisfies the generalized nonholonomic mechanics Чегаев definition,[12]

$$\sum_{s=1}^{n}\sum_{m=1}^{k}\frac{\partial f_\beta}{\partial\overset{(m)}{q_s}}\delta\overset{(m-1)}{q_s} = 0, \quad (m = 1, \cdots, k)\,, \tag{2}$$

$$\delta t = 0, \delta\overset{(m-1)}{q_s} \neq 0, \quad (m = 1, \cdots, k)\,. \tag{3}$$

In accordance with Ref. [12], the Routh equation of the system can be written in the form

$$\sum_{m=1}^{k}(-1)^{m+1}\frac{\mathrm{d}^m}{\mathrm{d}t^m}\left(\frac{\partial L}{\partial\overset{(m)}{q_s}}\right) - \frac{\partial L}{\partial q_s} = \Lambda_s, \tag{4}$$

*Project supported by the Natural Science Foundation of Weifang University, China (Grant No. 2008Z03).

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$$\Lambda_s = \sum_{m=1}^{k} \Lambda_s^m = \sum_{\beta=1}^{g} \sum_{m=1}^{k} \lambda_\beta^m \frac{\partial f_\beta}{\partial \overset{(m)}{q_s}}, \quad (s = 1, \ldots, n). \tag{5}$$

Now we introduce generalized momentum and the Hamiltonian[15]

$$p_{s/m} = \sum_{j=0}^{k-m} (-1)^j \frac{\mathrm{d}^j}{\mathrm{d}t^j} \frac{\partial L}{\partial \overset{(j+m)}{q_s}}, \tag{6}$$

$$H\left(\overset{(m-1)}{q_s}, p_{s/m}, t\right) = \sum_{s=1}^{n} \sum_{m=1}^{k} p_{s/m} \overset{(m)}{q_s} - L. \tag{7}$$

The motion equation of the system in canonical form is given as

$$\overset{(m)}{q_s} = \frac{\partial H}{\partial P_{s/m}}, \quad \dot{p}_{s/m} = -\frac{\partial H}{\partial \overset{(m-1)}{q_s}} + \tilde{\Lambda}_s, \quad (s = 1, \cdots, n;\ m = 1, \cdots, k), \tag{8}$$

$$\tilde{\Lambda}_s\left(\overset{(m-1)}{q_s}, p_{s/m}, t\right) = \sum_{\beta=1}^{g} \sum_{m=1}^{k} \lambda_\beta^m \left(\overset{(m-1)}{q_s}, \overset{(m)}{q_s}\left(\overset{(m-1)}{q_s}, p_{s/m}, t\right), t\right) \left(\frac{\partial \tilde{f}_\beta}{\partial \overset{(m)}{q_s}}\right), \tag{9}$$

where $\tilde{\Lambda}_s$ is the result obtained by substituting $\overset{(m)}{q_s}$ into $\left(\overset{(m-1)}{q_s}, p_{s/m}, t\right)$ of Λ_s, which represents the generalized nonholonomic constraint reaction force and λ_β^m is the generalized Lagrange constraint multiplier.

Let[16,17]

$$a^\mu = \begin{cases} \overset{(m-1)}{q_s}, & (\mu = (2m-2)n + s;\ m = 1, \ldots, k;\ s = 1, \ldots, n), \\ p_{s/m}, & (\mu = (2m-1)n + s;\ m = 1, \ldots, k;\ s = 1, \ldots, n), \end{cases} \tag{10}$$

$$R_\mu = \begin{cases} p_{s/m}, & (\mu = (2m-2)n + s;\ m = 1, \ldots, k;\ s = 1, \ldots, n), \\ 0, & (\mu = (2m-1)n + s;\ m = 1, \ldots, k;\ s = 1, \ldots, n). \end{cases} \tag{11}$$

Hence, Eqs. (1), (2) and (3) and the $\tilde{\Lambda}_s$ which appears in Eq. (8) can be expressed as follows:

$$\tilde{f}_\beta^*(t, a^\mu) = 0, \quad (\mu = (2m-2)n + s;\ m = 1, \ldots, k;\ s = 1, \ldots, n), \tag{12}$$

$$\sum_{\mu=1}^{2nk} \frac{\partial \overleftarrow{f}_\beta^*}{\partial a^{\mu+2n}} \delta a^\mu = 0, \quad (\mu = (2m-2)n + s;\ m = 1, \ldots, k;\ s = 1, \ldots, n), \tag{13}$$

$$\delta t = 0, \quad \delta a^\mu \neq 0, \quad (\mu = (2m-2)n + s;\ m = 1, \ldots, k;\ s = 1, \ldots, n), \tag{14}$$

$$\tilde{\Lambda}_s^* = \sum_{\beta=1}^{g} \sum_{m=1}^{k} \lambda_\beta^m \frac{\partial \tilde{f}_\beta^*}{\partial a^{\mu+2n}}, \quad (\mu = (2m-2)n + s;\ m = 1, \ldots, k;\ s = 1, \ldots, n). \tag{15}$$

Choose

$$\tilde{\Lambda}_s = -\Omega_{s/m} \frac{\partial H}{\partial P_{s/m}}, \tag{16}$$

$$X_\mu = \begin{cases} \Omega_{s/m} \frac{\partial H}{\partial P_{s/m}}, & (\mu = (2m-2)n + s;\ m = 1, \ldots, k;\ s = 1, \ldots, n), \\ 0, & (\mu = (2m-1)n + s;\ m = 1, \ldots, k;\ s = 1, \ldots, n), \end{cases} \tag{17}$$

canonical equation (8) can be further expressed as

$$\left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right) \dot{a}^\nu - \frac{\partial H}{\partial a^\mu} = -X_\mu \quad (\mu, \nu = 1, \ldots, 2kn), \tag{18}$$

and the contravariant equation corresponding to Eq. (18) is

$$\dot{a}^\mu - s^{\mu\nu} \frac{\partial H}{\partial a^\nu} = 0, \quad (\mu, \nu = 1, \ldots, 2kn), \tag{19}$$

where

$$s^{\mu\nu} = \omega^{\mu\nu} + T^{\mu\nu}, \tag{20}$$

$$\omega^{\mu\nu} = \frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}, \quad (\omega_{\mu\nu}) = (\omega^{\mu\nu})^{-1}, \tag{21}$$

and

$$\begin{cases} (\omega^{\mu\nu}) = \begin{bmatrix} M_1 & & \\ & \ddots & \\ & & M_k \end{bmatrix}_{2nk\times 2nk}, \\ M_i = \begin{bmatrix} 0_{n\times n} & I_{n\times n} \\ -I_{n\times n} & 0_{n\times n} \end{bmatrix}, \end{cases} \quad (i = 1, \ldots, k), \tag{22}$$

$$\begin{cases} (T^{\mu\nu}) = \begin{bmatrix} 0_{n\times n} & & & & & & \\ & H_1 & & & & & \\ & & \ddots & & & & \\ & & & 0_{n\times n} & & & \\ & & & & H_i & & \\ & & & & & \ddots & \\ & & & & & & H_k \end{bmatrix}_{2nk\times 2nk}, \\ H_i = \begin{bmatrix} \Omega_{1/m} & & & \\ & \Omega_{2/m} & & \\ & & \ddots & \\ & & & \Omega_{s/m} \end{bmatrix}, \end{cases} \quad (i = m = 1, 2, \ldots, k;\ s = 1, 2, \ldots, n). \tag{23}$$

Here, $0_{n\times n}$ is zero matrix with order n and $\boldsymbol{I}_{n\times n}$ is unit matrix with order n.

3. Noether's theorem

We can introduce an r-parameter group of the finite transformation $\mathrm{G_r}$

$$\begin{aligned} t^* &= g_0(t, a^\nu, b_\alpha), \\ a^{\mu *} &= g_\mu(t, a^\nu, b_\alpha), \quad (\mu, \nu = 1, \ldots, 2nk;\ \alpha = 1, \ldots, r), \end{aligned} \tag{24}$$

where $b_\alpha\,(\alpha = 1, \ldots, r)$ are independent parameters, the corresponding infinitesimal group transformations are

$$\begin{aligned} t^* &= t + \Delta t = t + \sum_{\alpha=1}^{r} \varepsilon_\alpha \xi_0^\alpha(t, a), \\ a^{\mu *} &= a^\mu + \Delta a^\mu = a^\mu + \sum_{\alpha=1}^{r} \varepsilon_\alpha \xi_\mu^\alpha(t, a), \quad (\mu = 1, \ldots, 2nk), \end{aligned} \tag{25}$$

with ε_α being infinitesimal parameters that have first-order infinitesimals, and ξ_0^α and ξ_μ^α the generators of the r-parameter finite transformation group.

The canonical action of the system is

$$S(\gamma) = \int_{t_1}^{t_2} \left[\sum_{s=1}^{n} \sum_{m=1}^{k} p_{s/m} \overset{(m)}{q_s} - H\left(\overset{(m)}{q_s}, p_{s/m}, t \right) \right] \mathrm{d}t = \int_{t_1}^{t_2} \left[\sum_{\mu=1}^{2nk} R_\mu(t, a)\, \mathrm{d}a^\mu - H(t, a)\, \mathrm{d}t \right],$$

and the difference before and after the transformation is[7]

$$S(\gamma^*) - S(\gamma) = \int_{t_1^*}^{t_2^*} \left\{ \sum_{\mu=1}^{2nk} R_\mu(t^*, a^*)\, \mathrm{d}a^{\mu *} - H(t^*, a^*)\, \mathrm{d}t^* \right\} - \int_{t_1}^{t_2} \left\{ \sum_{\mu=1}^{2nk} R_\mu(t, a)\, \mathrm{d}a^\mu - H(t, a)\, \mathrm{d}t \right\}, \tag{26}$$

where γ and γ^* are the given curve and the nearby curve respectively. Let ΔS denote the main linear part corresponding to ε of the difference that is accurate to the first-order infinitesimals, then we will have

$$\Delta S=\int_{t_1}^{t_2}\sum_{\alpha=1}^{r}\varepsilon_\alpha\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{\mu=1}^{2nk}R_\mu\xi_\mu^\alpha-H\xi_0^\alpha\right)+\sum_{\mu=1}^{2nk}\left[\sum_{\nu=1}^{2nk}\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu-\frac{\partial H}{\partial a^\mu}\right]\bar{\xi}_\mu^\alpha\right\}\mathrm{d}t, \tag{27}$$

where

$$\bar{\xi}_\mu^\alpha=\xi_\mu^\alpha-\dot{a}^\mu\xi_0^\alpha,\quad(\mu=1,\ldots,2nk)\,.$$

Hence, we obtain the theorem as follows.

Theorem 1 For generalized linear nonholonomic mechanics systems (l) and (4), if the infinitesimal transformations (25) of the finite transformation group G_r are quasi-symmetric transformations and satisfy the relations

$$\sum_{\mu=1}^{2nk}\frac{\partial\overleftarrow{f_\beta^*}}{\partial a^{\mu+2n}}\bar{\xi}_\mu^\alpha=0,\quad(\beta=1,\ldots,g;\ \alpha=1,\ldots,r)\,, \tag{28}$$

then the system has r linear independent first integrals

$$\sum_{\mu=1}^{2nk}R_\mu\xi_\mu^\alpha-H\xi_0^\alpha+G^\alpha=C^\alpha,\quad(\alpha=1,\ldots,r). \tag{29}$$

Proof Since the infinitesimal transformations (25) are quasi-symmetric, we have

$$\Delta S+\int_{t_1}^{t_2}\left[\frac{\mathrm{d}}{\mathrm{d}t}(\Delta G)\right]\mathrm{d}t=0, \tag{30}$$

where $\Delta G=\sum\limits_{\alpha=1}^{r}\varepsilon_\alpha G^\alpha$.

Substituting Eq. (27) into Eq. (30), we obtain

$$\int_{t_1}^{t_2}\sum_{\alpha=1}^{r}\varepsilon_\alpha\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{\mu=1}^{2nk}R_\mu\xi_\mu^\alpha-H\xi_0^\alpha+G^\alpha\right)+\sum_{\mu=1}^{2nk}\left[\sum_{\nu=1}^{2nk}\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu-\frac{\partial H}{\partial a^\mu}\right]\bar{\xi}_\mu^\alpha\right\}\mathrm{d}t=0, \tag{31}$$

because the infinitesimal group transformations (25) satisfy relations (28), then

$$\int_{t_1}^{t_2}\sum_{\alpha=1}^{r}\varepsilon_\alpha\sum_{\beta=1}^{g}\lambda_\beta^m\sum_{\mu=1}^{2nk}\frac{\partial\overleftarrow{f_\beta^*}}{\partial a^{\mu+2n}}\bar{\xi}_\mu^\alpha\mathrm{d}t=-\int_{t_1}^{t_2}\sum_{\alpha=1}^{r}\varepsilon_\alpha\sum_{\mu=1}^{2nk}X_\mu\bar{\xi}_\mu^\alpha\mathrm{d}t=0. \tag{32}$$

Adding Eq. (31) to Eq. (32) yields

$$\int_{t_1}^{t_2}\sum_{\alpha=1}^{r}\varepsilon_\alpha\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{\mu=1}^{2nk}R_\mu\xi_\mu^\alpha-H\xi_0^\alpha+G^\alpha\right)+\sum_{\mu=1}^{2nk}\left[\sum_{\nu=1}^{2nk}\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu-\frac{\partial H}{\partial a^\mu}-X_\mu\right]\bar{\xi}_\mu^\alpha\right\}\mathrm{d}t=0. \tag{33}$$

Making use of Eq. (8) and considering that the integral interval $[t_1,t_2]$ is arbitrary and ε_α are independent of each other, we have either

$$\frac{\mathrm{d}}{\mathrm{d}t}(\sum_{\mu=1}^{2nk}R_\mu\xi_\mu^\alpha-H\xi_0^\alpha+G^\alpha)=0,\quad(\alpha=1,\ldots,r), \tag{34}$$

or

$$\sum_{\mu=1}^{2nk}R_\mu\xi_\mu^\alpha-H\xi_0^\alpha+G^\alpha=C^\alpha,\quad(\alpha=1,\ldots,r). \tag{35}$$

Theorem 1 could be called Noether's theorem of generalized linear nonholonomic mechanical systems. Using this theorem, one can find the conserved quantities of systems from given symmetries.

If $r=1$, we can easily prove that theorem 1 reduces to the following corollary.

Corollary 1 For generalized linear nonholonomic mechanical systems (l) and (4), if the generators ξ_μ and ξ_0 of the infinitesimal group transformation and the gauge function G satisfy the following conditions:

$$\sum_{\mu=1}^{2nk} \frac{\partial \overleftarrow{f_\beta^*}}{\partial a^{\mu+2n}} \left(\xi_\mu - \dot{a}^\mu \xi_0\right) = 0, \quad (\beta = 1, \ldots, g)\,, \tag{36}$$

$$R_\mu \dot{\xi}_\mu + \left(\frac{\partial R_\nu}{\partial a^\mu}\dot{a}^\nu - \frac{\partial H}{\partial a^\mu}\right)\xi_\mu - \frac{\partial H}{\partial t}\xi_0 - H\dot{\xi}_0 + \dot{G} = 0, \tag{37}$$

the system has conversed quantities, such as

$$\sum_{\mu=1}^{2nk} R_\mu \xi_\mu - H\xi_0 + G = \text{const.} \tag{38}$$

Next, expanding Eq. (37), and letting the terms with and without $\dot{a}^\mu$ vanish respectively, we obtain the following equations:

$$\begin{aligned} &R_\mu \frac{\partial \xi_\mu}{\partial a^\nu} + \frac{\partial R_\nu}{\partial a^\mu}\xi_\mu - H\frac{\partial \xi_0}{\partial a^\nu} = -\frac{\partial G}{\partial a^\nu}, \\ &R_\mu \frac{\partial \xi_\mu}{\partial t} - \frac{\partial H}{\partial a^\mu}\xi_\mu - \frac{\partial H}{\partial t}\xi_0 - H\frac{\partial \xi_0}{\partial t} = -\frac{\partial G}{\partial t}. \end{aligned} \tag{39}$$

Equation (39) is called a Killing equation for generalized linear nonholonomic mechanical systems. By using either Eqs. (36) and (37), or Eqs. (36) and (39), the conserved quantities of the system can be solved.

In a generalized classical mechanical system with non-retaining constraints, $\Lambda_s = 0$, we obtain the following corollary according to theorem 1.

Corollary 2 In a generalized classical mechanical system, if the infinitesimal transformations (25) of the finite transformation group G_r are quasi-symmetric, then the system has r linear independent first integrals in the form of Eq. (29).

Similarly, for a non-retaining nonholonomic constraint classical mechanical system, $\Lambda_s = Q_l(t, q_s, \dot{q}_s, \ldots, \overset{(k)}{q_s})$, we obtain the following corollary according to theorem 1.

Corollary 3 In a generalized nonconservative mechanical system, if the infinitesimal transformations (25) of the finite transformation group G_r are generalized quasi-symmetric ones, the system has r linear independent first integrals in the form of Eq. (29).

Corollaries 2 and 3 have been presented by Refs. [16] and [18].

4. Noether's inverse theorem of the system

Now we find the infinitesimal generalized quasi-symmetric transformations according to the given first integrals of the system.

Assume that for given generalized linear nonholonomic mechanical system (1), Eq. (4) has r linear first integrals in the form of

$$I^\alpha = I^\alpha(t, a) = C^\alpha, \quad (\alpha = 1, \ldots, r)\,, \tag{40}$$

thereby

$$\frac{\mathrm{d}I^\alpha}{\mathrm{d}t} = \frac{\partial I^\alpha}{\partial t} + \sum_{\nu=1}^{2nk} \frac{\partial I^\alpha}{\partial a^\nu}\dot{a}^\nu = 0, \quad (\alpha = 1, \ldots, r)\,. \tag{41}$$

Multiplying both sides of Eq. (18) by $\overline{\xi_\mu^\alpha}$ and adding them up over μ, then combining the result with Eq. (41), we have

$$\frac{\partial I^\alpha}{\partial t} + \sum_{\nu=1}^{2nk} \frac{\partial I^\alpha}{\partial a^\nu}\dot{a}^\nu + \sum_{\mu=1}^{2nk}\left[\left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu - \frac{\partial H}{\partial a^\mu} - X_\mu\right]\overline{\xi_\mu^\alpha} = 0. \tag{42}$$

Obviously, in Eq. (42), the term which contains $\dot{a}^\mu \dot{a}^\nu$ is zero. Owing to the fact that the coefficients of $\dot{a}^\nu$ in Eq. (42) are zero, we obtain

$$\frac{\partial I^\alpha}{\partial a^\nu} + \sum_{\mu=1}^{2nk}\left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\xi_\mu^\alpha + \frac{\partial H}{\partial a^\nu}\xi_0^\alpha = 0, \quad (\nu = 1, \ldots, 2kn;\ \alpha = 1, \ldots, r)\,. \tag{43}$$

Considering formula (21), we obtain from Eq. (43)

$$\xi_\mu^\alpha = \sum_{\nu=1}^{2nk} \omega^{\mu\nu} \left(\frac{\partial I^\alpha}{\partial a^\nu} + \frac{\partial H}{\partial a^\nu} \xi_0^\alpha \right), \quad (\mu = 1, \ldots, 2kn; \alpha = 1, \ldots, r), \tag{44}$$

where

$$\sum_{\nu=1}^{2nk} \omega_{\mu\nu}\omega_{\nu\rho} = \delta_{\mu\rho}, \quad (\mu, \rho = 1, \ldots, 2nk). \tag{45}$$

Let integral (40) equal conserved quantity (29), i.e.,

$$I^\alpha = \sum_{\mu=1}^{2nk} R_\mu \xi_\mu^\alpha - H\xi_0^\alpha + G^\alpha, \quad (\alpha = 1, \ldots, r), \tag{46}$$

then we can calculate the $(2nk+1)r$ generators ξ_0^α and ξ_μ^α from formulae (44) and (46) when the gauge function G is given.

If an infinitesimal transformation group of the given generalized linear nonholonomic mechanical systems satisfies expression (28), substituting Eqs. (28) and (46) into Eq. (42) and arranging them, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\sum_{\mu=1}^{2nk} R_\mu \xi_\mu^\alpha - H\xi_0^\alpha \right) + \sum_{\mu=1}^{2nk} \left[\left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu} \right) \dot{a}^\nu - \frac{\partial H}{\partial a^\mu} - X_\mu \right] \overline{\xi_\mu^\alpha} = -\frac{\mathrm{d}G^\alpha}{\mathrm{d}t}. \tag{47}$$

According to the criterion in Ref. [7], it should be a quasi-symmetric transformation and we can establish following theorem.

Theorem 2 If we know the r linear independent first integrals in (40), then the infinitesimal group transformations determined by formulae (24), (44) and (46) satisfy relation (28), and they must therefore be the quasi-symmetric transformations of the above system.

Theorem 2 may be called Noether's inverse theorem of generalized linear nonholonomic mechanical systems.

For a generalized classical mechanical system, we obtain the following corollary according to theorem 1.

Corollary 4 If we know the r linear independent first integrals from generalized classical mechanics, then the infinitesimal group transformations determined by formulae (24), (44) and (46) must be the generalized quasi-symmetric transformations of the above system.

Obviously, for a generalized nonconservative mechanical system, theorem 2 gives the following corollary.

Corollary 5 If we know the r linear independent first integrals of a generalized nonconservative mechanical system, then the infinitesimal group transformations determined by formulae (24), (44) and (46) must be the quasi-symmetric transformations of the above system.

Corollaries 4 and 5 have been presented by Refs. [16] and [18].

5. Example

In the generalized electrodynamics of two-order derivatives, assume that the Lagrangian function is

$$L = \frac{1}{2}\dot{q}_1^2 + \frac{1}{2}\ddot{q}_2^2, \tag{48}$$

and is subjected to generalized linear nonholonomic constraints

$$f = q_1 - tq_2 + \dot{q}_2 + \ddot{q}_2 = 0. \tag{49}$$

Find the system symmetry and conserved quantities using Noether's theorem.

Firstly, the conserved quantities of the system are deduced using theorem 1. According to the known system symmetry, from Eqs. (6), (7) and (8), we can obtain

$$P_{1/1} = \dot{q}_1, \ P_{1/2} = 0, \ P_{2/1} = -\dddot{q}_2, \ P_{2/2} = \ddot{q}_2, \tag{50}$$

$$H = P_{1/1}\dot{q}_1 + P_{2/1}\dot{q}_2 + P_{2/2}\ddot{q}_2 - \frac{1}{2}\left(\dot{q}_1^2\right) - \frac{1}{2}\left(\ddot{q}_2^2\right), \tag{51}$$

$$\dot{q}_1 = \frac{\partial H}{\partial P_{1/1}}, \ \ddot{q}_1 = \frac{\partial H}{\partial P_{1/2}}, \ \dot{q}_2 = \frac{\partial H}{\partial P_{2/1}}, \ \ddot{q}_2 = \frac{\partial H}{\partial P_{2/2}}, \tag{52}$$

$$\begin{cases} \dot{P}_{1/1} = -\dfrac{\partial H}{\partial q_1} + \lambda^1 \dfrac{\partial \tilde{f}}{\partial \dot{q}_1}, \\ \dot{P}_{1/2} = -\dfrac{\partial H}{\partial \dot{q}_1} + \lambda^1 \dfrac{\partial \tilde{f}}{\partial \dot{q}_1} + \lambda^2 \dfrac{\partial \tilde{f}}{\partial \ddot{q}_1}, \\ \dot{P}_{2/1} = -\dfrac{\partial H}{\partial q_2} + \bar{\lambda}^1 \dfrac{\partial \tilde{f}}{\partial \dot{q}_2}, \\ \dot{P}_{2/2} = -\dfrac{\partial H}{\partial \dot{q}_2} + \bar{\lambda}^1 \dfrac{\partial \tilde{f}}{\partial \dot{q}_2} + \bar{\lambda}^2 \dfrac{\partial \tilde{f}}{\partial \ddot{q}_2}, \end{cases} \tag{53}$$

where $\tilde{f} = q_1 - tq_2 + \dot{q}_2 + P_{2/2} = 0$.

Considering expressions (50)–(53), we can have

$$\lambda^1 = \lambda^2 = 0, \quad \bar{\lambda}^1 = \bar{\lambda}^2 = \dot{q}_1 - 2\dot{q}_2 - t\ddot{q}_2 - \dddot{q}_2. \tag{54}$$

Select

$$\begin{aligned} &a^1 = q_1,\ a^2 = q_2,\ a^3 = P_{1/1},\ a^4 = P_{2/1},\ a^5 = \dot{q}_1, \\ &a^6 = \dot{q}_2,\ a^7 = P_{1/2},\ a^8 = P_{2/2},\ a^9 = \ddot{q}_1,\ a^{10} = \ddot{q}_2, \\ &R_1 = a^3,\ R_2 = a^4,\ R_3 = 0,\ R_4 = 0, \\ &R_5 = a^7,\ R_6 = a^8,\ R_7 = 0, \quad \text{and} \quad R_8 = 0, \end{aligned} \tag{55}$$

then we will obtain

$$H = a^3 a^5 + a^4 a^6 + \frac{1}{2}\left(a^8\right)^2 - \frac{1}{2}\left(a^5\right)^2, \tag{56}$$

$$\begin{cases} X_1 = 0,\ X_2 = a^4 - a^3 + 2a^6 + ta^8, \\ X_3 = X_4 = X_5 = X_6 = X_7 = X_8 = 0. \end{cases} \tag{57}$$

Similarly, from Eqs. (21)–(23), we have

$$S^{\mu\nu} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & a^4 - a^3 + 2a^6 + ta^8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}. \tag{58}$$

Then expression (19) gives

$$\begin{cases} \dot{a}^1 = a^5,\ \dot{a}^2 = a^6,\ \dot{a}^3 = 0, \\ \dot{a}^4 = \left(a^4 - a^3 + 2a^6 + ta^8\right) a^6, \\ \dot{a}^5 = 0,\ \dot{a}^6 = a^8,\ \dot{a}^7 = a^5,\ \dot{a}^8 = -a^4, \end{cases} \tag{59}$$

and expressions (36) and (37)give

$$\begin{aligned} &\left(\xi_2 - \dot{a}^2\xi_0\right) + \left(\xi_6 - \dot{a}^6\xi_0\right) = 0, \\ &a^3\dot{\xi}_1 + a^4\dot{\xi}_2 + a^7\dot{\xi}_5 + a^8\dot{\xi}_6 + \left(\dot{a}^1 - a^5\right)\xi_3 + \left(\dot{a}^2 - a^6\right)\xi_4 - \left(a^3 - a^5\right)\xi_5 \\ &\quad -a^4\xi_6 + \dot{a}^5\xi_7 + \left(\dot{a}^6 - a^8\right)\xi_8 - H\dot{\xi}_0 = -\dot{G}. \end{aligned} \tag{60}$$

So we have

$$\xi_0 = \xi_1 = \xi_2 = \xi_3 = \xi_4 = \xi_5 = \xi_6 = 0,\ \xi_7 = 1, \xi_8 = 0, G_1 = -a^5, \tag{61}$$

$$\xi_0 = 0,\ \xi_1 = t,\ \xi_2 = 0,\ \xi_3 = 1,\ \xi_4 = 0,\ \xi_5 = 1,\ \xi_6 = \xi_7 = \xi_8 = 0,\ G_2 = -a^1. \tag{62}$$

Equation (38) gives the conservation quantities of the system as follows:

$$I_1 = -a^5 = \text{const}, \tag{63}$$

$$I_2 = a^3 t + a^7 - a^1 = \text{const}. \tag{64}$$

Secondly, we use theorem 2 from the given first integral to implement the corresponding quasi-symmetric transformation. Assume that the system has an integral (64), then from Eqs. (44) and (46) we will obtain

$$\begin{aligned} &\xi_1 = t + a^5\xi_0, \quad \xi_2 = a^6\xi_0, \quad \xi_3 = 1, \quad \xi_4 = 0, \quad \xi_5 = 1, \quad \xi_6 = a^8\xi_0, \\ &\xi_7 = \left(a^5 - a^3\right)\xi_0, \quad \xi_8 = -a^4\xi_0, \quad a^3 t + a^7 - a^1 = a^3\xi_1 + a^4\xi_2 + a^7\xi_5 + a^8\xi_8 - H\xi_0 + G. \end{aligned} \tag{65}$$

Let

$$G = -a^1, \tag{66}$$

then we will have

$$\xi_0 = 0, \quad \xi_1 = t, \quad \xi_2 = 0, \quad \xi_3 = 1, \quad \xi_4 = 0, \quad \xi_5 = 1, \quad \xi_6 = \xi_7 = \xi_8 = 0. \tag{67}$$

From theorem 2, we can conclude that the transformation determined by expression (67) must be the generalized quasi-symmetric transformation of the system.

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