Special Lie symmetry and Hojman conserved quantity of Appell equations in a dynamical system of relative motion*

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Special Lie symmetry and the Hojman conserved quantity for Appell equations in a dynamical system of relative motion are investigated. The definition and the criterion of the special Lie symmetry of Appell equations in a dynamical system of relative motion under infinitesimal group transformation are presented. The expression of the equation for the special Lie symmetry of Appell equations and the Hojman conserved quantity, deduced directly from the special Lie symmetry in a dynamical system of relative motion, are obtained. An example is given to illustrate the application of the results.

Keywords: dynamics of relative motion, Appell equations, special Lie symmetry, Hojman conserved quantity

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1. Introduction

Symmetries and conserved quantities in dynamical systems play an important role in modern mathematical science.[1−15] In 1899, Appell equations for first-order linear nonholonomic constraint systems were developed. The have now become an important kind of equation for analytical mechanics. In recent decades, studies based on Appell equations and their application have been carried out and important results have been obtained.[16−26] The present paper studies the special Lie symmetry and the Hojman conserved quantity deduced directly from the special Lie symmetry of Appell equations in a dynamical system of relative motion. Also presented is the structural equation and the expression for the Hojman conserved quantity of Appell equations in a dynamical system of relative motion. Finally, an example is given to illustrate the application of the results.

2. The Appell equation and differential equations of motion in a relative motion dynamics system

Suppose that the velocity of a base-point O in a carrier is $\boldsymbol{v}$ and its angular velocity is $\boldsymbol{\omega}$, that a mechanical system consists of N particles each with mass m_i and position vector $\boldsymbol{r}_i$, and that the configuration of the mechanical system is governed by n generalized coordinates $q_s\,(s=1,\ldots,n)$, then the Appell equations will be written as follows:[16]

$$\frac{\partial S_r}{\partial \ddot{q}_s}=Q_s-\frac{\partial}{\partial q_s}\left(V^0+V^{\omega}\right)+Q_s^{\dot{\omega}}+\Gamma_s,\quad (s=1,\cdots,n), \tag{1}$$

where $S_r=(1/2)m_i\boldsymbol{a}_{ir}\cdot\boldsymbol{a}_{ir}$ is the energy of acceleration for the system; $Q_s=Q_s\,(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ are the generalized forces;

$$V^0=M\left(\boldsymbol{v}_0^*+\boldsymbol{\omega}\times\boldsymbol{v}_0\right)\cdot\boldsymbol{r}_{\rm c}' \tag{2}$$

is the potential energy in an uniform field, with M being the total mass for the system, $\boldsymbol{r}_{\rm c}'$ the vector of the mass point of the relative motion system with respect to the base point O, $\boldsymbol{v}_0^*$ the relative derivative of $\boldsymbol{v}_0$;

$$V^{\omega}=-\frac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{\theta}^0\cdot\boldsymbol{\omega} \tag{3}$$

is the potential energy of centrifugal force, with $\boldsymbol{\theta}^0$ being the inertial tensor of the system at point O;

$$Q_S^{\dot{\omega}}=-\left(\dot{\boldsymbol{\omega}}\times m_i\boldsymbol{r}_i'\right)\cdot\frac{\partial\boldsymbol{r}_i'}{\partial q_s} \tag{4}$$

is the generalized Coriolis inertial force;

$$\Gamma_s=\gamma_{sk}\dot{q}_k,\quad \gamma_{sk}=2\boldsymbol{\omega}\cdot\left(m_i\frac{\partial\boldsymbol{r}_i'}{\partial q_s}\times\frac{\partial\boldsymbol{r}_i'}{\partial q_k}\right) \tag{5}$$

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is the generalized gyroscopic force.

From Eq. (1), we can solve all $\ddot{q}_s$ as

$$\ddot{q}_s = \alpha_s\left(t, \boldsymbol{q}, \dot{\boldsymbol{q}}\right), \quad (s = 1, \ldots, n). \tag{6}$$

3. Structural equation of special Lie symmetry of Appell equations in a dynamical system of relative motion

Taking the special infinitesimal transformation in which the time is invariable as

$$t^* = t, q_s^*\left(t^*\right) = q_s\left(t\right) + \Delta q_s, \quad (s = 1, \ldots, n). \tag{7}$$

Equation (7) can be extended into

$$t^* = t, q_s^*\left(t^*\right) = q_s\left(t\right) + \varepsilon\xi_s\left(t, \boldsymbol{q}, \dot{\boldsymbol{q}}\right), \quad (s = 1, \ldots, n), \tag{8}$$

where ε is an infinitesimal parameter, and ξ_s is an infinitesimal generator.

The invariance of the differential equations under the special infinitesimal transformations (Eq. (8)) is called special Lie symmetry. The structural equation of the special Lie symmetry of Appell equations (Eq. (1)) in a dynamical system of relative motion can be written as

$$\begin{aligned} & X^{(2)}\left\{\frac{\partial S_r}{\partial \ddot{q}}\right\} \\ = \; & X^{(1)}\left[Q_s - \frac{\partial}{\partial q_s}\left(V^0 + V^\omega\right) + Q_s^{\dot{\omega}} + \Gamma_s\right], \end{aligned} \tag{9}$$

where

$$X^{(1)} = \xi_s\frac{\partial}{\partial q_s} + \frac{\bar{\mathrm{d}}\xi_s}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_s}, \tag{10}$$

$$X^{(2)} = X^{(1)} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}\xi_s}{\mathrm{d}t}\frac{\partial}{\partial \ddot{q}_k}. \tag{11}$$

The structural equation for the special Lie symmetry of Eq. (6) under special infinitesimal transformations (Eq. (8)) can be expressed as

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s = \frac{\partial \alpha_s}{\partial q_k}\xi_k + \frac{\partial \alpha_s}{\partial \dot{q}_k}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k, \tag{12}$$

where

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{q}_k\frac{\partial}{\partial q_k} + \alpha_k\frac{\partial}{\partial \dot{q}_k}. \tag{13}$$

If the generators ξ_s of Eq. (6) under the special infinitesimal transformations (Eq. (8)) satisfy Eq. (12), then the symmetry is called special Lie symmetry of Appell equations in a dynamical system of relative motion.

4. Hojman conserved quantity deduced directly from the special Lie symmetry of Appell equations in a dynamical system of relative motion

In a dynamical system of relative motion, the Hojman conserved quantity can be deduced directly from the special Lie symmetry.

Proposition If the generators ξ_s under special infinitesimal transformations (Eq. (8)) satisfy Eq. (12) and there is a function $\mu = \mu\left(t, \boldsymbol{q}, \dot{\boldsymbol{q}}\right)$ such that the following equation holds:

$$\frac{\partial \alpha_s}{\partial \dot{q}_s} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu = 0, \tag{14}$$

then the Hojman conserved quantity deduced directly from the special Lie symmetry of Appell equations in a dynamical system of relative motion is

$$I_{\mathrm{H}} = \frac{1}{\mu}\frac{\partial}{\partial q_s}\left(\mu\xi_s\right) + \frac{1}{\mu}\frac{\partial}{\partial \dot{q}_s}\left(\mu\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) = \text{const}. \tag{15}$$

Proof Substituting Eq. (15) into Eq. (6), and deriving the resulting equation with respect to t, we have

$$\begin{aligned} \frac{\bar{\mathrm{d}}}{\mathrm{d}t}I_H = \; & \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial \mu}{\partial q_s}\xi_s\right) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial \xi_s}{\partial q_s} \\ & + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial \mu}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s + \frac{\partial}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right). \end{aligned} \tag{16}$$

And we can prove easily

$$\begin{aligned} \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s = \; & \frac{\partial}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) - \frac{\partial}{\partial q_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s \\ & - \frac{\partial \alpha_k}{\partial \dot{q}_s}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right), \end{aligned}$$

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial \xi_s}{\partial q_s} = \frac{\partial}{\partial q_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s - \frac{\partial \alpha_k}{\partial q_s}\frac{\partial \xi_s}{\partial \dot{q}_k}. \tag{17}$$

Taking the partial derivative of Eq. (12) with respect to $\dot{q}_s$ and the summation of terms with subscript s, we have

$$\begin{aligned} & \frac{\partial}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) \\ = \; & \frac{\partial}{\partial \dot{q}}\left(\frac{\partial \alpha_s}{\partial q_k}\xi_k\right) + \frac{\partial}{\partial \dot{q}_s}\left(\frac{\partial \alpha_s}{\partial \dot{q}_k}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k\right). \end{aligned} \tag{18}$$

Substituting Eqs. (17) and (18) into Eq. (16), we have

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}I_H = \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial q_s}\xi_s\right) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial\dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) + \frac{\partial^2\alpha_s}{\partial q_k\partial\dot{q}_s}\xi_k + \frac{\partial^2\alpha_s}{\partial\dot{q}_k\partial\dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k. \quad (19)$$

Using Eq. (14), we obtain

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial q_s}\xi_s\right) = -\frac{\partial^2\alpha_s}{\partial q_k\partial\dot{q}_s}\xi_k + \frac{1}{\mu}\frac{\partial\mu}{\partial q_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s - \frac{1}{\mu}\frac{\partial\mu}{\partial\dot{q}_k}\frac{\partial\alpha_k}{\partial q_s}\xi_s, \quad (20)$$

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial\dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) = -\frac{\partial^2\alpha_s}{\partial\dot{q}_k\partial\dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k - \frac{1}{\mu}\frac{\partial\alpha_k}{\partial\dot{q}_s}\frac{\partial\mu}{\partial\dot{q}_k}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s + \frac{1}{\mu}\frac{\partial\mu}{\partial\dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) - \frac{1}{\mu}\frac{\partial\mu}{\partial q_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s. \quad (21)$$

Substituting Eqs. (20) and (21) into Eq. (19), we have

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}I_H = \frac{1}{\mu}\frac{\partial\mu}{\partial\dot{q}_s}\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) - \frac{\partial\alpha_s}{\partial\dot{q}_k}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k - \frac{\partial\alpha_s}{\partial q_k}\xi_k\right] = 0. \quad (22)$$

5. Illustrative example

Consider the Appell function for the system, expressed as

$$S_r = \frac{1}{2}m\left(\ddot{q}_1^2 + \ddot{q}_2^2 + \ddot{q}_3^2\right). \quad (23)$$

The generalized forces are given as

$$Q_1 = m\dot{q}_2, \quad Q_2 = -m\dot{q}_1, \quad \text{and} \quad Q_3 = 0, \quad (24)$$

and the potential energies as

$$V^0 = Q_s^{\ddot{\omega}} = \Gamma_s = 0, \quad (25)$$

$$V^{\omega} = -\frac{1}{2}m\left(q_1^2 + q_2^2\right)\omega^2. \quad (26)$$

Now, we study the special Lie symmetry and Hojman conserved quantity deduced directly from the special Lie symmetry of Appell equations.

Substituting Eqs. (23)–(26) into Eq. (1), we obtain

$$\begin{aligned}\ddot{q}_1 &= \omega^2 q_1 + \dot{q}_2,\\ \ddot{q}_2 &= \omega^2 q_2 - \dot{q}_1,\\ \ddot{q}_3 &= 0.\end{aligned} \quad (27)$$

Substituting Eq. (27) into Eq. (12), we have

$$\begin{aligned}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_1 &= \omega^2\xi_1 + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_2,\\ \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_2 &= \omega^2\xi_2 - \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_1,\\ \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_3 &= 0.\end{aligned} \quad (28)$$

Take the infinitesimal generators

$$\xi_1 = \xi_2 = 0, \xi_3 = 1, \quad (29)$$

$$\xi_1 = \xi_2 = 0, \xi_3 = \left(\dot{q}_3 t - q_3\right)^2, \quad (30)$$

$$\xi_1 = \xi_2 = 0, \xi_3 = \dot{q}_3\left(\dot{q}_3 t - q_3\right), \quad (31)$$

$$\xi_1 = \xi_2 = 0, \xi_3 = \left(\dot{q}_3 t - q_3\right)\left[\dot{q}_1^2 + \dot{q}_2^2 - \omega^2\left(q_1^2 + q_2^2\right)\right]. \quad (32)$$

Obviously, they all satisfy Eq. (28) and they are all infinitesimal generators of special Lie symmetry. The Appell equations of the system are, then, Lie symmetric.

From Eq. (14), we obtain

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu = 0. \quad (33)$$

Obviously

$$\mu = 1, \quad (34)$$

$$\mu = \dot{q}_3 t - q_3, \quad (35)$$

$$\mu = \left(\dot{q}_3 t - q_3\right)\left[\dot{q}_1^2 + \dot{q}_2^2 - \omega^2\left(q_1^2 + q_2^2\right)\right] \quad (36)$$

are all the solutions of Eq. (33).

Substituting Eqs. (29) and (35) into Eq. (15), we have a Hojman conserved quantity

$$I_{\mathrm{H}} = \left(q_3 - \dot{q}_3 t\right)^{-1} = \text{const.}$$

Substituting Eqs. (30) and (34) into Eq. (15), we have a Hojman conserved quantity

$$I_{\mathrm{H}} = 2\left(q_3 - \dot{q}_3 t\right) = \text{const.}$$

Substituting Eqs. (31) and (34) into Eq. (15), we have a Hojman conserved quantity

$$I_{\mathrm{H}} = -\dot{q}_3 = \text{const.}$$

Substituting Eqs. (32) and (34) into Eq. (15), we have a Hojman conserved quantity

$$I_{\mathrm{H}} = \omega^2\left(q_1^2 + q_2^2\right) - \dot{q}_1^2 - \dot{q}_2^2 = \text{const.}$$

Substituting Eqs. (29) and (36) into Eq. (15), we have a Hojman conserved quantity

$$I_{\mathrm{H}} = \left\{\left(q_3 - \dot{q}_3 t\right)\left[\dot{q}_1^2 + \dot{q}_2^2 - \omega^2\left(q_1^2 + q_1^2\right)\right]\right\}^{-1} = \text{const.}$$

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