Reductions and conserved quantities for discrete compound KdV–Burgers equations*

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We present two methods to reduce the discrete compound KdV–Burgers equation, which are reductions of the independent and dependent variables: the translational invariant method has been applied in order to reduce the independent variables; and a discrete spectral matrix has been introduced to reduce the number of dependent variables. Based on the invariance of a discrete compound KdV–Burgers equation under infinitesimal transformation with respect to its dependent and independent variables, we present the determining equations of transformation Lie groups for the KdV–Burgers equation and use the characteristic equations to obtain new forms of invariants.

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1. Introduction

The theories of symmetry for continuous differential equations are becoming more and more perfect.[1−4] There have been relatively few studies focusing on high-order discrete differential equations, or even differential–difference equations.[5−9] Whereas, finite difference analysis in mathematics, force and physical fields has attracted a lot of attention.[10−12] The main reason is that when the symmetry properties of a discrete difference system are defined on a time and space lattice, the system becomes easily solvable.

It has taken a long time to apply Lie group theory in the study differential equations. Lie group theory is an efficient tool for solving differential equations, classifying differential equations and determining the property of solution space.[13−16] The methods of Lie group analysis of differential equations are generalised so as to provide an infinitesimal formalism for calculating symmetries of difference equations.

Here, we introduce two methods to study compound KdV–Burgers equations: the first method is the reduction of independent variables and the other one is the reduction of dependent variables. These two methods are complementry to each other, for example, in a case where there are too many independent variables in the equation, we should choose to reduce of independent variables; if there are too many dependent variables, we should choose to reduce the dependent variables. Here, a translational invariant solution is presented for the reduction of independent variables. We also show how to use this method to obtain the corresponding solution of the compound KdV–Burgers equations present in a variety of physical contexts and studied by many authors.[17−37] In addition, the discrete matrix spectral problem can be introduced to reduce the number of dependent variables. Here, the equation for the physical mechanism of collision between solitons has been taken as an example, where a nonlinear difference equation with four dependent variables can be transformed into a difference equation with two dependent variables.

The KdV equation can describe many phenomena in the physical field and it has attracted the attention of many authors. It is not only a very important equation in theory but also in practice. It can describe small amplitude shallow water waves, the cold plasma MHD wave, an ion acoustic wavelet and wave processes in biological and physical systems. It is well known that studies on reductions and conserved quantities for discrete compound KdV–Burgers equations are still open. The infinitesimal transformation is an effective technique to obtain the invariants.

We present a new form of infinitesimal transformation that leads to new forms of invariants. In this paper, the time and space steps can be made to be independent variables which appear in an infinitesimal

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operator. Furthermore, infinitesimal ξ_i and η_j contain time and space steps. From this viewpoint, we can derive many new invariants.

2. KdV–Burgers equation

We express the compound KdV–Burgers equation in nonlinear terms of any order

$$u_t + puu_x + qu^2u_x + ru_{xx} - su_{xxx} = 0, \quad (1)$$

where p, q, r, and s are constants, which can be considered as a composition of the KdV, mKdV, and Burgers equations, involving nonlinear, dispersion and dissipation effects. When nonlinear wave problems in physics and hydrodynamics are studied, we use a compound KdV–Burgers equation, which contains the following particular important cases.

If $p \neq 0$, $q \neq 0$, $r = 0$, $s \neq 0$, then Eq. (1) becomes the compound KdV equation

$$u_t + puu_x + qu^2u_x - su_{xxx} = 0.$$

If $p = 0$, $q \neq 0$, $r \neq 0$, $s \neq 0$, then Eq. (1) becomes the mKdV–Burgers equation

$$u_t + qu^2u_x + ru_{xx} - su_{xxx} = 0.$$

If $p \neq 0$, $q = 0$, $r \neq 0$, $s \neq 0$, then Eq. (l) becomes the KdV–Burgers equation

$$u_t + puu_x + ru_{xx} - su_{xxx} = 0.$$

If $r = 0$, $p = 0$, $s \neq 0$, $q \neq 0$, then we obtain the mKdV equation

$$u_t + qu^2u_x - su_{xxx} = 0.$$

3. Reduction of the independent variables in a discrete compound KdV–Burgers equation

3.1. Discrete derivatives

Instead of partial derivatives of $u(x)$, the equations contain finite difference (variations) of the form

$$\Delta_{x_i}u = \frac{1}{\sigma_{x_i}}\{u(x_1, x_2, \ldots, x_{i-1}, x_i + \sigma_x, x_{i+1}, \ldots, x_q) - u(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_q)\}, \quad (2)$$

and also shift operator T_{x_i}

$$T_{x_i}u = u(x_1, x_2, \ldots, x_i, x_i + \sigma_x, x_{i+1}, \ldots, x_q).$$

The spacing, variations, and shift operators are related with one another

$$\Delta_{x_i} = \frac{1}{\sigma_i}(T_{x_i} - 1),$$
$$\Delta_{x_ix_i} = \frac{1}{\sigma_i^2}(T_{x_i}^2 - 2T_{x_i} + 1),$$
$$\Delta_{x_ix_ix_i} = \frac{1}{\sigma_i^3}(T_{x_i}^3 - 3T_{x_i}^2 + 3T_{x_i} - 1).$$

Then discrete compound KdV equation can be transformed to

$$\frac{u_{m,n+1} - u_{m,n}}{\sigma_t} + pu_{m,n}\frac{u_{m+1,n} - u_{m,n}}{\sigma_x} + qu_{m,n}^2\frac{u_{m+1,n} - u_{m,n}}{\sigma_x} + r\frac{u_{m+2,n} - 2u_{m+1,n} + u_{m,n}}{\sigma_x^2} - s\frac{u_{m+3,n} - 3u_{m+2,n} + 3u_{m+1,n} - u_{m,n}}{\sigma_x^3} = 0, \quad (3)$$

where

$$\sigma_t = t_{m,n+1} - t_{m,n}, \sigma_x = x_{m+1,n} - x_{m,n}. \quad (4)$$

3.2. Reduction of the KdV–Burgers equation using a translational invariant

To reduce Eq. (3), we shall consider the following lattice equation

$$x_{m+3,n} - 3x_{m+2,n} + 3x_{m+1,n} - x_{m,n} = 0,$$
$$x_{m,n+1} - x_{m,n} = 0,$$
$$t_{m+1,n} - t_{m,n} = 0,$$
$$t_{m,n+1} - t_{m,n} = c(x_{m+1,n} - x_{m,n})^3. \quad (5)$$

A solution of Eq. (3), invariant under a translation generated by $P_0 - aP_1$, $P_0 = \partial_t$, $P_1 = \partial_x$ will have the form

$$u = u(x,t) = u(z), z = x + at. \quad (6)$$

Then, the compound KdV–Burgers equation only relies on one independent variable z rather than two x and t, and then Eq. (3) can be given as

$$u(z_{m,n+1}) - u(z_{m,n}) + p(\sigma_x)^2u(z_{m,n})[u(z_{m+1,n}) - u(z_{m,n})] + q(\sigma_x)^2u(z_{m,n})^2[u(z_{m+1,n}) - u(z_{m,n})] + r\sigma_x[u(z_{m+2.n}) - 2u(z_{m+1,n}) - u(z_{m,n})] - s[u(z_{m+3,n}) - 3u(z_{m+2,n}) + 3u(z_{m+1,n}) - u(z_{m,n})] = 0. \quad (7)$$

Furthermore, the lattice equation is transformed to

$$z_{m+3,n} - 3z_{m+2,n} + 3z_{m+1,n} - z_{m,n} = 0,$$
$$z_{m,n+1} - z_{m,n} = c(z_{m+1,n} - z_{m,n})^3. \quad (8)$$

The particular solution of Eq. (8) is shown to be

$$z_{m,n} = am + a^3cn + c,$$

so $z_{m,n}$ really depends on just one vabiable $N = m + ca^2n$ (c and a are integration constants). When the reduced equation (8) is supposed to be a difference equation on some lattices, N must vary over integer values. Then Eq. (7) can be given as

$$u(z + ca^3) - u(z) + p(\sigma_x)^2[u(z + a) - (z)]$$
$$+ qu(z)^2(\sigma_x)^2[u(z + a) - (z)]$$
$$+ r\sigma_x(u(z + 2a) - 2u(z + a) - (z))$$
$$- s[u(z + 3a) - 3u(z + 2a)$$
$$+ 3u(z + a) - u(z)] = 0. \quad (9)$$

The particular solution of Eq. (9) can be given as

$$u = c_1 e^{\alpha z} + c_2, \quad (10)$$

where c_1, $c_2 = \text{const}$, and α should satisfy

$$e^{\alpha ca^3} - 1 + p(c_1 e^{\alpha z} + c_2)(\sigma_x)^2(e^{\alpha z} - 1)$$
$$+ q(c_1 e^{\alpha z} + c_2)^2(e^{\alpha z} - 1) + r(e^{2\alpha z} - 2e^{\alpha z} + 1)$$
$$- s(e^{3\alpha z} - 3e^{2\alpha z} + 3e^{\alpha z} - 1) = 0. \quad (11)$$

We can say that by means of a translational invariant, we can obtain the exact solution of a compound KdV–Burgers equation.

3.3. Invariants of a KdV–Burgers equation

For the general form of difference equation

$$F(x_{m,n}, x_{m+1,n}, \ldots, x_{m+N_1}, t_{m,n}, t_{m,n+1}, \ldots, t_{m,n+N_2}, u_{m,n}, \ldots, u_{m+N_3,n+N_4}) = 0, \quad (12)$$

the infinitesimal operator X will verify the invariance condition

$$XF(x_{m,n}, x_{m+1,n}, \ldots, x_{m+N_1}, t_{m,n}, t_{m,n+1}, \ldots, t_{m,n+N_2}, u_{m,n}, \ldots, u_{m+N_3,n+N_4}) = 0. \quad (13)$$

With Eq. (4), we can write

$$u_{m,n} = u(x_{m,n}, t_{m,n}) = u(x_{m+1,n} - \sigma_x, t_{m,n+1} - \sigma_t) = u(x_{m+1,n}, \sigma_x, t_{m,n+1}, \sigma_t),$$
$$u_{m+1,n} = u(x_{m+1,n}, t_{m+1,n}) = u(x_{m,n}, t_{m,n}, \sigma_x),$$
$$u_{m+2,n} = u(x_{m+2,n}, t_{m+2,n}) = u(x_{m+1,n}, t_{m+1,n}, \sigma_t, \sigma_x),$$
$$u_{m+3,n} = u(x_{m+3,n}, t_{m+3,n}) = u(x_{m,n}, t_{m,n}, \sigma_x, \sigma_t),$$
$$u_{m,n+1} = u(x_{m,n+1}, t_{m,n+1}) = u(x_{m,n}, t_{m,n}, \sigma_t). \quad (14)$$

To study the symmetry, we give the following infinitesimal transformation

$$\begin{aligned}
t^*_{m,n} &= X_1(t_{m,n}, x_{m,n}, x_{m+1,n}, t_{m,n+1}, \sigma_t, \sigma_x, u_{m,n+1}, u_{m,n}, u_{m+1,n}, u_{m+2,n}, u_{m+3,n}, \varepsilon)\\
&= t_{m,n} + \varepsilon\xi_1(t_{m,n}, x_{m,n}, x_{m+1,n}, t_{m,n+1}, \sigma_t, \sigma_x, u_{m,n+1}, u_{m,n}, u_{m+1,n}, u_{m+2,n}, u_{m+3,n}) + o(\varepsilon^2),\\
x^*_{m,n} &= X_2(t_{m,n}, x_{m,n}, x_{m+1,n}, t_{m,n+1}, \sigma_t, \sigma_x, u_{m,n+1}, u_{m,n}, u_{m+1,n}, u_{m+2,n}, u_{m+3,n}, \varepsilon)\\
&= x_{m,n} + \varepsilon\xi_2(t_{m,n}, x_{m,n}, x_{m+1,n}, t_{m,n+1}, \sigma_t, \sigma_x, u_{m,n+1}, u_{m,n}, u_{m+1,n}, u_{m+2,n}, u_{m+3,n}) + o(\varepsilon^2),\\
t^*_{m,n+1} &= X_3(t_{m,n}, x_{m,n}, t_{m,n+1}, x_{m+1,n}, \sigma_t, \sigma_x, u_{m,n}, u_{m,n+1}, u_{m+1,n}, u_{m+2,n}, u_{m+3,n}, \varepsilon)\\
&= t_{m,n+1} + \varepsilon\xi_3(t_{m,n}, x_{m,n}, t_{m,n+1}, x_{m+1,n}, \sigma_t, \sigma_x, u_{m,n}, u_{m,n+1}, u_{m+1,n}, u_{m+2,n}, u_{m+3,n}) + o(\varepsilon^2),\\
x^*_{m+1,n} &= X_4(t_{m,n}, x_{m,n}, t_{m,n+1}, x_{m+1,n}, \sigma_t, \sigma_x, u_{m,n}, u_{m,n+1}, u_{m+1,n}, u_{m+2,n}, u_{m+3,n}, \varepsilon)\\
&= x_{m+1,n} + \varepsilon\xi_4(t_{m,n}, x_{m,n}, t_{m,n+1}, x_{m+1,n}, \sigma_t, \sigma_x, u_{m,n}, u_{m,n+1}, u_{m+1,n}, u_{m+2,n}, u_{m+3,n}) + o(\varepsilon^2),\\
\sigma^*_t &= t^*_{m,n+1} - t^*_{m,n} = \sigma_t + \varepsilon\xi_5(t_{m,n}, x_{m,n}, u_{m,n}) + o(\varepsilon^2)\\
&= t_{m,n+1} + \varepsilon\xi_3(t_{m,n+1}, x_{m,n+1}, u_{m,n+1}, \varepsilon) - t_{m,n} - \varepsilon\xi_1(t_{m,n}, x_{m,n}, u_{m,n}) + o(\varepsilon^2)\\
\sigma^*_x &= x^*_{m+1,n} - x^*_{m,n} = \sigma_x + \varepsilon\xi_6(t_{m,n}, x_{m,n}, u_{m,n}) + o(\varepsilon^2)\\
&= x_{m+1,n} + \varepsilon\xi_4(t_{m,n}, x_{m,n}, u_{m,n}) - x_{m,n} - \varepsilon\xi_2(t_{m,n}, x_{m,n}, u_{m,n}) + o(\varepsilon^2),
\end{aligned}$$

$$
\begin{aligned}
u^*_{m,n} &= Y_1(t_{m,n}, x_{m,n}, x_{m+1,n}, t_{m,n+1}, \sigma_t, \sigma_x, u_{m,n}, u_{m,n+1}, u_{m+1,n}, u_{m+2,n}, u_{m+3,n}, \varepsilon)\\
&= u_{m,n} + \varepsilon\eta_1(t_{m,n}, x_{m,n}, x_{m+1,n}, t_{m,n+1}, \sigma_t, \sigma_x, u_{m,n}, u_{m,n+1}, u_{m+1,n}, u_{m+2,n}, u_{m+3,n}) + o(\varepsilon^2),\\
u^*_{m+1,n} &= Y_2(t_{m,n}, x_{m,n}, x_{m+1,n}, t_{m+1,n}, \sigma_t, \sigma_x, u_{m,n}, u_{m,n+1}, u_{m+1,n}, u_{m+2,n}, u_{m+3,n}, \varepsilon)\\
&= u_{m+1,n} + \varepsilon\eta_2(t_{m,n}, x_{m,n}, x_{m+1,n}, t_{m,n+1}, \sigma_t, \sigma_x, u_{m,n}, u_{m,n+1}, u_{m+1,n}, u_{m+2,n}, u_{m+3,n}) + o(\varepsilon^2),\\
u^*_{m,n+1} &= Y_3(t_{m,n}, x_{m,n}, t_{m+1,n}, x_{m+1,n}, \sigma_t, \sigma_x, u_{m,n}, u_{m+1,n}, u_{m+1,n}, u_{m+2,n}, u_{m+3,n}, \varepsilon)\\
&= u_{m,n+1} + \varepsilon\eta_3(t_{m,n}, x_{m,n}, x_{m+1,n}, t_{m,n+1}, \sigma_t, \sigma_x, u_{m,n}, u_{m,n+1}, u_{m+1,n}, u_{m+2,n}, u_{m+3,n}) + o(\varepsilon^2),\\
u^*_{m+2,n} &= Y_4(t_{m,n}, x_{m,n}, x_{m+1,n}, t_{m,n+1}, \sigma_t, \sigma_x, u_{m,n}, u_{m,n+1}, u_{m+1,n}, u_{m+2,n}, u_{m+3,n}, \varepsilon)\\
&= u_{m+2,n} + \varepsilon\eta_4(t_{m,n}, x_{m,n}, x_{m+1,n}, t_{m,n+1}, \sigma_t, \sigma_x, u_{m,n}, u_{m,n+1}, u_{m+1,n}, u_{m+2,n}, u_{m+3,n}) + o(\varepsilon^2),\\
u^*_{m+3,n} &= Y_5(t_{m,n}, x_{m,n}, x_{m+1,n}, t_{m,n+1}, \sigma_t, \sigma_x, u_{m,n}, u_{m,n+1}, u_{m+1,n}, u_{m+2,n}, u_{m+3,n}, \varepsilon)\\
&= u_{m+3,n} + \varepsilon\eta_5(t_{m,n}, x_{m,n}, x_{m+1,n}, t_{m,n+1}, \sigma_t, \sigma_x, u_{m,n}, u_{m,n+1}, u_{m+1,n}, u_{m+2,n}, u_{m+3,n}) + o(\varepsilon^2).
\end{aligned}
$$

We write a vector field which covers all points involved in the difference scheme. In the case of system (3), we have

$$
\begin{aligned}
X =\ & \xi_1\frac{\partial}{\partial t_{m,n}} + \xi_2\frac{\partial}{\partial x_{m,n}} + \xi_3\frac{\partial}{\partial t_{m,n+1}} + \xi_4\frac{\partial}{\partial x_{m+1,n}} + \xi_5\frac{\partial}{\partial \sigma_t} + \xi_6\frac{\partial}{\partial \sigma_x}\\
&+ \eta_1\frac{\partial}{\partial u_{m,n+1}} + \eta_2\frac{\partial}{\partial u_{m,n}} + \eta_3\frac{\partial}{\partial u_{m+1,n}} + \eta_4\frac{\partial}{\partial u_{m+2,n}} + \eta_5\frac{\partial}{\partial u_{m+3,n}}. \qquad (15)
\end{aligned}
$$

Let X act on Eq. (3) and obtain the coefficients of

$$
\frac{\partial u_{m,n}}{\partial \sigma_t},\quad \frac{\partial u_{m,n}}{\partial t_{m,n}},\quad \frac{\partial u_{m,n}}{\partial x_{m+1,n}},\quad \frac{\partial u_{m,n}}{\partial \sigma_x},\quad \frac{\partial u_{m+1,n}}{\partial t_{m,n}},
$$

and $\dfrac{\partial u_{m+1,n}}{\partial x_{m,n}}$, and then we can conclude

$$
\xi_1 = \xi_2 = \xi_3 = \xi_4 = \xi_5 = \xi_6 = 0,
$$

$$
\begin{aligned}
&\frac{\eta_1}{\sigma_t} + \eta_2\bigg\{ -\frac{1}{\sigma_t} + p\frac{u_{m+1,n} - 2u_{m,n}}{\sigma_x}\\
&+ \frac{q}{\sigma_x}(2u_{m,n}u_{m+1,n} - 3u^2_{m,n}) + \frac{r}{\sigma_x^2} + \frac{s}{\sigma_x^3}\bigg\}\\
&+ \eta_3\bigg\{\frac{pu_{m,n}}{\sigma_x} + q\frac{u^2_{m,n}}{\sigma_x} - 2\frac{r}{\sigma_x^2} - 3\frac{s}{\sigma_x^3}\bigg\}\\
&+\eta_4\bigg(-\frac{s}{\sigma_x^3}\bigg) = 0. \qquad (16)
\end{aligned}
$$

When $\eta_1 = \eta_2 = 0$, we can obtain the corresponding invariants through the characteristic equation, which can be shown to be

$$
\frac{\mathrm{d}u_{m+1,n}}{\eta_3} = \frac{\mathrm{d}u_{m+2,n}}{\eta_4}. \qquad (17)
$$

Integrating both sides of Eq. (17), to say

$$
\begin{aligned}
I_1 =\ & (pu_{m,n}\sigma_x^2 + qu^2_{m,n}\sigma_x^2\\
&- 2r\sigma_x - 3s)u_{m+2,n} - u_{m+3,n}\\
=\ & \text{const.} \qquad (18)
\end{aligned}
$$

When $\eta_2 = \eta_4 = 0$, the invariant will be of a new form

$$
\begin{aligned}
I_2 =\ & \sigma_t\bigg(-\frac{p}{\sigma_t}u_{m,n} - \frac{q}{\sigma_x}u^2_{m,n}\\
&+ \frac{2r}{\sigma_x^2} + \frac{3s}{\sigma_x^3}\bigg)u_{m+1,n} - u_{m,n+1} = \text{const.} \qquad (19)
\end{aligned}
$$

4. Reduction of the dependent variables in a discrete compound KdV–Burgers equation

If we fix $p = 6,\ q = 0,\ r = 0$, and $s = -1$, then Eq. (1) can be transformed to

$$
u_t + 6uu_x + u_{xxx} = 0, \qquad (20)
$$

which was presented by Russel in 1834. It is a very unusual wave movement found in natural phenomena. However, when limited to mathematical theory and on a scientific level, one cannot give a satisfactory explanation for its solitary waves. When Korteweg and de Vries studied the wave of the movement using long-wave approximation and the assumption of small amplitude, the establishment of a one-dimensional mathematical model could successfully explain the phenomena of solitary waves, until 1895. Equation (20) can model the phenomena of the physical mechanism of collision between solitons. With Eq. (4), Eq. (20) will be transformed to

$$
\begin{aligned}
&\frac{u_{m,n+1} - u_{m,n}}{\sigma_t} + 6u_{m,n}\frac{u_{m+1,n} - u_{m,n}}{\sigma_x}\\
&+ \frac{u_{m+3,n} - 3u_{m+2,n} + 3u_{m+1,n} - u_{m,n}}{(\sigma_x)^3} = 0. \qquad (21)
\end{aligned}
$$

Equation (21), together with Eq. (5), will be of another form

$$u_{m,n+1} - u_{m,n} + 6(\sigma_x)^2 u_{m,n}(u_{m+1,n} - u_{m,n}) + u_{m+3,n} - 3u_{m+2,n} + 3u_{m+1,n} - u_{m,n} = 0. \quad (22)$$

In order to simplify Eq. (22), we would use a transformation. Let $u_{m,n}$ be a lattice function, and the shift E is defined as

$$Eu_{m-1,n} = u_{m,n}. \quad (23)$$

The discrete matrix spectral problem is written as

$$E\phi_n = U_n(u_n, \lambda)\phi_n, \quad (24)$$

where

$$U_n(u_n, \lambda) = \begin{pmatrix} 0 & \alpha v_n^2 + \beta v_n \\ -1 & \lambda - \omega_n \end{pmatrix}, \quad \phi_n = \begin{pmatrix} \phi_n^1 \\ \phi_n^2 \end{pmatrix}, \quad (25)$$

and

$$u_n = \begin{pmatrix} v_n \\ \omega_n \end{pmatrix}, \quad (26)$$

where λ is a parameter, the two constants α and β satisfy the non-degenerated condition $\alpha^2 + \beta^2 \neq 0$. We assume

$$\phi_n = \begin{pmatrix} u_{m,n} \\ u_{m+2,n} \end{pmatrix}. \quad (27)$$

With this assumption, Eq. (24) will be transformed to

$$\begin{pmatrix} u_{m+1,n} \\ u_{m+3,n} \end{pmatrix} = E\begin{pmatrix} u_{m,n} \\ u_{m+2,n} \end{pmatrix} = \begin{pmatrix} 0 & \alpha v_n^2 + \beta v_n \\ -1 & \lambda - \omega_n \end{pmatrix}\begin{pmatrix} u_{m,n} \\ u_{m+2,n} \end{pmatrix}. \quad (28)$$

Then we give the equivalent form of Eq. (28)

$$u_{m+1,n} = (\alpha v_n^2 + \beta v_n)u_{m+2,n}, \quad u_{m+3,n} = -u_{m,n} + (\lambda - \omega_n)u_{m+2,n}. \quad (29)$$

From this result, we can conclude that $u_{m+1,n}$ and $u_{m+3,n}$ can be replaced by $u_{m,n}$ and $u_{m+2,n}$. So we can eliminate them to simplify Eq. (22). Then Eq. (22) will be transformed to

$$u_{m,n+1} + 6(\sigma_x)^2 u_{m,n}[(\alpha v_n^2 + \beta v_n)u_{m+2,n} - u_{m,n}] - 3u_{m,n} + u_{m+2,n} \times [3(\alpha v_n^2 + \beta v_n) + \lambda - \omega_n - 3] = 0. \quad (30)$$

The three dependent variables $u_{m,n}$, $u_{m,n+1}$, and $u_{m+2,n}$ can be written as

$$u_{m,n} = u(t_{m,n}, x_{m,n}),$$
$$u_{m,n+1} = u(x_{m,n+1}, t_{m,n+1}) = u(x_{m,n}, t_{m,n}, \sigma_t),$$
$$u_{m+2,n} = u(x_{m+2,n}, t_{m+2,n}) = u(x_{m,n}, t_{m,n}, \sigma_x). \quad (31)$$

Because $u_{m,n}$, $u_{m,n+1}$, and $u_{m+2,n}$ depend on $x_{m,n}$, $t_{m,n}$, σ_t and σ_x, $x_{m,n}$, $t_{m,n}$, σ_t, and σ_x will be chosen as independent variables. At the same time $u_{m,n}$, $u_{m,n+1}$ and $u_{m+2,n}$ will be dependent variables. In the standard procedure, the infinitesimal transformation for the problem can be written as

$$\begin{aligned} x^*_{m,n} &= x_{m,n} + \varepsilon\xi_1(t_{m,n}, x_{m,n}, \sigma_t, \sigma_x) + o(\varepsilon^2), \\ t^*_{m,n} &= t_{m,n} + \varepsilon\xi_2(t_{m,n}, x_{m,n}, \sigma_t, \sigma_x) + o(\varepsilon^2), \\ \sigma^*_t &= \sigma_t + \varepsilon\xi_3(t_{m,n}, x_{m,n}, \sigma_t, \sigma_x) + o(\varepsilon^2), \\ \sigma^*_x &= \sigma_x + \varepsilon\xi_4(t_{m,n}, x_{m,n}, \sigma_t, \sigma_x) + o(\varepsilon^2), \\ u^*_{m,n+1} &= u_{m,n+1} + \varepsilon\eta_1(t_{m,n}, x_{m,n}, \sigma_t, \sigma_x) + o(\varepsilon^2), \\ u^*_{m,n} &= u_{m,n} + \varepsilon\eta_2(t_{m,n}, x_{m,n}, \sigma_t, \sigma_x) + o(\varepsilon^2), \\ u^*_{m+2,n} &= u_{m+2,n} + \varepsilon\eta_3(t_{m,n}, x_{m,n}, \sigma_t, \sigma_x) + o(\varepsilon^2). \end{aligned} \quad (32)$$

Then the corresponding infinitesimal operator will be given as

$$X = \xi_1\frac{\partial}{\partial x_{m,n}} + \xi_2\frac{\partial}{\partial t_{m,n}} + \xi_3\frac{\partial}{\partial \sigma_t} + \xi_4\frac{\partial}{\partial \sigma x} + \eta_1\frac{\partial}{\partial u_{m,n+1}} + \eta_2\frac{\partial}{\partial u_{m,n}} + \eta_3\frac{\partial}{\partial u_{m+2,n}}. \quad (33)$$

Let Eq. (33) act on Eq. (30) and draw the coefficients of

$$\frac{\partial u_{m,n}}{\partial x_{m,n}}, \quad \frac{\partial u_{m,n}}{\partial t_{m,n}}, \quad \frac{\partial u_{m,n}}{\partial \sigma_t}, \quad \text{and} \quad \frac{\partial u_{m,n}}{\partial \sigma_x},$$

and then set them to zero. We can obtain that following equations

$$\xi_1 = \xi_2 = \xi_3 = \xi_4 = 0,$$

$$\eta_1 + \eta_2[6(\sigma_x)^2(\alpha v_n^2 + \beta v_n)u_{m+2,n} - 12(\sigma_x)^2] + \eta_3[6(\sigma_x)^2(\alpha v_n^2 + \beta v_n)u_{m,n} + 3(\alpha v_n^2 + \beta v_n) + \lambda - \omega_n - 3] = 0. \quad (34)$$

When we fix $\eta_1 = 0$, we can give the corresponding characteristic equation

$$\frac{\mathrm{d}u_{m,n}}{\eta_2} = \frac{\mathrm{d}u_{m+2,n}}{\eta_3}. \quad (35)$$

Integrating both sides of Eq. (35), we obtain the conserved quantity

$$\begin{aligned} I_1 &= [6(\sigma_x)^2(\alpha v_n^2 + \beta v_n) \\ &\quad + 3(\alpha v_n^2 + \beta v_n) + \lambda - \omega_n - 3]u_{m+2,n} \\ &\quad + [6(\sigma_x)^2(\alpha v_n^2 + \beta v_n) - 12(\sigma_x)^2]u_{m,n} \\ &= \text{const.} \end{aligned} \quad (36)$$

From Eq. (36), we can obtain the result that if $I_1 = \text{const}\ t = C$, then

$$u_{m+2,n} = \frac{12\sigma_x^2 - 6\sigma_x^2(\alpha v_n^2 + \beta v_n)}{6\sigma_x^2(\alpha v_n^2 + \beta v_n) + 3(\alpha v_n^2 + \beta v_n) + \lambda - \omega_n - 3} u_{m,n} + \frac{C}{6\sigma_x^2(\alpha v_n^2 + \beta v_n) + 3(\alpha v_n^2 + \beta v_n) + \lambda - \omega_n - 3}. \tag{37}$$

We can derive the value of $u_{m+2,n}$ from $u_{m,n}$. If we set the initial value of $u(x_{m,n}, t_{m,n}) = 0$, we obtain

$$u_{m+2,n} = \frac{C}{6\sigma_x^2(\alpha \nu_n^2 + \beta \nu_n) + 3(\alpha \nu_n^2 + \beta \nu_n) + \lambda - \omega_n - 3}.$$

Furthermore, we have

$$u_{m+4,n} = \frac{C(12\sigma_x^2 - 6\sigma_x^2(\alpha \nu_n^2 + \beta \nu_n))}{[6\sigma_x^2(\alpha \nu_n^2 + \beta \nu_n) + 3(\alpha \nu_n^2 + \beta \nu_n) + \lambda - \omega_n - 3]^2} + \frac{C}{6\sigma_x^2(\alpha \nu_n^2 + \beta \nu_n) + 3(\alpha \nu_n^2 + \beta \nu_n) + \lambda - \omega_n - 3}. \tag{38}$$

So we can derive $u_{m+6,n}, u_{m+8,n}, \ldots$. If $\eta_2 = 0$, from the characteristic equation given as below

$$\frac{\mathrm{d}u_{m,n+1}}{\eta_1} = \frac{\mathrm{d}u_{m+3,n}}{\eta_3},$$

then we can obtain another conserved quantity written as

$$I_2 = u_{m+2,n} + [6(\sigma_x)^2(\alpha v_n^2 + \beta v_n)u_{m,n} + 3(\alpha v_n^2 + \beta v_n) + \lambda - \omega_n - 3]u_{m,n+1} = \text{const}. \tag{39}$$

If we set the initial value $u(x_{m,n}, t_{m,n}) = 0$, $u(x_{m,n+1}, t_{m,n+1}) = C_1$, and $I_2 = \text{const} = C$, then we obtain

$$u_{m+2,n} = C + [3(\alpha v_n^2 + \beta v_n) + \lambda - \omega_n - 3]C_1.$$

Under the condition that $\eta_3 = 0$, the corresponding conserved quantity is

$$I_3 = u_{m,n}[12(\sigma_x)^2 - 6(\sigma_x)^2(\alpha v_n^2 + \beta v_n)u_{m+2,n}] - u_{m,n+1} = \text{const}. \tag{40}$$

If we set different values for arbitrary function η, then we will gain a lot of different invariants.

5. Conclusion

We present a systemic method to reduce a discrete compound KdV–Burgers equation and obtain the invariants for it in order to extend the method into other fields.

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