Second-order nonlinear differential operators possessing invariant subspaces of submaximal dimension*

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The invariant subspace method is used to construct the explicit solution of a nonlinear evolution equation. The second-order nonlinear differential operators that possess invariant subspaces of submaximal dimension are described. There are second-order nonlinear differential operators, including cubic operators and quadratic operators, which preserve an invariant subspace of submaximal dimension. A full description of the second-order cubic operators with constant coefficients admitting a four-dimensional invariant subspace is given. It is shown that the maximal dimension of invariant subspaces preserved by a second-order cubic operator is four. Several examples are given for the construction of the exact solutions to nonlinear evolution equations with cubic nonlinearities. These solutions blow up in a finite time.

Keywords: nonlinear evolution equations, cubic operators, invariant subspace method, submaximal dimension, blow-up solution

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1. Introduction

The invariant subspace method was proposed in Refs. [1] and [2] to construct explicit solutions for nonlinear differential equations and is related to conditional Lie–Bäcklund symmetry and the differential constraints method.[3−10] The partial differential equation (PDE) can be reduced to an ordinary differential equation (ODE) using that method.[10,11] Svirshchevskii made an important observation regarding the nonlinear ordinary differential operators that admit a given subspace.[12−16] Recently, a systematic account of the approach, including some new results and many potential applications, was given in Ref. [10]. It shows that many important solutions with both physical and geometrical applications can be obtained through this method. Using the method, Qu and Zhu successfully constructed solutions for generalized thin film equations[17] and systems of nonlinear diffusion equations[18]. Structure theorems for linear and nonlinear differential operators admitting invariant subspaces of a polynomial form were obtained in Ref. [19], which can be used to construct the polynomial solutions of nonlinear evolution equations. Indeed, it is noted that the fundamental solutions of heat equations and porous medium equations belong to polynomial invariant subspaces in a variable transformation sense.[10]

The purpose of this paper is to describe nonlinear second-order operators that possess invariant subspaces of a submaximal dimension. We can then construct explicit solutions for the corresponding nonlinear evolution equations. It has been proven that the dimension of an invariant subspace admitting nonlinear k-th order operators cannot be greater than $2k+1$.[10,12] In Ref. [12], the first- and second-order nonlinear operators that admit a subspace of maximal dimension were studied, and it was shown that they were quadratic operators. Svirshchevskii obtained a full description of the second-order quadratic operators with constant coefficients, which admit five-dimensional invariant subspaces. It is implied that there are second-order nonlinear differential operators that admit invariant subspaces of maximal dimension. The second-order quadratic operators that admit lower-dimensional invariant subspaces are described in Ref. [10]. In the present paper, it is shown that there are second-order operators, including quadratic operators and cubic operators, which possess invariant subspaces of four dimensions. A full description of the cubic operators that have constant coefficients possessing four-dimensional invariant sub-

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spaces is presented, proving that the maximal dimension of invariant subspaces preserved by a cubic operator is four. The exact solutions of the corresponding nonlinear evolution equations can then be obtained. Some of these solutions blow up in a finite time.

The remainder of the present paper is organized as follows. In Section 2, we give a brief discussion of the invariant subspace method. In Section 3, we present the expression of the second-order ordinary operators possessing invariant subspaces of submaximal dimension. In Section 4, we fully investigate cubic operators with constant coefficients that admit four-dimensional invariant subspaces. Several examples are presented for the construction of exact solutions to nonlinear partial differential equations with cubic nonlinearities. These solutions blow up in a finite time.

2. Invariant subspace method

Consider the general evolution equation

$$u_t = F[u], \tag{1}$$

where $F[u]$ is a kth-order ordinary differential operator and $F[u] \equiv F(x, u, u_x, u_{xx}, \ldots)$ is a sufficiently smooth function of indicated variables. The essence of the approach is presented by the following lemma as expressed in Refs. [2], [10], and [12].

Lemma 1 Let the n-dimensional linear subspace

$$\begin{aligned} W_n &= \mathcal{L}\{f_1(x), \ldots, f_n(x)\} \\ &\equiv \left\{ \sum_{i=1}^{n} C_i f_i(x) \Big| C_i \in \mathbb{R} \right\} \end{aligned} \tag{2}$$

(a linear hull of linearly independent functions $f_1(x), \ldots, f_n(x)$) be invariant with respect to the operator F, *i.e.*, $F[W_n] \subseteq W_n$ (the operator F is said to preserve or admit W_n), which means

$$F\left(\sum_{i=1}^{n} C_i f_i(x)\right) = \sum_{i=1}^{m} \tilde{F}_i(C_1, \ldots, C_n) f_i(x),$$
for any $C_i \in \mathbb{R}$,

where $\{\tilde{F}_i\}$ are expansion coefficients of $F[u]$ in the basis $\{f_i(x)\}$. Then Eq. (1) has solutions in the form of

$$u(x,t) = \sum_{i=1}^{n} C_i(t) f_i(x). \tag{3}$$

By substituting Eq. (3) into Eq. (1), Eq. (1) is reduced into the system of ODEs

$$C_i'(t) = \tilde{F}(C_1(t), \ldots, C_n(t)), \quad i = 1, \ldots, n.$$

Invariant subspace W_n is defined as the space of solutions of a linear nth-order ordinary differential equation (ODE)

$$\begin{aligned} L[y] \equiv\ & y^{(n)} + a_{n-1}(x) y^{(n-1)} + \ldots \\ & + a_1(x) y' + a_0(x) y = 0. \end{aligned} \tag{4}$$

If the operator F admits the invariant subspace defined by the linear ODE (4), then the invariant condition with respect to F takes the form of

$$L[F[u]]\big|_{[H]} \equiv 0, \tag{5}$$

where $[H]$ denotes the equation $L[u] = 0$ and its differential consequences with respect to x. As pointed in Ref. [10], condition (5) is actually the invariance criterion for Eq. (4) with respect to the Lie–Bäcklund operator $X = F[y]\partial/\partial y$. Therefore, all the results can be interpreted in terms of symmetries of linear ODEs.[13,14] On the other hand, invariant condition (5) implies that Eq. (1) has conditional Lie-Bäcklund symmetry or conditional symmetry with the characteristic $\sigma = L[y]$,[13,14] and linear differential constraint $L[y] = 0$.[6–8] Invariant condition (5) leads to the following theorem on the maximal dimension of the invariant subspaces preserved by $F[u]$, which plays an important role in the approach.[1,2,10,12–17]

Theorem 1 Assume that a linear subspace W_n is invariant under a nonlinear ordinary operator F of order k, then $n \leqslant 2k + 1$.

The nonlinear operators preserving five-dimensional invariant subspaces defined by ODE (4) are described by the following theorem.[10,12]

Theorem 2 Any operator $F(x, u, u_x, u_{xx})$ possessing a five-dimensional invariant subspace can be represented in the form of

$$F[u] = F_2[u] + F_1[u] + F_0[u],$$

where

$$\begin{aligned} F_2[u] &= M_{22} u_{xx}^2 + M_{12} u u_{xx} + M_{11} u_x^2 \\ &\quad + M_{01} u u_x + M_{00} u^2, \\ F_1[u] &= M_2 u_{xx} + M_1 u_x + M_0 u, \quad F_0[u] = M, \end{aligned}$$

with M_{ij}, M_i, and M $(i, j = 0, 1, 2)$ being functions of x.

There are second-order quadratic operators with constant coefficients that admit five-dimension invariant subspaces.[12] Therefore, for $k = 2$, the maximal dimension of the invariant subspaces W_n admitted by a nonlinear operator is five. The quadratic operators,

which admit invariant subspaces of lower dimension, are given in Ref. [10]. In Sections 3 and Section 4, we study the second-order nonlinear differential operators preserving invariant subspaces of submaximal (four) dimension.

3. Second-order operators admitting invariant subspaces of submaximal dimension

In this section, we give the expression of nonlinear operator of the second order

$$F[u] \equiv F(x, u, u_x, u_{xx}),$$

which admits an invariant subspace of submaximal dimension defined by the ODE

$$\begin{aligned} L[y] \equiv\ & y^{(4)} + a_3(x)y''' + a_2(x)y'' \\ & + a_1(x)y' + a_0(x)y = 0. \end{aligned} \tag{6}$$

In this case, the invariant condition takes the form of

$$\begin{aligned} & D^4F + a_3(x)D^3F + a_2(x)D^2F \\ & + a_1(x)DF + a_0(x)F \equiv 0 \end{aligned} \tag{7}$$

holding for the solution of manifold of Eq. (6). From the invariant condition (7), we have the following theorem.

Theorem 3 Any operator $F[u] \equiv F(x,\ u,\ u_x,\ u_{xx})$ possessing a four-dimensional invariant subspace defined by (6) can be expressed in the form of

$$F[u] = F_3[u] + F_2[u] + F_1[u] + F_0[u],$$

where

$$\begin{aligned} F_3[u] =\ & F_{30}u_{xx}^3 + F_{21}u_xu_{xx}^2 + F_{201}uu_{xx}^2 + F_{12}u_x^2u_{xx} \\ & + F_{111}uu_xu_{xx} + F_{102}u^2u_{xx} + F_{030}u_x^3 \\ & + F_{021}uu_x^2 + F_{012}u^2u_x + F_{003}u^3, \\ F_2[u] =\ & F_{200}u_{xx}^2 + F_{110}u_xu_{xx} + F_{101}uu_{xx} \\ & + F_{020}u_x^2 + F_{011}uu_x + F_{002}u^2, \\ F_1[u] =\ & F_{100}u_{xx} + F_{01}u_x + F_{001}u, \\ F_0[u] =\ & F_{000}, \end{aligned}$$

with F_{ijk}, and F_{ij} being functions of x.

Proof following notation is used throughout the paper

$$u_j = \frac{\partial^j u}{\partial x^j} \ \text{ for all } j \geqslant 0\ (u_0 = u).$$

The left-hand side of identity (7) is a polynomial in variable u_3, while the coefficient of u_3^4 equals $\partial^4F/\partial u_3^4$. Equating this coefficient to zero and solving the resulting equation lead to

$$\begin{aligned} & F(x, u_0, u_1, u_2) \\ =\ & F_3(x, u_0, u_1)u_2^3 + F_2(x, u, u_1)u_2^2 \\ & + F_1(x, u_0, u_1)u_2 + F_0(x, u_0, u_1). \end{aligned} \tag{8}$$

Substituting Eq. (8) into Eq. (7), then the left-hand side of identity (7) is a polynomial in variables u_3 and u_2. Collecting the coefficients of $u_2u_3^3$ and $u_2^2u_3^2$, we obtain

$$\begin{aligned} u_2u_3^3 :\ & \frac{\partial F_3}{\partial u_1}(x, u_0, u_1) = 0, \\ u_2^2u_3^2 :\ & -75a_3(x)\frac{\partial F_3}{\partial u_1}(x, u_0, u_1) + 39\frac{\partial^2 F_2}{\partial u_1^2}(x, u_0, u_1) \\ & + 48\frac{\partial F_3}{\partial u_0}(x, u_0, u_1) + 108\frac{\partial^2 F_3}{\partial x \partial u_1}(x, u_0, u_1) \\ & + 108u_1\frac{\partial^2 F_3}{\partial u_0 \partial u_1}(x, u_0, u_1) = 0, \end{aligned}$$

which imply that

$$\frac{\partial F_3}{\partial u_1}(x, u_0, u_1) = 0, \quad \frac{\partial^3 F_2}{\partial u_1^3}(x, u_0, u_1) = 0.$$

Thus $F(x, u, u_1, u_2)$ can be expressed as follows:

$$\begin{aligned} & F(x, u_0, u_1, u_2) \\ =\ & F_3(x, u_0)u_2^3 + [F_{22}(x, u_0)u_1^2 \\ & + F_{21}(x, u_0)u_1 + F_{20}(x, u_0)]u_2^2 \\ & + F_1(x, u_0, u_1)u_2 + F_0(x, u_0, u_1). \end{aligned} \tag{9}$$

Substituting Eq. (9) into Eq. (7) and equating the coefficients of $u_2^2u_3^2$ and u_3^3 to zero, we obtain

$$\begin{aligned} u_2^2u_3^2 :\ & 48\frac{\partial F_3}{\partial u_0}(x, u_0) + 78F_{22}(x, u_0) = 0, \\ u_3^3 :\ & 12F_{21}(x, u_0) + 24\frac{\partial F_3}{\partial u_0}(x, u_0)u_1 + 24F_{22}(x, u_0)u_1 \\ & + 24\frac{\partial F_3}{\partial x}(x, u_0) - 30a_3(x)F_3(x, u_0) = 0, \end{aligned}$$

which leads to

$$\frac{\partial F_3}{\partial u_0}(x, u_0) = 0, \quad F_{22}(x, u_0) = 0, \quad \frac{\partial F_{21}}{\partial u_0}(x, u_0) = 0.$$

Expressing $F_3(x, u_0)$ and $F_{21}(x, u_0)$ as follows:

$$F_3(x, u_0) = F_{30}(x),\ F_{21}(x, u_0) = F_{21}(x),$$

taking into account the above results, collecting the coefficient of $u_2u_3^2$ and differentiating in u_1 lead to

$$15\frac{\partial^3 F_1}{\partial u_1^3}(x, u_0, u_1) = 0, \tag{10}$$

which implies that

$$F_1(x,u_0,u_1)=F_{12}(x,u_0)u_1^2+F_{11}(x,u_0)u_1+F_{10}(x,u_0).$$

Then equating the coefficient of u_2^4 to zero gives

$$\frac{\partial^4 F_0}{\partial u_1^4}(x,u_0,u_1)+12\frac{\partial F_{12}}{\partial u_0}(x,u_0)+3\frac{\partial^2 F_{20}}{\partial u_0^2}(x,u_0)=0,$$

which implies that $F_0(x,u_0,u_1)$ can be expressed as follows:

$$\begin{aligned}&F_0(x,u_0,u_1)\\ =\;&F_{04}(x,u_0)u_1^4+F_{03}(x,u_0)u_1^3+F_{02}(x,u_0)u_1^2\\ &+F_{01}(x,u_0)u_1+F_{00}(x,u_0).\end{aligned}$$

In view of the above results, the left-hand side of identity (7) is a polynomial in variables u_3, u_2 and u_1. Coefficients of $u_1u_2^2u_3$, $u_2u_3^2$ and $u_1^2u_3$ yield

$$\begin{aligned}u_1u_2^2u_3:\;&28\frac{\partial^2 F_{20}}{\partial u_0^2}(x,u_0)+92\frac{\partial F_{12}}{\partial u_0}(x,u_0)\\ &+144F_{04}(x,u_0)=0,\\ u_2u_3^2:\;&20\frac{\partial F_{20}}{\partial u_0}(x,u_0)+30F_{12}(x,u_0)+B_1(x)=0,\\ u_1^2u_3^2:\;&12\frac{\partial^2 F_{20}}{\partial u_0^2}(x,u_0)+28\frac{\partial F_{12}}{\partial u_0}(x,u_0)\\ &+36F_{04}(x,u_0)=0,\end{aligned}\tag{11}$$

where $B_1(x)$ is a function of x dependent on F_3, F_{21}, and a_3. From Eq. (11), we obtain

$$\frac{\partial^2 F_{20}}{\partial u_0^2}(x,u_0)=0,\ \frac{\partial F_{12}}{\partial u_0}(x,u_0)=0,\ F_{04}(x,u_0)=0,$$

which yield

$$F_{20}(x,u_0)=F_{201}(x)u_0+F_{200}(x),\ F_{12}(x,u_0)=F_{12}(x).$$

Then collecting the coefficient of $u_1u_3^2$, equating it to zero, differentiating with respect to u_0, and equating the coefficient of $u_1u_2^3$ to zero, we have

$$16\frac{\partial^2 F_{11}}{\partial u_0^2}(x,u_0)+18\frac{\partial F_{03}}{\partial u_0}(x,u_0)=0,$$

and

$$15\frac{\partial^2 F_{11}}{\partial u_0^2}(x,u_0)+60\frac{\partial F_{03}}{\partial u_0}(x,u_0)=0,$$

which implies that we can obtain the following formula:

$$F_{11}(x,u_0)=F_{111}(x)u_0+F_{110}(x),\ \ F_{03}(x,u_0)=F_{03}(x).$$

Consider the above results. Equating the coefficient of $u_1u_2u_3$ to zero, differentiating with respect to u_0 and equating the coefficient of $u_1^2u_2^2$ to zero, we obtain

$$44\frac{\partial^2 F_{02}}{\partial u_0^2}(x,u_0)+16\frac{\partial^3 F_{10}}{\partial u_0^3}(x,u_0)=0,$$

$$39\frac{\partial^2 F_{02}}{\partial u_0^2}+6\frac{\partial^3 F_{10}}{\partial u_0^3}(x,u_0)=0,$$

which yields

$$\begin{aligned}&F_{10}(x,u_0)=F_{102}(x)u_0^2+F_{101}(x)u_0+F_{100}(x),\\ &F_{02}(x,u_0)=F_{021}(x)u_0+F_{020}(x).\end{aligned}$$

In view of the above results, equating the coefficient of $u_1^2u_3$ to zero, differentiating with respect to u_0 and collecting the coefficient of u_1^4, we obtain

$$\begin{aligned}&\frac{\partial^3 F_{01}}{\partial u_0^3}(x,u_0)=0,\\ &a_3(x)\frac{\partial^3 F_{01}}{\partial u_0^3}(x,u_0)+\frac{\partial^4 F_{00}}{\partial u_0^4}(x,u_0)\\ &+\frac{\partial^4 F_{01}}{\partial u_0^3\partial x}(x,u_0)=0,\end{aligned}$$

which leads to

$$\begin{aligned}F_{01}(x,u_0)\;=\;&F_{012}(x)u_0^2+F_{011}(x)u_0+F_{010}(x),\\ F_{00}(x,u_0)\;=\;&F_{003}(x)u_0^3+F_{002}(x)u_0^2\\ &+F_{001}(x)u_0+F_{000}(x).\end{aligned}$$

By the above computation, the theorem 3 follows.

4. Cubic operators admitting four-dimensional invariant subspaces

Theorem 2 implies that the second-order operator with cubic nonlinearities does not preserve invariant subspaces of five dimensions. In this section, we consider the cubic operators with constant coefficients

$$\begin{aligned}F[u]\equiv\;&b_1u_{xx}^3+b_2u_xu_{xx}^2+b_3uu_{xx}^2\\ &+b_4u_x^2u_{xx}+b_5uu_xu_{xx}\\ &+b_6u^2u_{xx}+b_7u_x^3+b_8uu_x^2+b_9u^2u_x+b_{10}u^3,\\ &b_i\in\mathbb{R}\ (i=1,\ldots,10),\end{aligned}\tag{12}$$

which admits the four-dimensional invariant subspaces defined by Eq. (6). We find that there are second-order cubic operators that admit four-dimensional invariant subspaces. Thus the maximal dimension of the invariant subspace admitted by cubic operators is four.

Firstly, we obtain the following lemma.

Lemma 2 If the subspace defined by linear ODE (6) is invariant under operator (12), then the coefficients of ODE (6) do not depend on x.

Proof If the subspace defined by linear ODE (6) is invariant under operator (12), then the left-hand side of the invariant condition (7) is a polynomial in variables u_3, u_2, u_1 and u_0. Equating the coefficients of u_3^3, $u_2u_3^2$, $u_1u_3^2$ and $u_0u_3^2$ to zero yields

$$-30b_1a_3(x)+12b_2=0, \tag{13}$$

$$-32b_2a_3(x)-54b_1a_2(x)-24b_1a_3'(x)+24b_1a_3^2(x)+20b_3+30b_4=0, \tag{14}$$

$$-14b_4a_3(x)-18b_3a_3(x)-8b_2a_3'(x)+8b_2a_3(x)^2-6b_2a_2(x)-36b_1a_1(x)+18b_7+16b_5=0, \tag{15}$$

$$-8b_3a_3'(x)-7b_5a_3(x)+8b_3a_3^2(x)-6b_3a_2(x)-36b_1a_0(x)+8b_6+6b_8=0. \tag{16}$$

Formula (13) implies that $a_3(x)$ does not depend on x. Assume that $a_3(x)=\tilde{a}_3$, where $\tilde{a}_3$ is a constant. Substituting it into Eq. (14) leads to $a_2(x)=\tilde{a}_2$, where $\tilde{a}_2$ is a constant. Similarly, from Eqs. (15) and (16), we obtain that $a_1(x)$ and $a_0(x)$ do not depend on x. The lemma is proved.

The next theorem provides a full description of the cubic operators with constant coefficients that admit the invariant subspaces defined by Eq. (6).

Theorem 4 Operators (12) possessing four-dimensional invariant subspaces defined by (6) with constant coefficients belong to the following list:

$$\begin{cases} F_1[u]=u_{xx}^3,\ F_2[u]=u_{xx}\left(uu_{xx}-\dfrac{2}{3}u_x^2\right), \\ W_4=\mathcal{L}\{1,x,x^2,x^3\}\ (L[y]=y^{(4)}=0), \end{cases} \tag{17}$$

$$\begin{cases} F_1[u]=(2u_{xx}+3u_x+u)(12u_{xx}^2+12u_xu_{xx}-6uu_{xx}-u_x^2-5uu_x+2u^2), \\ F_2[u]=(2u_{xx}+3u_x+u)(6uu_{xx}-4u_x^2+uu_x^2-u^2), \\ W_4=\mathcal{L}\left\{1,\exp\left(-\dfrac{x}{2}\right),\exp\left(\dfrac{x}{2}\right),\exp(-x)\right\} \\ \left(L[y]=y^{(4)}+y'''-\dfrac{1}{4}y''-\dfrac{1}{4}y'=0\right), \end{cases} \tag{18}$$

$$\begin{cases} F[u]=(6u_{xx}+7u_x+u)(3u_{xx}+2u_x-u)^2, \\ W_4=\mathcal{L}\left\{1,\exp\left(-\dfrac{x}{3}\right),\exp\left(\dfrac{x}{3}\right),\exp(-x)\right\} \\ \left(L[y]=y^{(4)}+y'''-\dfrac{1}{9}y''-\dfrac{1}{9}y'=0\right), \end{cases} \tag{19}$$

$$\begin{cases} F[u]=(6u_{xx}+5u_x+u)^3, \\ W_4=\mathcal{L}\left\{1,\exp\left(-\dfrac{x}{2}\right),\exp\left(-\dfrac{x}{3}\right),\exp\left(-\dfrac{x}{6}\right)\right\} \\ \left(L[y]=y^{(4)}+y'''+\dfrac{11}{36}y''+\dfrac{1}{36}y'=0\right), \end{cases} \tag{20}$$

$$\begin{cases} F_1[u]=(10u_{xx}+9u)(10u_{xx}^2-9uu_{xx}+18u_x^2), \\ F_2[u]=(10u_{xx}+9u)(30uu_{xx}-20u_x^2+9u^2), \\ W_4=\mathcal{L}\left\{\cos\left(\dfrac{\sqrt{10}}{10}x\right),\sin\left(\dfrac{\sqrt{10}}{10}x\right),\cos\left(\dfrac{3\sqrt{10}}{10}x\right),\sin\left(\dfrac{3\sqrt{10}}{10}x\right)\right\} \\ \left(L[y]=y^{(4)}+y''+\dfrac{9}{100}y=0\right), \end{cases} \tag{21}$$

$$\begin{cases} F_1[u]=(10u_{xx}-9u)(10u_{xx}^2+9uu_{xx}-18u_x^2), \\ F_2[u]=(10u_{xx}-9u)(30uu_{xx}-20u_x^2-9u^2), \\ W_4=\mathcal{L}\left\{\exp\left(\dfrac{\sqrt{10}}{10}x\right),\exp\left(-\dfrac{\sqrt{10}}{10}x\right),\exp\left(\dfrac{3\sqrt{10}}{10}x\right),\exp\left(-\dfrac{3\sqrt{10}}{10}x\right)\right\} \\ \left(L[y]=y^{(4)}-y''+\dfrac{9}{100}y=0\right), \end{cases} \tag{22}$$

$$\begin{cases} F[u]=(64u_{xx}+32u_x-5u)(64u_{xx}+64u_x^2+15u), \\ W_4=\mathcal{L}\left\{\exp\left(-\dfrac{5}{8}x\right),\exp\left(\dfrac{x}{8}\right),\exp\left(-\dfrac{x}{8}\right),\exp\left(-\dfrac{3}{8}x\right)\right\} \\ \left(L[y]=y^{(4)}+y'''+\dfrac{7}{32}y''-\dfrac{1}{64}y'-\dfrac{15}{4096}y=0\right). \end{cases} \tag{23}$$

Proof In view of lemma 2, we consider operator (12) admitting invariant subspaces defined by (6) with constant coefficients. In this case, the left-hand side of the invariant condition (7) is a polynomial in variables u_3, u_2, u_1 and u_0,

$$
\begin{aligned}
&(-30a_3b_1+12b_2)u_3^3+(30b_4-54a_2b_1+24b_1a_3^2+20b_3-32a_3b_2)u_2u_3^2\\
&+(18a_3b_3+8a_3^2b_2+16b_5-6a_2b_2-14a_3b_4+18b_7-36b_1a_1)u_1u_3^2\\
&+(-7a_3b_5+6b_8-6a_2b_3-36a_0b_1+8a_3^2b_3+8b_6)u_0u_3^2\\
&+(-6a_3b_3+42a_3b_1a_2-48a_2b_2-24b_1a_1+2a_3^2b_2-8a_3b_4+22b_5+36b_7)u_2^2u_3\\
&+(-52a_1b_2-24a_2b_4+14a_3b_2a_2+2a_3^2b_3+2a_3^2b_4-6a_3b_7-28a_2b_3+42b_1a_3a_1\\
&+44b_8-10a_3b_5+32b_6-24a_0b_1)u_1u_2u_3+(42b_1a_3a_0-8a_1b_3-6a_3b_6-12a_2b_5\\
&-2a_3b_8+14a_3b_3a_2+20b_9+a_3^2b_5-44b_2a_0)u_0u_2u_3\\
&+(-24a_1b_3-20a_1b_4-8b_2a_0-2a_3b_8+14a_3b_2a_1-6a_3b_6+a_3^2b_5-2a_2b_5+20b_9)u_1^2u_3\\
&+(-2a_3b_9-32b_3a_0-10a_1b_5+14a_3b_2a_0+24b_{10}+14a_3b_3a_1-20b_4a_0+2a_3^2b_6\\
&-4a_2b_6)u_0u_1u_3+(-10b_5a_0+14a_3b_3a_0)u_0^2u_3+(6b_6+18b_1a_2^2+2a_3b_2a_2+12b_8\\
&-2a_0b_1-12a_2b_3-18a_2b_4+3a_3b_5+6a_3b_7-8a_1b_2)u_2^3+(6a_3b_6-20a_2b_5+6b_2a_2^2\\
&+2a_3b_3a_2-20a_1b_3-18a_2b_7+12a_3b_8+2a_3b_2a_1-28a_1b_4+2a_3b_4a_2+36b_1a_2a_1\\
&-10b_2a_0+30b_9)u_1u_2^2+(36b_1a_2a_0+6a_3b_9-12a_2b_6-6a_2b_8-14b_3a_0+2a_3b_2a_0\\
&+a_3b_5a_2-20b_4a_0+6a_2^2b_3+18b_{10}-4a_1b_5)u_0u_2^2+(-4a_2b_8-26a_1b_5-8b_3a_0+36b_{10}\\
&-10a_2b_6-10b_4a_0-24a_1b_7+12a_3b_9+12b_2a_2a_1+18b_1a_1^2+a_3b_5a_2+2a_3b_3a_1\\
&+2b_4a_3a_1)u_1^2u_2+(a_3b_5a_1-4a_2b_9+2a_3b_6a_2+2a_3b_3a_0+2a_3b_4a_0+18a_3b_{10}-20a_1b_6\\
&+36b_1a_1a_0-8a_1b_8+12b_2a_2a_0+12a_2b_3a_1-24b_7a_0-28b_5a_0)u_0u_1u_2\\
&+(-8b_8a_0+12a_2b_3a_0+18b_1a_0^2+a_3b_5a_0-14a_0b_6)u_0^2u_2\\
&+(2a_2b_9-8a_1b_8+6b_2a_1^2+6a_3b_{10}+a_3b_5a_1-2b_7a_0-4b_5a_0-12a_1b_6)u_1^3\\
&+(2a_3b_6a_1+6b_3a_1^2+6a_2b_{10}-10b_8a_0+a_3b_5a_0-8a_1b_9+12b_2a_1a_0-20a_0b_6)u_0u_1^2\\
&+(12b_3a_1a_0-10a_0b_9+6b_2a_0^2+2a_3b_6a_0)u_0^2u_1+(6b_3a_0^2-2a_0b_{10})u_0^3.
\end{aligned}
$$

Equating all the coefficients to zero, we obtain an algebraic system for $a_i, b_j,\ (i=0,\ldots,3, j=1,\ldots,10)$. Solving the system, we obtain the following five solutions:

$$
\begin{aligned}
&a_0=0,\ a_1=-\frac{1}{4}a_3^3,\ a_2=-\frac{1}{4}a_3^2,\ b_2=\frac{5}{2}a_3b_1,\ b_4=\frac{17}{12}b_1a_3^2-\frac{2}{3}b_3,\\
&b_5=\frac{1}{3}(5a_3b_3-2a_3^3b_1),\ b_6=-\frac{1}{12}a_3^4b_1+\frac{1}{3}a_3^2b_3,\ b_7=-a_3b_3-\frac{1}{8}a_3^3b_1,\\
&b_8=-\frac{1}{12}a_3^2b_3-\frac{2}{3}a_3^4b_1,\ b_9=\frac{1}{24}a_3^5b_1-\frac{1}{6}a_3^3b_3,\ b_{10}=\frac{1}{12}(a_3^6b_1-a_3^4b_3),
\end{aligned}
\tag{24}
$$

with arbitrary constants a_3, b_1 and b_3;

$$
\begin{aligned}
&a_0=0,\ a_1=-\frac{1}{9}a_3^3,\ a_2=-\frac{1}{9}a_3^2,\ b_2=\frac{5}{2}a_3b_1,\ b_3=-\frac{1}{2}b_1a_3^2,\\
&b_4=2b_1a_3^2,\ b_5=-a_3^3b_1,\ b_6=0,\ b_7=\frac{14}{27}a_3^3b_1,\ b_8=-\frac{4}{9}a_3^4b_1,\ \ b_9=\frac{1}{18}a_3^5b_1,\ b_{10}=\frac{1}{54}a_3^6b_1,
\end{aligned}
\tag{25}
$$

with arbitrary constants a_3 and b_1;

$$
\begin{aligned}
&a_0=0,\ a_1=\frac{1}{36}a_3^3,\ a_2=\frac{11}{36}a_3^2,\ b_2=\frac{5}{2}a_3b_1,\ b_3=\frac{1}{2}b_1a_3^2,\ b_4=\frac{25}{12}b_1a_3^2,\\
&b_5=\frac{5}{6}a_3^3b_1,\ b_6=\frac{1}{12}a_3^4b_1,\ b_7=\frac{125}{216}a_3^3b_1,\ b_8=\frac{25}{72}a_3^4b_1,\ b_9=\frac{5}{72}a_3^5b_1,\ \ b_{10}=\frac{1}{216}a_3^6b_1,
\end{aligned}
\tag{26}
$$

with arbitrary constants a_3 and b_1 and b_3;

$$
a_0=\frac{9}{100}a_2^2,\ a_1=a_3=b_2=b_5=b_7=b_9=0,\ b_4=\frac{9}{5}a_2b_1-\frac{2}{3}b_3,
$$

$$b_6 = -\frac{81}{100}b_1a_2^2 + \frac{6}{5}a_2b_3,\ b_8 = -\frac{3}{5}a_2b_3 + \frac{81}{50}b_1a_2^2,\ b_{10} = \frac{27}{100}a_2^2b_3, \tag{27}$$

with arbitrary constants a_2, b_1 and b_3;

$$\begin{aligned} &a_0 = -\frac{15}{4096}a_3^4,\ a_1 = -\frac{1}{64}a_3^3,\ a_2 = \frac{7}{32}a_3^2,\ b_2 = \frac{5}{2}a_3b_1,\ b_3 = \frac{25}{64}b_1a_3^2,\\ &b_4 = 2b_1a_3^2,\ b_5 = \frac{35}{64}a_3^3b_1,\ b6 = \frac{75}{4096}a_3^4b_1,\ b_7 = \frac{1}{2}a_3^3b_1,\ b8 = \frac{5}{32}a_3^4b_1,\\ &b_9 = -\frac{75}{8192}a_3^5b_1,\ b_{10} = -\frac{1125}{262144}a_3^6b_1, \end{aligned} \tag{28}$$

with arbitrary constants a_3 and b_1.

Substituting the first solution (24) into Eqs. (12) and (6), we obtain

$$F[u] = b_1F_1[u] + b_3F_2[u], \quad L[y] \equiv y^{(4)} + a_3y''' - \frac{a_3^2}{4}y'' - \frac{a_3^3}{4}y' = 0, \tag{29}$$

where

$$\begin{aligned} F_1[u] &\equiv \frac{1}{24}(2u_{xx} + 3a_3u_x + a_3^2u)(12u_{xx}^2 + 12a_3u_xu_{xx} - 6a_3^2uu_{xx} - a_3^2u_x^2 - 5a_3^3uu_x + 2a_3^4u^2),\\ F_2[u] &\equiv \frac{1}{12}(2u_{xx} + 3a_3u_x + a_3^2u)(6uu_{xx} - 4u_x^2 + a_3uu_x - a_3^2u^2), \end{aligned} \tag{30}$$

b_1 and b_3 are arbitrary constants. Therefore there exist two operators $F_1[u]$ and $F_2[u]$ admitting the invariant subspaces defined by ODE (29). For $a_3 = 0$, $F_1[u] = u_{xx}^3$ and $F_2[u] = uu_{xx}^2 - (2/3)u_x^2u_{xx}$, which admit the invariant subspace $W_4 = \mathcal{L}\{1, x, x^2, x^3\}$ ($L[y] \equiv y^{(4)} = 0$). Then we obtain result (17) in theorem 4. For $a_3 \neq 0$, performing the variable transformation $X = a_3x$ in Eqs. (29) and (30) yields the second result (18) in theorem 4. Without loss of generality, we denote by u_x the u_X in Eqs. (17)–(23). By a similar computation to the above, inserting the solutions (25)–(28) into Eqs. (6) and (12) yields the other results (19)–(23) respectively. Thus the theorem 4 is proved.

Example 1 Theorem 4 implies that the equation

$$u_t = u_{xx}^3 + \frac{9}{5}u_x^2u_{xx} - \frac{81}{100}u^2u_{xx} + \frac{81}{50}uu_x^2 \tag{31}$$

has an exact solution of the form

$$\begin{aligned} u =\ & c_1(t)\cos\left(\frac{\sqrt{10}}{10}x\right) + c_2(t)\sin\left(\frac{\sqrt{10}}{10}x\right)\\ &+ c_3(t)\cos\left(\frac{3\sqrt{10}}{10}\right)x\\ &+ c_4(t)\sin\left(\frac{3\sqrt{10}}{10}x\right). \end{aligned} \tag{32}$$

Substituting Eq. (32) into Eqs. (31), then Eq. (31) is reduced into the following system of ODEs:

$$\begin{aligned} c_1'(t) =\ & \frac{54}{125}c_3(c_1^2 - c_2^2) + \frac{12}{125}c_1(c_1^2 + c_2^2)\\ &+ \frac{162}{125}c_1(c_3^2 + c_4^2) + \frac{108}{125}c_1c_2c_4,\\ c_2'(t) =\ & \frac{12}{125}c_2(c_1^2 + c_2^2) + \frac{54}{125}c_4(c_1^2 - c_2^2)\\ &+ \frac{162}{125}c_2(c_3^2 + c_4^2) - \frac{108}{125}c_1c_2c_3,\\ c_3'(t) =\ & \frac{54}{125}c_3(c_1^2 + c_2^2) + \frac{2}{125}c_1(3c_2^2 - c_1^2),\\ c_4'(t) =\ & \frac{54}{125}c_4(c_1^2 + c_2^2) + \frac{2}{125}c_2(c_2^2 - 3c_1^2). \end{aligned}$$

For $c_2(t) = c_4(t) = 0$ and $c_1(t) = 3c_3(t)$, solving the system yields the $2\sqrt{10}\pi$-periodic separate variables similarity solution of Eq. (31), i.e.,

$$u(x,t) = \frac{100}{\sqrt{T - 4320t}}\cos\left(\frac{\sqrt{10}}{10}x\right)^3, \tag{33}$$

where T is an arbitrary positive constant. Noting that solution (33) is equal to zero at the points $x_k = \sqrt{10}(2k+1)\pi/2$, we take the one-hump

$$\bar{u}(x,t) = \begin{cases} \dfrac{100}{\sqrt{T - 4320t}}\cos\left(\dfrac{\sqrt{10}}{10}x\right)^3, & \text{for } |x| \le \dfrac{\sqrt{10}}{2}\pi,\\ 0, & \text{for } |x| > \dfrac{\sqrt{10}}{2}\pi, \end{cases} \tag{34}$$

which is a localized standing wave (a kind of blow-up standing compacton). It exhibits regional blow-up in the interval $(-\sqrt{10}\pi/2, \sqrt{10}\pi/2)$. The compactly supported function $\bar{u}(x,t)$ is a standard weak solution of the Cauchy problem for Eq. (31) in $\mathbb{R} \times (0, T/4320)$ with corresponding initial data $\bar{u}(x,0)$.

Similarly, we can obtain the following examples.

Example 2 The dual porous medium equation[20]

$$u_t = u_{xx}^3$$

has a solution

$$u(x,t) = \frac{(C-3x)^3}{27\sqrt{T-432t}} \in W_4 = \mathcal{L}\{1, x, x^2, x^3\}, (35)$$

where C and T are arbitrary positive constants. Solution (35) is positive for $x < C/3$ and blows up at the time $t = T/432$.

Example 3 The full nonlinear equation

$$u_t = \frac{1}{216}(6u_{xx} + 5u_x + u)^3$$

has a positive solution

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{T-3t}}\left[\frac{C^3}{8748}\exp\left(-\frac{x}{2}\right) + \frac{C^2}{54}\right. \\ &\quad \left. \times \exp\left(-\frac{x}{3}\right) + C\exp\left(-\frac{x}{6}\right) + 18\right] \\ &\in W_4 = \mathcal{L}\left\{\exp\left(-\frac{x}{2}\right), \exp\left(-\frac{x}{3}\right), \exp\left(-\frac{x}{6}\right), 1\right\}, \end{aligned}$$

where C and T are arbitrary positive constants. This solution blows up at the time $t = T/3$.

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