# Elastic properties of spherically anisotropic piezoelectric composites*

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Effective elastic properties of spherically anisotropic piezoelectric composites, whose spherically anisotropic piezoelectric inclusions are embedded in an infinite non-piezoelectric matrix, are theoretically investigated. Analytical solutions for the elastic displacements and the electric potentials under a uniform external strain are derived exactly. Taking into account of the coupling effects of elasticity, permittivity and piezoelectricity, the formula is derived for estimating the effective elastic properties based on the average field theory in the dilute limit. An elastic response mechanism is revealed, in which the effective elastic properties increase as inclusion piezoelectric properties increase and inclusion dielectric properties decrease. Moreover, a piezoelectric response mechanism, of which the effective piezoelectric response vanishes due to the symmetry of spherically anisotropic composite, is also disclosed.

**Keywords:** elasticity, piezoelectric composites, spherically anisotropy
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## 1. Introduction

Effective elastic properties of composites have attracted much attention due to their technological importance.[1−3] Because the elastic properties of composites are related to their constituents, microstructures, physical properties and external fields, such as the elastic, piezoelectric and thermoelastic composites, it is hard to find an analytical method of calculating the effective elastic properties. Theoretically, it becomes a major research field in material science to predict and analyse the composite elasticity. For pure elastic composites, many theoretical methods were developed for investigating the effective elastic properties. For example, based on the Eshelby solutions for a composite having an isolated ellipsoidal particle embedded in an infinite medium,[4] self-consistent methods,[5,6] the Mori–Tanaka model,[7] the Bruggeman method,[8] the interaction direct derivative method and the phase–field microelasticity method were proposed under different effective medium approximations and homogenization processes.[9−11] Moreover, to deal with the problem of complex inclusion structure composites, a transformation field method was developed for estimating the elastic properties of periodic composites.[12] Recently, the perturbation expansion method for the effective stiffness tensors and Causius–Mossotti-type approximations for the effective elastic modulus of cubic array of spheres were also presented.[13−15] Although these investigations have solved some problems of pure elastic composites, they are not suitable for directly estimating the effective elastic responses of coupled composites, such as the piezoelectric and thermoelastic composites, due to the coupling effects of other physical properties. Piezoelectric composites are abundant in nature, from the tissue of live-forms to the piezoelectric ceramics (and some crystals).[16] Because piezoelectric material can induce electric polarization under external mechanical stress and mechanical deformation under external electric field, they have wide applications in ultrasonic transducers, underwater acoustics, biologics, etc. Thus, theoretical and experimental investigations of the effects of coupling phenomena between elastic, dielectric and piezoelectric properties of the constituents on the effec-

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tive properties become very important. Recently, the effective responses of piezoelectric composites were investigated with various methods for different research interests.[17,18] With the effective-medium approximations, explicit formulas were obtained for the effects of electric conductivity on the effective piezoelectric coefficients of 0–3 composites.[19,20] For fibrous piezoelectric composites, universal relations between effective response and local field were obtained by means of virtual work theorems.[21,22] Green's functions and Eshelby's tensor methods were also developed to investigate the effective properties of piezoelectric composites.[23,24] For random piezoelectric periodic composites having arbitrary inclusion shapes, a transformation field method was developed to estimate the effective elastic tensors, dielectric and piezoelectric constants.[25] In the present paper, in order to further investigate the bulk effective elastic responses of piezoelectric composites analytically, we attempt to derive analytical solutions for a spherically anisotropic piezoelectric composite consisting of a piezoelectric spherical inclusion embedded in an infinite non-piezoelectric matrix, under the action of a uniform external strain.

Many piezoelectric materials, for example, piezoceramics, exhibit transverse isotropy after poling, with the unique axis aligned along the poling direction. As a special case, if a ceramic spherical inclusion is poled along its radial direction, it will possess a special kind of transverse isotropy on the spherical surface, which is called spherically anisotropy. For a transversely isotropic piezoelectric inclusion sphere embedded in an infinite non-piezoelectric matrix, analytical solution has been obtained based on the Green's function.[26] For spherically anisotropic piezoelectric composites, the effective piezoelectric and dielectric coefficients have been derived recently.[27] However, the effective elastic response problem for this kind of spherically anisotropic piezoelectric composite is not yet solved, although the vibration property was studied.[28,29] Therefore, according to our previous work, for solving the effective elastic response of radially polarized cylindrical piezoelectric composite,[30] it is worth investigating and discussing the effective elastic responses of this kind of spherically anisotropic piezoelectric composite and disclosing its coupling response mechanism. In this theoretical research, our aim is to derive analytical solutions of the local elastic displacements and the electric potentials in the inclusion and the matrix regions and give out formulas for the effective elastic coefficients. In addition, we also discuss, through numerical calculations, the effective elastic response mechanism in the dilute inclusion limit based on our analytical solutions.

## 2. Analytical solutions

Consider an infinite isotropic non-piezoelectric matrix, which is assumed to be elastically and dielectrically isotropic, containing a spherically anisotropic piezoelectric particle, with its centre acting as the origin of a spherical coordinate system. The linear constitutive equations in inclusion $\Omega_{\rm i}$ and host $\Omega_{\rm h}$ regions are, respectively,

$$\begin{aligned} &\sigma_{ij}^{\rm i} = c_{ijkl}^{\rm i}\gamma_{kl}^{\rm i} - e_{kij}^{\rm i}E_k^{\rm i}, \\ &D_i^{\rm i} = e_{ikl}^{\rm i}\gamma_{kl}^{\rm i} + \varepsilon_{ik}^{\rm i}E_k^{\rm i}, \quad \text{in} \quad \Omega_{\rm i}, \end{aligned} \tag{1}$$

$$\sigma_{ij}^{\rm h} = c_{ijkl}^{\rm h}\gamma_{kl}^{\rm h}, \quad D_i^{\rm h} = \varepsilon_{ik}^{\rm h}E_k^{\rm h}, \quad \text{in} \quad \Omega_{\rm h}, \tag{2}$$

where subscripts $i, j, k, l = 1, 2, 3$ denote the $\theta$, $\varphi$, $r$ directions, respectively, and superscripts i and h denote the quantities of the inclusion and the host regions, respectively. $\sigma$, $\gamma$, $D$ and $E$ are the stress, the strain, the electric displacement and the electric field, respectively, and $c$, $e$, $\varepsilon$ are the elastic stiffness, the piezoelectric coefficient, and the dielectric constant, respectively. In the absence of both body force and free electric charge, the governing equations for the inclusion and the host regions are $\sigma_{ij,j}^{\alpha} = 0$ and $D_{i,i}^{\alpha} = 0$ ($\alpha$=i, h), respectively. The boundary conditions at the interface between inclusion and matrix are the continuity of the elastic displacement, the electric potential, the normal traction, and the electric displacement. In spherical coordinates, the governing equations are

$$\begin{aligned} &\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\sigma_{\varphi r}}{\partial\varphi} + \frac{2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\varphi\varphi} + \sigma_{r\theta}\cot\theta}{r} = 0, \\ &\frac{\partial\sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\sigma_{\theta\varphi}}{\partial\varphi} + \frac{(\sigma_{\theta\theta} - \sigma_{\varphi\varphi})\cot\theta + 3\sigma_{r\theta}}{r} = 0, \\ &\frac{\partial\sigma_{\varphi r}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\varphi}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\sigma_{\varphi\varphi}}{\partial\varphi} + \frac{2\sigma_{\theta\varphi}\cot\theta + 3\sigma_{\varphi r}}{r} = 0, \end{aligned}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 D_r\right)+\frac{1}{r\sin\theta}\frac{\partial\left(\sin\theta D_\theta\right)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial D_\varphi}{\partial\varphi}=0. \tag{3}$$

For a spherical particle with a radial polarization, which is termed a spherically anisotropic piezoelectric particle, the constitutive relations inside the particle are given in spherical coordinates as follows:[16]

$$\begin{aligned}
&\sigma_{\theta\theta}=c_{11}\gamma_{\theta\theta}+c_{12}\gamma_{\varphi\varphi}+c_{13}\gamma_{rr}-e_{31}E_r,\\
&\sigma_{\varphi\varphi}=c_{12}\gamma_{\theta\theta}+c_{11}\gamma_{\varphi\varphi}+c_{13}\gamma_{rr}-e_{31}E_r,\\
&\sigma_{rr}=c_{13}\gamma_{\theta\theta}+c_{13}\gamma_{\varphi\varphi}+c_{33}\gamma_{rr}-e_{33}E_r,\\
&\sigma_{\varphi r}=2c_{44}\gamma_{\varphi r}-e_{15}E_\varphi,\\
&\sigma_{r\theta}=2c_{44}\gamma_{r\theta}-e_{15}E_\theta,\\
&\sigma_{\theta\varphi}=(c_{11}-c_{12})\gamma_{\varphi\theta},\\
&D_\theta=2e_{15}\gamma_{r\theta}+\varepsilon_{11}E_\theta,\\
&D_\varphi=2e_{15}\gamma_{\varphi r}+\varepsilon_{11}E_\varphi,\\
&D_r=e_{31}\gamma_{\theta\theta}+e_{31}\gamma_{\varphi\varphi}+e_{33}\gamma_{rr}+\varepsilon_{33}E_r.
\end{aligned} \tag{4}$$

In this section, we have omitted superscript i (inclusion) from the above equations for convenience. Subscripts $i,j$ in coefficients $c_{ij}$ and $e_{ij}$ are replacement indices according to Nye's rule. Because the matrix is a non-piezoelectric and isotropic material, the constitutive relations in the host region can be obtained from Eq.(3) by omitting the piezoelectric coefficients and letting $c_{ijkl}=\lambda\delta_{ij}\delta_{kl}+\mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})$ and $\varepsilon_{11}=\varepsilon_{22}=\varepsilon_{33}$, where $\lambda$ and $\mu$ are Lame constants and $\delta_{ij}$ is the Kronecker delta. Strain tensor $\gamma$ are expressed by the elastic displacements as follows:

$$\begin{aligned}
&\gamma_{rr}=\frac{\partial u_r}{\partial r},\\
&\gamma_{\theta\theta}=\frac{1}{r}\left[\frac{\partial u_\theta}{\partial\theta}+u_r\right],\\
&\gamma_{\varphi\varphi}=\frac{u_r}{r}+\frac{u_\theta\cot\theta}{r}+\frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial\varphi},\\
&\gamma_{r\theta}=\frac{1}{2}\left[\frac{1}{r}\frac{\partial u_r}{\partial\theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right],\\
&\gamma_{\theta\varphi}=\frac{1}{2}\left[\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\varphi}+\frac{1}{r}\frac{\partial u_\varphi}{\partial\theta}-\frac{u_\varphi\cot\theta}{r}\right],\\
&\gamma_{\varphi r}=\frac{1}{2}\left[\frac{\partial u_\varphi}{\partial r}-\frac{u_\varphi}{r}+\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\varphi}\right].
\end{aligned} \tag{5}$$

In the following, we will derive analytical solutions for the elastic displacement and the electric potentials of the composite under an external uniform strain acting along the $\hat{z}$-direction.

In order to solve the governing equations in the inclusion region, we introduce three unknown functions $\psi(\theta,\varphi,r)$, $G(\theta,\varphi,r)$ and $W(\theta,\varphi,r)$ for the displacement components,[28,29] i.e.

$$\begin{aligned}
&u_\theta=-\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\varphi}-\frac{\partial G}{\partial\theta},\\
&u_\varphi=\frac{\partial\psi}{\partial\theta}-\frac{1}{\sin\theta}\frac{\partial G}{\partial\varphi},\quad\text{and}\quad u_r=W.
\end{aligned} \tag{6}$$

Substituting Eq. (6) into Eq.(4), it is easy to obtain four partial differential equations about $\psi(\theta,\varphi,r)$, $G(\theta,\varphi,r)$, $W(\theta,\varphi,r)$ and the unknown electric potential function $\Phi(\theta,\varphi,r)$. Furthermore, the general solutions of these four unknown functions $\psi$, $G$, $W$ and $\Phi$ are assumed to take the following forms:

$$\begin{aligned}
&\psi=\sum_{n=0}^{\infty}\psi_n(r)Y_n^m(\theta,\varphi),\\
&G=\sum_{n=0}^{\infty}G_n(r)Y_n^m(\theta,\varphi),\\
&W=\sum_{n=0}^{\infty}W_n(r)Y_n^m(\theta,\varphi),\\
&\Phi=\frac{e_{33}^0}{\varepsilon_{33}^0}\sum_{n=0}^{\infty}\Phi_n(r)Y_n^m(\theta,\varphi),
\end{aligned} \tag{7}$$

where $Y_n^m(\theta,\varphi)$ are spherical harmonics, and $\psi_n(r)$, $G_n(r)$, $W_n(r)$ and $\Phi_n(r)$ are unknown functions. Substituting Eqs. (7) and (6) into Eq.(3), we have a set of linear differential equations for the unknown functions $\psi_n(r)$, $G_n(r)$, $W_n(r)$ and $\Phi_n(r)$ as follows:

$$\begin{aligned}
&-f_1W_n-f_2r\frac{\partial W_n}{\partial r}-r^2\frac{\partial^2W_n}{\partial r^2}+f_3r\frac{\partial\Phi_n}{\partial r}+r^2\frac{\partial^2\Phi_n}{\partial r^2}-f_4\Phi_n-f_5\frac{r\partial G_n}{\partial r}-f_6G_n=0,\\
&-f_7\psi_n+f_3r\frac{\partial\psi_n}{\partial r}+r^2\frac{\partial^2\psi_n}{\partial r^2}=0,\\
&-f_8G_n+f_3r\frac{\partial G_n}{\partial r}+r^2\frac{\partial^2G_n}{\partial r^2}-f_9r\frac{\partial W_n}{\partial r}-f_{10}W_n-f_{11}r\frac{\partial\Phi_n}{\partial r}-f_{12}\Phi_n=0,\\
&f_{13}r\frac{\partial\Phi_n}{\partial r}+f_{14}r^2\frac{\partial^2\Phi_n}{\partial r^2}-f_{15}\Phi_n+f_{16}W_n+f_3r\frac{\partial W_n}{\partial r}+r^2\frac{\partial^2W_n}{\partial r^2}+f_{17}G_n+f_{18}r\frac{\partial G_n}{\partial r}=0,
\end{aligned} \tag{8}$$

where

$$
\begin{aligned}
&f_1 = [2e_{31} - n(n+1)e_{15}]/e_{33},\\
&f_2 = 2 + 2e_{31}/e_{33}, \quad f_3 = 2,\\
&f_4 = n(n+1)\varepsilon_{11}/\varepsilon_{33},\\
&f_5 = n(n+1)(e_{31} + e_{15})/e_{33},\\
&f_6 = n(n+1)[e_{31} - e_{15}]/e_{33},\\
&f_7 = 2 + (c_{11}/c_{44} - c_{12}/c_{44})(n^2 + n - 2)/2,\\
&f_8 = n(n+1)c_{11}/c_{44} + 2 - c_{11}/c_{44} + c_{12}/c_{44},\\
&f_9 = 1 + c_{13}/c_{44},\\
&f_{10} = 2 + (c_{12} + c_{11})/c_{44},\\
&f_{11} = e_{33}e_{15}(1 + e_{31}/e_{15})/(\varepsilon_{33}c_{44}),\\
&f_{12} = 2e_{33}e_{15}/(\varepsilon_{33}c_{44}),\\
&f_{13} = 2e_{33}e_{33}(1 - e_{31}/e_{33})/(\varepsilon_{33}c_{33}),\\
&f_{14} = e_{33}e_{33}/(\varepsilon_{33}c_{33}),\\
&f_{15} = n(n+1)e_{33}e_{15}/(\varepsilon_{33}c_{33}),\\
&f_{16} = [2(c_{13} - c_{11} - c_{12}) - n(n+1)c_{44}]/c_{33},\\
&f_{17} = n(n+1)(c_{13} - c_{11} - c_{12} - c_{44})/c_{33},\\
&f_{18} = n(n+1)(c_{44} + c_{13})/c_{33}.
\end{aligned}
$$

We assume that the unknown functions $\psi_n(r)$, $G_n(r)$, $W_n(r)$ and $\Phi_n(r)$ can be expressed in the following forms:

$$
\begin{aligned}
&G_n(r) = A_n r^{\nu_n}, \quad W_n(r) = B_n r^{\nu_n},\\
&\Phi_n(r) = C_n r^{\nu_n}, \quad \psi_n(r) = D_n r^{\lambda_n},
\end{aligned} \tag{9}
$$

where $A_n$, $B_n$, $C_n$, $D_n$ and $\nu_n$ are unknown constants to be determined. Substituting Eq. (9) into Eq. (8), we obtain a set of linear equations

$$
F_{ij}X_j = 0, \tag{10}
$$

where

$$
\begin{aligned}
&X_j = (A_n, B_n, C_n)^{\mathrm{T}}, \quad F_{11} = -f_5\nu_n - f_6,\\
&F_{12} = -f_2\nu_n - f_1 - \nu_n(\nu_n - 1),\\
&F_{13} = f_3\nu_n - f_4 + \nu_n(\nu_n - 1),\\
&F_{21} = f_3\nu_n - f_8 + \nu_n(\nu_n - 1),\\
&F_{22} = -f_9\nu_n - f_{10},\\
&F_{23} = -f_{11}\nu_n - f_{12}, \quad F_{31} = f_{18}\nu_n + f_{17},\\
&F_{32} = f_3\nu_n + f_{16} + \nu_n(\nu_n - 1),\\
&F_{33} = f_{13}\nu_n - f_{15} + f_{14}\nu_n(\nu_n - 1).
\end{aligned}
$$

For Eq. (10) to have non-zero solutions, eigenvalues $\nu_n$ should be determined from eigen-equation $|F_{ij}| = 0$. Also, for a stable material, eigenvalues $\nu_n$ should have non-negative real part solutions, and the valid solutions of $\nu_n$ inside the spherical inclusion, namely $\nu_{ni}$, must have $\mathrm{Re}(\nu_{ni}) \geq 0$ $(i = 1, 2, 3)$. Thus, the following relationships between $A_n$, $B_n$ and $C_n$ are obtained from Eq. (10):

$$
B_{ni} = K^1_{ni}A_{ni}, \quad C_{ni} = K^2_{ni}A_{ni}, \quad (i = 1, 2, 3), \tag{11}
$$

where the matrices $K^1_{ni}$ and $K^2_{ni}$ are obtained from Eq. (10), corresponding to eigenvalue $\nu_{ni}$. It is noted that eigenvalue $\lambda_n$ can be determined from equation $-f_7 + f_3\lambda_n + \lambda_n(\lambda_n - 1) = 0$ for non-zero $D_n$. However, with the present boundary conditions, we can demonstrate that eigenvalue $\lambda_n$ and coefficient $D_n$ should have no effects on the stress and the electric fields in the composite under an external strain, i.e. coefficients $D_n$ should be zero. In addition, we should note from governing Eq. (3) that Eq. (10) will reduce to Eq. (12) if parameter $n = 0$.

$$
\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} B_0 \\ C_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{12}
$$

where $a_1 = 2e_{31}/e_{33} + 2(1 + e_{31}/e_{33})\nu_0 + \nu_0(\nu_0 - 1)$, $a_2 = -\nu_0(\nu_0 + 1)$, $a_3 = 2(c_{13} - c_{11} - c_{12})/c_{33} + \nu_0(\nu_0 + 1)$, $a_4 = 2e_{33}(e_{33} - e_{31})\nu_0/(\varepsilon_{33}c_{33}) + e_{33}e_{33}\nu_0(\nu_0 - 1)/(\varepsilon_{33}c_{33})$, and $\nu_0$ is the eigenvalue of Eq. (12) for non-zero coefficients $B_0$ and $C_0$. Then, once eigenvalue $\nu_0$ is found, relationship $B_0 = T(\nu_0)C_0$ can be obtained from Eq. (12). For the two eigenvalues $\nu_{01}$ and $\nu_{02}$ of the inclusion region solutions, we have then two unknown constants $C_{01}$ and $C_{02}$ in the solutions, $W_0(r) = B_{01}r^{\nu_{01}} + B_{02}r^{\nu_{02}}$ and $\Phi_0(r) = C_{01}r^{\nu_{01}} + C_{02}r^{\nu_{02}}$, where

$$
B_{0i} = -\frac{a_2(v_{0i})}{a_1(v_{0i})}C_{0i}, \quad (i = 1, 2).
$$

If external strain $T_0$ is applied along the $\hat{z}$-direction, the solutions should be independent of variable $\varphi$ because of the axial symmetry of the spherical inclusion. Inside the inclusion region, the elastic displacement and the electric potentials can be expressed in terms of Legendre polynomials $P_n(\cos\theta)$ as follows:

$$
\begin{aligned}
&\Phi = \tfrac{e_{33}}{\varepsilon_{33}}\Phi_0(r) + \tfrac{e_{33}}{\varepsilon_{33}}\sum_{n=1}^{\infty}\sum_{i=1}^{3} K^2_{ni}A_{ni}r^{\nu_{ni}}P_n(\cos\theta),\\
&u_\theta = -\sum_{n=1}^{\infty}\sum_{i=1}^{3} A_{ni}r^{\nu_{ni}}\frac{\partial}{\partial\theta}P_n(\cos\theta),\\
&u_\varphi = \sum_{n=1}^{\infty} D_n r^{\lambda_n}\frac{\partial}{\partial\theta}P_n(\cos\theta),\\
&u_r = W_0(r) + \sum_{n=1}^{\infty}\sum_{i=1}^{3} K^1_{ni}A_{ni}r^{\nu_{ni}}P_n(\cos\theta),
\end{aligned} \tag{13}
$$

where unknown coefficients $A_{ni}$ $(i = 1, 2, 3)$, $C_{01}$ and $C_{02}$ should be determined by the boundary conditions.

For external uniform strain $T_0$ applied along the $\hat{z}$-direction, the general solutions of elastic displacements, $u_r^{\rm h}$, $u_\theta^{\rm h}$ and $u_\varphi^{\rm h}$, and electric potential $\Phi^{\rm h}$ in the isotropic host region can be derived from pure isotropic elastic and dielectric governing Eq. (3) with infinite boundary conditions $u_z = T_0 z$ and $u_r = rT_0[\frac{1}{3} + \frac{2}{3}P_2(\cos\theta)]$, respectively.

$$
\begin{aligned}
u_r^{\rm h} &= \left[ -\frac{B_1}{r^2} - \frac{1}{3}\left(\frac{5-4\nu^{\rm h}}{1-2\nu^{\rm h}}\frac{B_3}{r^2}\right) + u_r^0(0)r \right] \\
&\quad + \left[\frac{4}{3}\left(\frac{5-4\nu^{\rm h}}{1-2\nu^{\rm h}}\frac{B_3}{r^2} - 9\frac{B_2}{r^4}\right) + u_r^0(2)r\right] P_2(\cos\theta), \\
u_\theta^{\rm h} &= \left[\frac{4}{3}\left(\frac{B_3}{r^2} + 3\frac{B_2}{r^4}\right) + u_\theta^0(2)r\right] \partial P_2(\cos\theta)/\partial\theta, \\
u_\varphi^{\rm h} &= 0, \qquad \Phi^{\rm h} = \frac{B_4}{r} + \frac{4}{3}\frac{B_5}{r^3}P_2(\cos\theta), \qquad (14)
\end{aligned}
$$

where $u_r^0(0) = \frac{1}{3}T_0$, $u_r^0(2) = \frac{2}{3}T_0$ and $u_\theta^0(2) = \frac{1}{3}T_0$; $\nu^{\rm h} = \lambda^{\rm h}/2(\mu^{\rm h} + \lambda^{\rm h})$ is the Poisson ratio, and $\lambda^{\rm h}$ and $\mu^{\rm h}$ are Lame constants of the elastic isotropic host material; unknown coefficients $B_i$ $(i = 1, 2, 3, 4, 5)$ each can be determined by the boundary condition at the interface between spherical inclusion and the host material.

Based on the above solutions of the elastic displacements and the electric potentials, the stress tensors and the electric displacements can be expressed in terms of the unknown coefficients in Eq. (4). In the following, we use the inclusion interface boundary conditions to find unknown coefficients $A_{ni}$ $(i = 1, 2, 3)$, $B_{\rm i}$ $(i = 1, 2, 3, 4, 5)$, $C_{01}$ and $C_{02}$. The continuity boundary conditions for a spherical inclusion with radius $a$ are as follows:

$$
\begin{aligned}
&u_r^{\rm i}(r) = u_r^{\rm h}(r)\big|_{r=a}, \quad u_\theta^{\rm i}(r) = u_\theta^{\rm h}(r)\big|_{r=a}, \quad u_\varphi^{\rm i}(r) = u_\varphi^{\rm h}(r)\big|_{r=a}, \quad \sigma_{rr}^{\rm i}(r) = \sigma_{rr}^{\rm h}(r)\big|_{r=a}, \\
&\sigma_{r\theta}^{\rm i}(r) = \sigma_{r\theta}^{\rm h}(r)\big|_{r=a}, \quad \sigma_{r\varphi}^{\rm i}(r) = \sigma_{r\varphi}^{\rm h}(r)\big|_{r=a}, \quad \Phi^{\rm i}(r) = \Phi^{\rm h}(r)\big|_{r=a}, \quad D_r^{\rm i}(r) = D_r^{\rm h}(r)\big|_{r=a}.
\end{aligned}
$$

Here, note that the two boundary conditions about $\varphi$-direction $u_\varphi^{\rm i}(r) = u_\varphi^{\rm h}(r)\big|_{r=a}$ and $\sigma_{r\varphi}^{\rm i}(r) = \sigma_{r\varphi}^{\rm h}(r)\big|_{r=a}$ are satisfied automatically. A set of closed-form equations for the ten unknown coefficients is obtained as follows:

$$
\begin{aligned}
&C_{01}a^{\nu_{01}} + C_{02}a^{\nu_{02}} - B_4 a^{-1}\varepsilon_{33}/e_{33} = 0, \\
&\sum_{i=1}^{3}(K_{2i}^2 A_{2i}a^{\nu_{2i}}) - \frac{4}{3}a^{-3}B_5\varepsilon_{33}/e_{33} = 0, \\
&\sum_{i=1}^{3} A_{2i}a^{\nu_{2i}} + 4a^{-4}B_2 + \frac{4}{3}a^{-2}B_3 = -\frac{1}{3}aT_0, \\
&\frac{a_2(\nu_{02})}{a_1(\nu_{02})}C_0^2 a^{\nu_{02}} - a^{-2}B_1 - \frac{1}{3}a^{-2}B_3\left(\frac{5-4\nu^{\rm h}}{1-2\nu^{\rm h}}\right) = -\frac{1}{3}T_0 a, \\
&\sum_{i=1}^{3} K_{2i}^1 A_{2i}a^{\nu_{2i}} + 12a^{-4}B_2 - \frac{4}{3}a^{-2}B_3\left(\frac{5-4\nu^{\rm h}}{1-2\nu^{\rm h}}\right) = \frac{2}{3}T_0 a, \\
&e_{33}\frac{e_{33}}{\varepsilon_{33}}\nu_{01}C_{01}a^{\nu_{01}-1} + m_1 C_{02}a^{\nu_{02}-1} - 4\mu^{\rm h}a^{-3}B_1 - m_2 a^{-3}B_3 = m_3 T_0, \\
&\sum_{i=1}^{3} m_{4i}A_{2i}a^{\nu_{2i}-1} - 96a^{-5}\mu^{\rm h}B_2 + \frac{16}{3}a^{-3}m_5 B_3 = \frac{4}{3}T_0\mu^{\rm h}, \\
&\sum_{i=1}^{3} m_{6i}A_{2i}a^{\nu_{2i}-1} + 32a^{-5}\mu^{\rm h}B_2 - \frac{4}{3}a^{-3}\mu^{\rm h}m_7 B_3 = \frac{2}{3}\mu^{\rm h}T_0, \\
&\sum_{i=1}^{3} m_{8i}a^{\nu_{2i}-1}A_{2i} - 4a^{-4}\varepsilon^{\rm h}B_5 = 0, \\
&e_{33}\nu_{01}a^{\nu_{01}-1}C_{01} + m_9 a^{\nu_{02}-1}C_{02} + \varepsilon^{\rm h}a^{-2}B_4 = 0, \qquad (15)
\end{aligned}
$$

where $\varepsilon^{\rm h}$ is the dielectric constant of isotropic host material.

$$m_1 = e_{33}\frac{e_{33}}{\varepsilon_{33}}\nu_{02} - (2c_{13} + c_{33}\nu_{02})\frac{a_2(\nu_{02})}{a_1(\nu_{02})}, \quad m_2 = -\frac{4}{3}\frac{\mu^{\rm h}\left(5-4\nu^{\rm h}\right)}{1-2\nu^{\rm h}},$$
$$m_3 = \mu^{\rm h}(\frac{1}{1-2\nu^{\rm h}} - \frac{1}{3}), \quad m_{4i} = 6c_{13} + 2c_{13}K_{2i}^1 + c_{33}K_{2i}^1\nu_{2i} + e_{33}\frac{e_{33}}{\varepsilon_{33}}K_{2i}^2\nu_{2i},$$
$$m_5 = \frac{\mu^{\rm h}\left(5-\nu^{\rm h}\right)}{1-2\nu^{\rm h}}, \quad m_{6i} = c_{44}(K_{2i}^1 + 1 - \nu_{2i}) + e_{15}\frac{e_{33}}{\varepsilon_{33}}K_{2i}^2,$$
$$m_7 = \frac{2\left(1+\nu^{\rm h}\right)}{1-2\nu^{\rm h}}, \quad m_{8i} = (K_{2i}^1 - K_{2i}^2)e_{33}\nu_{2i} + 2e_{31}(K_{2i}^1 + 3),$$
$$m_9 = 2e_{31}\frac{a_2(\nu_{02})}{a_1(\nu_{02})} + e_{33}\nu_{02}\left[\frac{a_2(\nu_{02})}{a_1(\nu_{02})} + 1\right].$$

Solving Eq. (15) yields the analytical solutions in the host region and the inclusion region, under an external uniform strain acting along the $\hat{z}$-direction. The solutions in the inclusion region are as follows:

$$\Phi = \frac{e_{33}}{\varepsilon_{33}}\Phi_0(r) + \frac{e_{33}}{\varepsilon_{33}}\sum_{i=1}^{3}K_{2i}^2 A_{2i}r^{\nu_{2i}}P_2(\cos\theta), \quad u_\theta = -\sum_{i=1}^{3}A_{2i}r^{\nu_{2i}}\frac{\partial}{\partial\theta}P_2(\cos\theta),$$
$$u_r = W_0(r) + \sum_{i=1}^{3}K_{2i}^1 A_{2i}r^{\nu_{2i}}P_2(\cos\theta), \quad u_\varphi = 0. \tag{16}$$

These solutions will be used to find the effective elastic coefficient and the effective piezoelectric coefficient later.

## 3. Effective elastic and piezoelectric responses

The strain tensor and the electric field in the inclusion region can be derived by means of the solutions of the elastic displacements and the electric potentials obtained in Section 2. Effective elastic and piezoelectric responses, $c_{ijkl}^{\rm e}$ and $e_{ikl}^{\rm e}$ can be estimated from the effective constitutive equations, the equations that govern the effective properties, $\bar{\sigma}_{ij} = c_{ijkl}^{\rm e}\bar{\gamma}_{kl} - e_{kij}^{\rm e}\bar{E}_k$ and $\bar{D}_{\rm i} = e_{ikl}^{\rm e}\bar{\gamma}_{kl} + \varepsilon_{ik}^{\rm e}\bar{E}_k$, where $\bar{A} = V^{-1}\int_{\Omega_{\rm i}+\Omega_{\rm h}} A{\rm d}V$ and $V$ is the total volume occupied by inclusion $\Omega_{\rm i}$ and host $\Omega_{\rm h}$ regions. We have

$$\frac{1}{V}\int_{\Omega_{\rm i}}\left[\left(c_{ijkl}^{\rm i} - c_{ijkl}^{\rm h}\right)\gamma_{kl} - \left(e_{kij}^{\rm i} - e_{kij}^{\rm h}\right)E_k\right]{\rm d}V = \bar{\sigma}_{ij} - c_{ijkl}^{\rm h}\bar{\gamma}_{kl} + e_{kij}^{\rm h}\bar{E}_k, \tag{17}$$
$$\frac{1}{V}\int_{\Omega_{\rm i}}\left[\left(e_{ikl}^{\rm i} - e_{ikl}^{\rm h}\right)\gamma_{kl} + \left(\varepsilon_{ik}^{\rm i} - \varepsilon_{ik}^{\rm h}\right)E_k\right]{\rm d}V = \bar{D}_{\rm i} - e_{ikl}^{\rm h}\bar{\gamma}_{kl} - \varepsilon_{ik}^{\rm h}\bar{E}_k, \tag{18}$$

where superscripts i and h denote the quantities of the inclusion region and the host region, respectively. Combining the effective constitutive equations with Eqs. (17) and (18), we have

$$e_{ikl}^{\rm e}\bar{\gamma}_{kl} + \varepsilon_{ik}^{\rm e}\bar{E}_k = \varepsilon_{ik}^{\rm h}\bar{E}_k + \frac{1}{V}\int_{\Omega_{\rm i}}\left[e_{ikl}^{\rm i}\gamma_{kl} + \left(\varepsilon_{ik}^{\rm i} - \varepsilon_{ik}^{\rm h}\right)E_k\right]{\rm d}V, \tag{19}$$
$$c_{ijkl}^{\rm e}\bar{\gamma}_{kl} - e_{kij}^{\rm e}\bar{E}_k = c_{ijkl}^{\rm h}\bar{\gamma}_{kl} + \frac{1}{V}\int_{\Omega_{\rm i}}\left[\left(c_{ijkl}^{\rm i} - c_{ijkl}^{\rm h}\right)\gamma_{kl} - e_{kij}^{\rm i}E_k\right]{\rm d}V. \tag{20}$$

Because external strain $T_0$ is applied along the $\hat{z}$-direction (i.e. $\gamma_{zz}^0 = T_0$) and applied external electric field $E_0$ is zero, effective elastic response $c_{ijzz}^{\rm e}$ and piezoelectric responses $e_{zij}^{\rm e}$ can be estimated in the dilute limit as follows:

$$c_{ijzz}^{\rm e} = c_{ijzz}^{\rm h} + \frac{1}{VT_0}\int_{\Omega_{\rm i}}\left[\left(c_{ijkl}^{\rm i} - c_{ijkl}^{\rm h}\right)\gamma_{kl} - e_{kij}^{\rm i}E_k\right]{\rm d}V, \tag{21}$$
$$e_{izz}^{\rm e} = \frac{1}{VT_0}\int_{\Omega_{\rm i}}\left[e_{ikl}^{\rm i}\gamma_{kl} + \left(\varepsilon_{ik}^{\rm i} - \varepsilon_{ik}^{\rm h}\right)E_k\right]{\rm d}V, \quad (i,j,k,l = x,y,z), \tag{22}$$

where the quantities in Eqs. (21) and (22) are represented in Cartesian coordinates. Thus, based on the solutions of the strain tensor and the electric field in the inclusion region, the effective elastic coefficient and piezoelectric

coefficient, $c^{\rm e}_{ijzz}$ and $e^{\rm e}_{izz}$ $(i,j=x,y,z)$, can be calculated from Eqs. (21) and (22), respectively. For example, effective elastic coefficient $c^{\rm e}_{zzzz}$ and piezoelectric coefficient $e^{\rm e}_{zzz}$ are given by the following formulas:

$$c^{\rm e}_{zzzz} = c^{\rm h}_{zzzz} + \frac{1}{VT_0}\int_{\Omega_{\rm i}} \left[\left(c^{\rm i}_{zzkl} - c^{\rm h}_{zzkl}\right)\gamma_{kl} - e^{\rm i}_{kzz}E_k\right] {\rm d}V, \tag{23}$$

$$e^{\rm e}_{zzz} = \frac{1}{VT_0}\int_{\Omega_{\rm i}} \left[e^{\rm i}_{zkl}\gamma_{kl} + \left(\varepsilon^{\rm i}_{zk} - \varepsilon^{\rm h}_{zk}\right)E_k\right] {\rm d}V. \tag{24}$$

Using the tensor transformation from spherical coordinates to Cartesian coordinates, for example, $\sigma^{\rm i}_{zz} = c^{\rm i}_{zzkl}\gamma_{kl} - e^{\rm i}_{kzz}E_k = \sigma^{\rm i}_{rr}\cos^2\theta - 2\sigma^{\rm i}_{r\theta}\cos\theta\sin\theta + \sigma^{\rm i}_{\theta\theta}\sin^2\theta$, we have,

$$\begin{aligned}(c^{\rm i}_{zzkl} - c^{\rm h}_{zzkl})\gamma_{kl} - e^{\rm i}_{kzz}E_k = {} & \left(\tilde{c}_{13}\gamma_{\theta\theta} + \tilde{c}_{13}\gamma_{\varphi\varphi} + \tilde{c}_{33}\gamma_{rr} - e^{\rm i}_{33}E_r\right)\cos^2\theta \\ & - 2\left(2\tilde{c}_{44}\gamma_{r\theta} - e^{\rm i}_{15}E_\theta\right)\cos\theta\sin\theta + \left(\tilde{c}_{11}\gamma_{\theta\theta} + \tilde{c}_{12}\gamma_{\varphi\varphi} + \tilde{c}_{13}\gamma_{rr} - e^{\rm i}_{31}E_r\right)\sin^2\theta,\end{aligned}$$

where $\tilde{c}_{13} = c^{\rm i}_{13} - c^{\rm h}_{13}$, $\tilde{c}_{33} = c^{\rm i}_{33} - c^{\rm h}_{33}$, $\tilde{c}_{11} = c^{\rm i}_{11} - c^{\rm h}_{11}$, $\tilde{c}_{12} = c^{\rm i}_{12} - c^{\rm h}_{12}$, $\tilde{c}_{44} = c^{\rm i}_{44} - c^{\rm h}_{44}$, and host elastic coefficients $c^{\rm h}_{kj}$ are calculated from formula $c^{\rm h}_{ijkl} = \lambda^{\rm h}\delta_{ij}\delta_{kl} + \mu^{\rm h}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ according to Nye's rule. Substituting the above transformations into Eq. (23), we obtain the effective elastic formula in the dilute limit to be

$$\begin{aligned}c^{\rm e}_{zzzz} = {} & c^{\rm h}_{zzzz} - pT_0^{-1}(2N_{01} + N_{02} - N_{03})C_{02} + pT_0^{-1}N_{04}C_{01} \\ & + \frac{2}{5}pT_0^{-1}\sum_{j=1}^{3}(N_{1j} + 6N_{2j} - N_{3j} + 3N_{4j})A_{2j},\end{aligned} \tag{25}$$

where $p$ is the inclusion volume fraction,

$$\begin{aligned}
N_{01} &= \frac{(c^{\rm i}_{11} + c^{\rm i}_{12} + c^{\rm i}_{13} - c^{\rm h}_{11} - c^{\rm h}_{12} - c^{\rm h}_{13})}{\nu_{02} + 2}\frac{a_2(\nu_{02})}{a_1(\nu_{02})}a^{\nu_{02}-1}, \\
N_{02} &= \nu_{02}\frac{(2c^{\rm i}_{13} + c^{\rm i}_{33} - 2c^{\rm h}_{13} - c^{\rm h}_{33})}{\nu_{02} + 2}\frac{a_2(\nu_{02})}{a_1(\nu_{02})}a^{\nu_{02}-1}, \\
N_{03} &= (e^{\rm i}_{33} + 2e^{\rm i}_{31})\frac{e^{\rm i}_{33}}{\varepsilon^{\rm i}_{33}}\frac{\nu_{02}}{\nu_{02} + 2}a^{\nu_{02}-1}, \quad N_{04} = (e^{\rm i}_{33} + 2e^{\rm i}_{31})\frac{e^{\rm i}_{33}}{\varepsilon^{\rm i}_{33}}\frac{\nu_{01}}{\nu_{01} + 2}a^{\nu_{01}-1}, \\
N_{1j} &= \frac{a^{\nu_{2j}-1}}{\nu_{2j} + 2}[(2K^1_{2j} + 6)(c^{\rm i}_{13} - c^{\rm h}_{13}) + \nu_{2j}K^1_{2j}(c^{\rm i}_{33} - c^{\rm h}_{33}) + e^{\rm i}_{33}\frac{e^{\rm i}_{33}}{\varepsilon^{\rm i}_{33}}\nu_{2j}K^2_{2j}], \\
N_{2j} &= \frac{a^{\nu_{2j}-1}}{\nu_{2j} + 2}[(K^1_{2j} + 1 - \nu_{2j})(c^{\rm i}_{44} - c^{\rm h}_{44}) + e^{\rm i}_{15}\frac{e^{\rm i}_{33}}{\varepsilon^{\rm i}_{33}}K^2_{2j}], \\
N_{3j} &= \frac{a^{\nu_{2j}-1}}{\nu_{2j} + 2}[K^1_{2j}(c^{\rm i}_{11} + c^{\rm i}_{12} - c^{\rm h}_{11} - c^{\rm h}_{12}) + \nu_{2j}K^1_{2j}(c^{\rm i}_{13} - c^{\rm h}_{13}) + e^{\rm i}_{31}\frac{e^{\rm i}_{33}}{\varepsilon^{\rm i}_{33}}K^2_{2j}\nu_{2j}], \\
N_{4j} &= \frac{a^{\nu_{2j}-1}}{\nu_{2j} + 2}[(c^{\rm i}_{12} - 3c^{\rm i}_{11}) - (c^{\rm h}_{12} - 3c^{\rm h}_{11})].
\end{aligned}$$

To derive the effective piezoelectric coefficient, $e^{\rm e}_{zzz}$, the vector transformation formula, $D^{\rm i}_z = e^{\rm i}_{zkl}\gamma_{kl} + \varepsilon^{\rm i}_{zk}E_k = D^{\rm i}_r\cos\theta - D^{\rm i}_\theta\sin\theta$, from spherical coordinates to Cartesian coordinates is used. We have

$$\begin{aligned}
& e^{\rm i}_{zkl}\gamma_{kl} + (\varepsilon^{\rm i}_{zk} - \varepsilon^{\rm h}_{zk})E_k \\
= {} & [e^{\rm i}_{31}(\gamma_{\theta\theta} + \gamma_{\varphi\varphi}) + e^{\rm i}_{33}\gamma_{rr} + (\varepsilon^{\rm i}_{33} - \varepsilon^{\rm h})E_r]\cos\theta \\
& - [2e^{\rm i}_{15}\gamma_{r\theta} + (\varepsilon^{\rm i}_{11} - \varepsilon^{\rm h})E_\theta]\sin\theta.
\end{aligned}$$

Substituting this formula into Eq. (24) and considering the orthogonality, we can show that

$$e^{\rm e}_{zzz} = 0. \tag{26}$$

Similarly, we have $e^{\rm e}_{izz} = 0$ $(i = x, y)$. The results show that the vanishing piezoelectric response is due to the symmetry of the spherically anisotropic piezoelectric composite system, even though external strain is applied in the $\hat{x}$- and the $\hat{y}$-directions. If the spherically symmetry of the piezoelectric system is broken, effective piezoelectric properties of the composite will appear. For example, if a transversely isotropic piezoelectric spherical inclusion is employed in the non-piezoelectric matrix, there will be effective piezoelectric responses.[26]

In the following, we will discuss, through nu-

merical calculations, the coupling effects of dielectric, elastic and piezoelectric properties of the constituents on the effective elastic constants by using Eq. (25). In order to test the validity of our effective elastic formula, we have investigated the case of a pure elastic isotropic spherical composites by setting the inclusion piezoelectric coefficients to be zero in Eq. (25), and our results are compared with those obtained from the formulas found by Bruggeman for the first order inclusion fraction of isotropic spherical composites.[2,8] As shown in Fig. 1, it is clear that good agreement is, indeed, obtained in the dilute limit. Here, it should be noted that in the dilute limit, our results are also consistent with Causius–Mossotti-type approximations.[14,15]

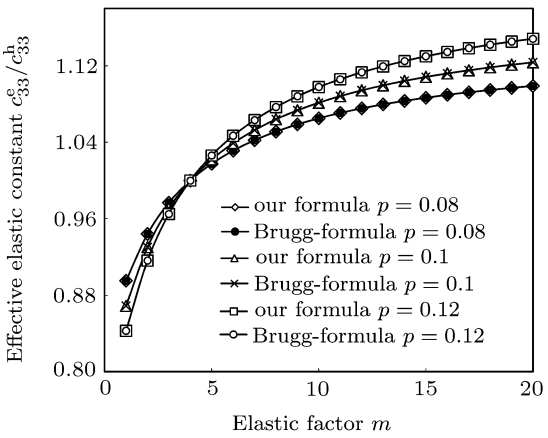



**Fig. 1.** Curves for effective elastic constant $c^{\rm e}_{33}/c^{\rm h}_{33}$ ($c^{\rm e}_{zzzz}/c^{\rm h}_{zzzz}$) versus elastic constant factor $m$ of the isotropic spherical inclusion, where the Poisson's ratios and the shear modulus of the host material and the inclusion material are $\nu^{\rm h} = \nu^{\rm i} = 0.30$, $\mu^{\rm h} = 32$ GPa and $\mu^{\rm i} = 8 \times m$ GPa, respectively; the inclusion elastic factor is noted by $m$;, the Bruggeman effective elastic formula of first order inclusion volume fraction $p$ of isotropic elastic spherical composite is denoted by 'Brugg-formula'.

Furthermore, from Fig. 2 to Fig. 4, we have calculated the effective elastic property $c^{\rm e}_{zzzz}$ [Eq. (25)] by taking a small inclusion volume fraction ($p = 0.1$), sphere radius $a = 1$, elastic parameters and the dielectric constants of host region $\nu^{\rm h} = 0.25$, $\mu^{\rm h} = 32$ GPa and $\varepsilon^{\rm h} = 3 \times 10^{-10}$ C$^2$·N$^{-1}$·m$^{-2}$. Also, inclusion elastic, dielectric and piezoelectric constants in spherical coordinates are taken to be $c^{\rm i}_{11} = 16.6 \times m$ GPa, $c^{\rm i}_{12} = 7.7 \times m$ GPa, $c^{\rm i}_{13} = 7.8 \times m$ GPa, $c^{\rm i}_{33} = 16.2 \times m$ GPa, $c^{\rm i}_{44} = 4.3 \times m$ GPa, $\varepsilon^{\rm i}_{11} = \varepsilon^{\rm i}_{22} = 1.13 \times n \times 10^{-10}$ C$^2$·N$^{-1}$·m$^{-2}$, $\varepsilon^{\rm i}_{33} = 1.4 \times n \times 10^{-10}$ C$^2$·N$^{-1}$·m$^{-2}$, $e_{33} = 0.186 \times s$ C · m$^{-2}$, $e_{31} = -0.044 \times s$ C · m$^{-2}$, and $e_{15} = 0.116 \times s$ C · m$^{-2}$, where $m$, $n$ and $s$ are multiplying factors for the elastic, the dielectric and the piezoelectric constants, respectively. For a given inclusion dielectric constant, the effects of inclusion piezoelectric coefficients on bulk effective elastic constant are shown in Fig. 2. As shown in the figure, the effective elastic responses increase with inclusion elastic constants increasing. The results indicate clearly that the inclusion piezoelectric coefficients can enhance the effective elastic property, especially for small inclusion elastic coefficients.

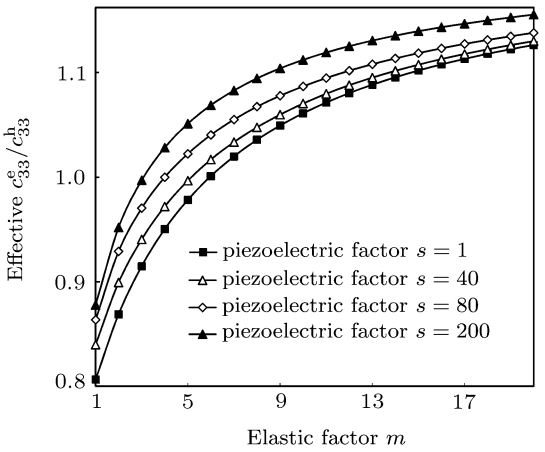



**Fig. 2.** Curves for effective elastic constant $c^{\rm e}_{33}/c^{\rm h}_{33}$ ($c^{\rm e}_{zzzz}/c^{\rm h}_{zzzz}$) versus elastic constant factor $m$ of the spherically anisotropic inclusion, where inclusion dielectric constant factor $n$ is fixed at 10, i.e. $n = 10$.

On the contrary, from Fig. 3, it can be seen that, in general, the inclusion dielectric constant can reduce the effective elastic coefficients for given inclusion piezoelectric and elastic coefficients. However,

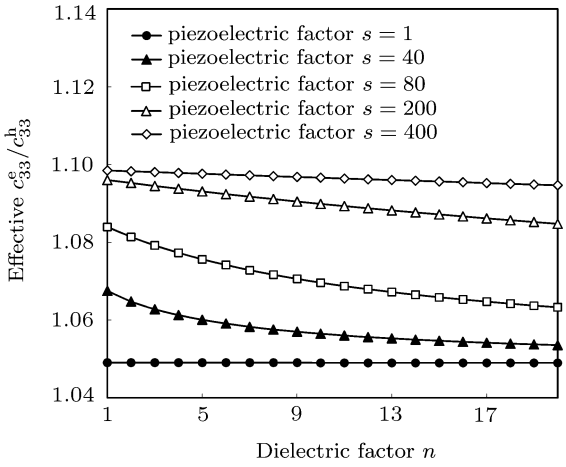



**Fig. 3.** Curves for effective elastic constant $c^{\rm e}_{33}/c^{\rm h}_{33}$ ($c^{\rm e}_{zzzz}/c^{\rm h}_{zzzz}$) versus dielectric constant factor $n$ of the spherically anisotropic inclusion, where elastic constant factor $m$ of inclusion is fixed at 10, i.e. $m = 10$.

for the case of small piezoelectric coefficients, the dielectric constant seems to have no significant effects

on the effective elastic constants. The above results show that the piezoelectric coefficients can play an important role in modulating the coupling effects of the dielectric and the elastic properties on bulk effective elastic response. Moreover, the effects of piezoelectric and dielectric coefficients on effective elastic constant are related to the inclusion elastic constant.

In order to further investigate the coupling effects, in Fig. 4, we have given an example showing how the effective elastic constants vary with inclusion piezoelectric constant for different, fixed dielectric and elastic constants. Also, it is clearly that the effects of inclusion piezoelectric, dielectric and dielectric constants on effective elastic properties are coupled. Therefore, we conclude that the effective elastic constant of this type of piezoelectric composite is affected by elastic, piezoelectric and dielectric properties of inclusion. In particular, piezoelectricity is an important parameter for modulating the effect of inclusion dielectric constant on effective elastic response. In general, the piezoelectric (or dielectric) constant enhances (or reduces) the effective elastic response of a piezoelectric composite system.

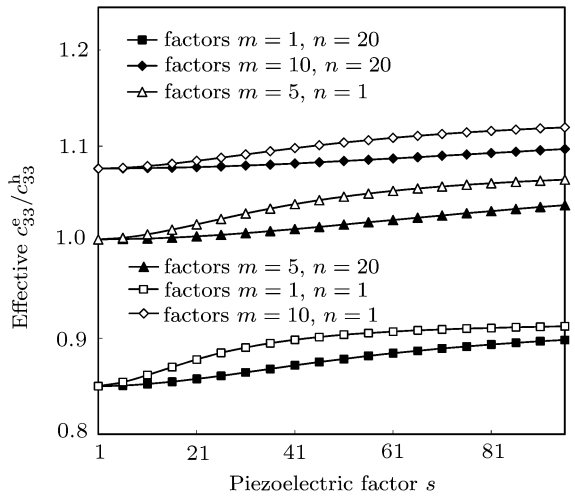



**Fig. 4.** Curves for effective elastic constant $c^{e}_{33}/c^{h}_{33}$ ($c^{e}_{zzzz}/c^{h}_{zzzz}$) versus piezoelectric constant factor $s$ of the spherically anisotropic inclusion, where elastic constant factor $m$ and dielectric constant factor $n$ of inclusion are given in this figure, respectively.

# 4. Conclusions

Spherically anisotropic piezoelectric composites, whose matrixes are each of isotropic non-piezoelectric material, are investigated theoretically. Analytical solutions of the local elastic displacements and the potentials are exactly derived each under the boundary condition of a uniform external strain acting along the $\hat{z}$-direction. The stress tensors and the electric fields are derived based on the constitutive relationships and the local analytical solutions of the elastic displacements and electric potentials obtained in Section 2. Furthermore, in the dilute limit, formulas for estimating effective elastic and piezoelectric coefficients are obtained by average field method. The results in this paper show that the spherically anisotropic piezoelectric composites do not have bulk effective piezoelectric behaviours due to the symmetries of the spherical composite systems and the radial polarizations of the spherical inclusions. The bulk effective elastic responses are demonstrated to be affected by the piezoelectric constant and the dielectric constant of inclusion. In general, the piezoelectric (or dielectric) constants can enhance (or reduce) the effective elastic coefficients and the coupled effects are also dependent on inclusion elastic constants. The inclusion piezoelectric constant plays an important role in the coupling effects and the dielectric constant seems to give rise to its effect on the effective elastic properties through the inclusion piezoelectric properties. Clearly, in the absence of the piezoelectric constants, the dielectric constant will not produce any effect on effective elastic behaviour. This means that the piezoelectric composites can be designed for controlling the elastic responses of composite materials in the technological applications by increasing (or decreasing) the piezoelectric constant and the dielectric constant of inclusion. Thus, we disclose the coupled effective elastic response mechanism of spherically anisotropic composites.

Based on the analytical solutions, the effective responses at high inclusion concentrations can be developed in the effective-medium approximation. Because the spherically anisotropy implies the gradient profiles of inclusion properties in Cartesian coordinates, the graded spherically anisotropic piezoelectric composites, for example, the radial power-law and the exponential profile of inclusion, can be studied.[31,32] In addition, it is possible to develop some existing methods of solving nonlinear problems of composites to further investigate the effective elastic responses of nonlinear piezoelectric composites due to the electroelastic interactions related to the nonlinear dielectric responses.[33−35]

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