Concurrence evolution of two qubits coupled with one-mode cavity separately*

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The concurrence evolution of two qubits coupled with one-mode cavity separately is investigated exactly without adopting the rotating-wave approximation. The results show that for the resonant case, the concurrence evolution behaviour of the system is similar to that of the Markovian case when the coupling strength is weak, while the concurrence vanishes in a finite time and might revive fractional initial entanglement before it permanently vanishes when the coupling strength is strong. And for the detuning case, the entanglement could periodically recover after complete disentanglement. These results are quite different from those of system subjected to Jaynes–Cummings model.

Keywords: decoherence, qubit, rotating-wave approximation, cavity
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1. Introduction

Recently, many studies have been devoted to the decoherence behaviour of entangled qubits, invoked by the phenomenon and termed 'entanglement sudden death' (ESD).[1−17] It is shown that the interaction between qubits with environment could lead to finite-time disentanglement, which is quite different from the case of continuous variable two-atom model discussed by Dodd and Halliwell.[2,18,19]

The decoherence of two atoms coupled with one-mode cavity separately, was investigated in Ref. [4] with the Jaynes–Cummings model, in which counter rotating terms corresponding to the emission and the absorption of virtual photon without energy conservation, are neglected, and is widely used in quantum optics.[20−22] Generally, the rotating-wave approximation (RWA) neglecting counter rotating is considered to be applicable for the case of small detuning, i.e. small ratio of the atom–field coupling to the atomic transition frequency. In atom–field cavity systems, this ratio is typically on the order of 10^{-7}–10^{-6}. Recently, cavity systems with very strong couplings have been discussed.[23] In solid state systems, the ratio may become so large that the effect of counter-rotating wave terms must be considered.[24]

In this paper, the counter-rotating wave terms are taken into account although the system is the same as that in Ref. [4], and the results are compared with those in Ref. [4]. The rest of the present paper is organised as follows. In Section 2, the reduced non-perturbative quantum master equation of atom is derived and its exact solution is obtained. In Section 3, the concurrence evolution of two initially entangled qubits is discussed. The conclusion will be given in Section 4.

2. Model and exact solution

2.1. Non-perturbative master equation and its exact solution

As done in Ref. [4], we restrict our attention to two noninteracting two-level atoms A and B, coupled with a perfect one-mode cavity, separately. The dynamics of two-qubit density matrix elements can be obtained on knowing the dynamics of the single qubit if the different parts, 'qudit+reservoir', are independent.[13] Thus, we first consider the subsystem Hamiltonian of one atom coupled to one cavity field mode, including counter–rotating wave terms as follows:

$$H = H_{\rm a} + H_{\rm f} + H_{\rm af}, \tag{1}$$

where

$$H_{\rm a} = \omega_0 \frac{\sigma_z}{2}, \tag{2}$$

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$$H_{\rm f} = \omega a^{\dagger} a, \tag{3}$$

$$H_{\rm af} = g(\sigma_+ + \sigma_-)(a^{\dagger} + a), \tag{4}$$

where ω_0 is the atomic transition frequency between ground state $|0\rangle$ and excited state $|1\rangle$; $\sigma_z = |1\rangle\langle 1| - |0\rangle\langle 0|$, $\sigma_+ = |1\rangle\langle 0|$ and $\sigma_- = |0\rangle\langle 1|$ are pseudo-spin operators of atom; $a^{\dagger}$ and a are the creation operator and the annihilation operator of the cavity field mode corresponding frequency ω; g is the coupling constant between the $|1\rangle - |0\rangle$ transition and the field mode.

Up to now, the subsystem described by Eq. (1), has not been investigated analytically and completely. Based on the latest results, the reduced non-perturbative quantum master equation of atom could be obtained by path integrals to be[25]

$$\frac{\partial}{\partial t}\rho_{\rm a} = -{\rm i}\mathcal{L}_{\rm a}\rho_{\rm a} - \int_0^t {\rm d}s\langle \mathcal{L}_{\rm af}\, {\rm e}^{-{\rm i}\mathcal{L}_0(t-s)}\mathcal{L}_{\rm af} \times {\rm e}^{-{\rm i}\mathcal{L}_0(s-t)}\rangle_{\rm f}\rho_{\rm a}, \tag{5}$$

where $\mathcal{L}_0$, $\mathcal{L}_a$ and $\mathcal{L}_{\rm af}$ are the Liouvillian operators defined respectively as

$$\mathcal{L}_0\rho \equiv [H_{\rm a} + H_{\rm f}, \rho],$$
$$\mathcal{L}_{\rm a}\rho \equiv [H_{\rm a}, \rho],$$
$$\mathcal{L}_{\rm af}\rho \equiv [H_{\rm af}, \rho],$$

and $\langle ... \rangle_{\rm f}$ denotes the partial trace of cavity mode ${\rm Tr}_{\rm f}\{...\rho_{\rm f}\}$, and $\rho_{\rm f}$ is the initial density matrix of the cavity field.[25]

The non-perturbative reduced master equation of the atom with the cavity field initially in vacuum state could be derived from Eq. (5) to be

$$\begin{aligned}\frac{\partial}{\partial t}\rho_{\rm a} =& -g^2\left(\alpha^{\rm R} + f(t)\right)\rho_{\rm a} - 2{\rm i}\left(\omega_0 - g^2\alpha^{\rm I} + g^2 f^{\rm I}(t)\right)J_0\rho_{\rm a} \\ &+ g^2\left(\alpha + f^*(t)\right)J_+\rho_{\rm a} + g^2\left(\alpha^* + f(t)\right)J_-\rho_{\rm a} \\ &+ 2g^2\alpha^{\rm R}K_+\rho_{\rm a} + 2g^2\left(\alpha^{\rm R} - f^{\rm R}(t)\right)K_0\rho_{\rm a} \\ &+ 2g^2 f^{\rm R}(t)K_-\rho_{\rm a},\end{aligned} \tag{6}$$

where J_0, J_+, J_-, K_0, K_+ and K_- are superoperators defined respectively as

$$J_0\rho_{\rm a} \equiv \left[\frac{\sigma_z}{4}, \rho_{\rm a}\right],$$
$$J_+\rho_{\rm a} \equiv \sigma_+\rho_{\rm a}\sigma_+,$$
$$J_-\rho_{\rm a} \equiv \sigma_-\rho_{\rm a}\sigma_-,$$
$$K_0\rho_{\rm a} \equiv (\sigma_+\sigma_-\rho_{\rm a} + \rho_{\rm a}\sigma_+\sigma_- - \rho_{\rm a})/2,$$
$$K_+\rho_{\rm a} \equiv \sigma_+\rho_{\rm a}\sigma_-,$$
$$K_-\rho_{\rm a} \equiv \sigma_-\rho_{\rm a}\sigma_+,$$

and

$$\alpha = \frac{1 - \exp(-{\rm i}\Delta t)}{{\rm i}\Delta}, \tag{7}$$

$$f(t) = \frac{\exp({\rm i}\delta t) - 1}{{\rm i}\delta}, \tag{8}$$

where $\Delta = \omega + \omega_0$; $\delta = \omega_0 - \omega$; $\alpha^{\rm R}$, $\alpha^{\rm I}$, α^* and $f^{\rm R}$, $f^{\rm I}$, f^* are real parts, image parts and conjugates of α and of $f(t)$, respectively; function $f(t)$ comes from the rotating–wave interaction, while α comes from the counter–rotating wave interaction.

Using the algebraic approach, the formal solution of Eq. (6) is obtained to be[21,26]

$$\begin{aligned}&\rho_{\rm a}(t) \\ &= \exp(-\Gamma_k)\hat{T}\exp\left[\int_0^t {\rm d}t(\varepsilon_0 J_0 + \varepsilon_+ J_+ + \varepsilon_- J_-)\right] \\ &\quad\times\hat{T}\exp\left[\int_0^t {\rm d}t(\nu_0 K_0 + \nu_+ K_+ + \nu_- K_-)\rho_{\rm a}(0)\right],\end{aligned} \tag{9}$$

where $\hat{T}$ is the time ordering operator, $\varepsilon_0 = -2{\rm i}\left(\omega_0 - g^2\alpha^{\rm I} + g^2 f^{\rm I}\right)$, $\varepsilon_+ = g^2\left(\alpha + f^*\right)$, $\varepsilon_- = g^2\left(\alpha^* + f\right)$, $\nu_0 = 2g^2\left(\alpha^{\rm R} - f^{\rm R}\right)$, $\nu_+ = 2g^2\alpha^{\rm R}$, $\nu_- = 2g^2 f^{\rm R}$, $\Gamma_k = g^2\left(\tilde{\alpha}^{\rm R} + F^{\rm R}\right)$ and

$$\begin{aligned}\tilde{\alpha} &= \int_0^t \alpha\,{\rm d}t \\ &= \frac{1 - \exp(-{\rm i}\Delta t) - {\rm i}\Delta t}{\Delta^2} \\ &\equiv \tilde{\alpha}^{\rm R} + {\rm i}\tilde{\alpha}^{\rm I}, \\ F &= \int_0^t f(t)\,{\rm d}t \\ &= \frac{1 + {\rm i}\delta t - \exp({\rm i}\delta t)}{\delta^2} \\ &\equiv F^{\rm R} + {\rm i}F^{\rm I},\end{aligned} \tag{10}$$

where $\tilde{\alpha}^{\rm R}$, $\tilde{\alpha}^{\rm I}$, $\tilde{\alpha}^*$ and $F^{\rm R}$, $F^{\rm I}$, F^* are real parts, image parts and conjugates of $\tilde{\alpha}$ and F, respectively.

Using the decomposition of SU(2) operator, the exact solution of master equation Eq. (6) is obtained to be

$$\rho_{\rm a}(t) = {\rm e}^{-\Gamma_k}\tilde{\rho}(t), \tag{11}$$

$$\tilde{\boldsymbol{\rho}}(t) = \begin{pmatrix} l\rho_{\rm a}^{11}(0) + m\rho_{\rm a}^{00}(0) & x\rho_{\rm a}^{10}(0) + y\rho_{\rm a}^{01}(0) \\ q\rho_{\rm a}^{01}(0) + r\rho_{\rm a}^{10}(0) & n\rho_{\rm a}^{00}(0) + p\rho_{\rm a}^{11}(0) \end{pmatrix}, \tag{12}$$

$$l = {\rm e}^{k_0/2} + {\rm e}^{-k_0/2}k_+k_-,\ m = {\rm e}^{-k_0/2}k_+, \tag{13}$$

$$n = {\rm e}^{-k_0/2},\ p = {\rm e}^{-k_0/2}k_-, \tag{14}$$

$$q = {\rm e}^{-j_0/2},\ r = {\rm e}^{-j_0/2}j_-, \tag{15}$$

$$x = {\rm e}^{j_0/2} + {\rm e}^{-j_0/2}j_+j_-,\ y = {\rm e}^{-j_0/2}j_+, \tag{16}$$

where j_+, j_0, j_- and k_+, k_0, k_- satisfy the following differential equations:[21]

$$\dot{X}_+ = \mu_+ - \mu_- X_+^2 + \mu_0 X_+, \tag{17}$$
$$\dot{X}_0 = \mu_0 - 2\mu_- X_+, \tag{18}$$
$$\dot{X}_- = \mu_- \exp(X_0), \tag{19}$$

where $\mu = \varepsilon$ for $X = j$ and $\mu = \nu$ for $X = k$.

2.2. Concurrence

Now, we start to investigate the concurrence evolution of two initially entangled two-level atoms A and B, coupled with a perfect one-mode cavity, separately. Throughout the paper, we use the concurrence of Wootters.[27] For simplicity, we assume that the two subsystems have the same parameters. The concurrence of the whole system could be obtained to be[7]

$$\begin{aligned} C_\xi &= \max\{0, c_1, c_2\}, (\xi = \Phi, \Psi), \\ c_1 &= 2\,\mathrm{e}^{-2\Gamma_k}(\sqrt{\rho_{23}\rho_{32}} - \sqrt{\rho_{11}\rho_{44}}), \\ c_2 &= 2\,\mathrm{e}^{-2\Gamma_k}(\sqrt{\rho_{14}\rho_{41}} - \sqrt{\rho_{22}\rho_{33}}), \end{aligned} \tag{20}$$

corresponding to the initial states of $|\Phi\rangle = \beta|01\rangle + \eta|10\rangle$ and $|\Psi\rangle = \beta|00\rangle + \eta|11\rangle$, respectively. Here, β is real and $0 < \beta < 1$, $\eta = |\eta|\,\mathrm{e}^{\mathrm{i}\varphi}$ and $\beta^2 + |\eta|^2 = 1$. If the different parts, 'atom+reservoir', are independent, the reduced joint density matrix of the two atoms, in the standard product basis $\mathcal{B} = \{|1\rangle \equiv |11\rangle, |2\rangle \equiv |10\rangle, |3\rangle \equiv |01\rangle, |4\rangle \equiv |00\rangle\}$, could be obtained by the method in Ref. [13] as follws:

$$\boldsymbol{\rho}^{\mathrm{AB}} = \mathrm{e}^{-2\Gamma_k} \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix}, \tag{21}$$

where the diagonal elements are

$$\begin{aligned} \rho_{11} &= l^2\rho_{11}(0) + lm\rho_{22}(0) + ml\rho_{33}(0) + m^2\rho_{44}(0), \\ \rho_{22} &= lp\rho_{11}(0) + lm\rho_{22}(0) + mp\rho_{33}(0) + mn\rho_{44}(0), \\ \rho_{33} &= lp\rho_{11}(0) + pm\rho_{22}(0) + nl\rho_{33}(0) + nm\rho_{44}(0), \\ \rho_{44} &= p^2\rho_{11}(0) + pn\rho_{22}(0) + np\rho_{33}(0) + n^2\rho_{44}(0), \end{aligned}$$

and the nondiagonal elements are

$$\begin{aligned} \rho_{14} &= x^2\rho_{14}(0) + xy\rho_{23}(0) + yx\rho_{32}(0) + y^2\rho_{41}(0), \\ \rho_{23} &= xr\rho_{14}(0) + xq\rho_{23}(0) + yr\rho_{32}(0) + yq\rho_{41}(0), \\ \rho_{32} &= rx\rho_{14}(0) + ry\rho_{23}(0) + qx\rho_{32}(0) + qy\rho_{41}(0), \\ \rho_{41} &= r^2\rho_{14}(0) + rq\rho_{23}(0) + qr\rho_{32}(0) + q^2\rho_{41}(0). \end{aligned}$$

3. Numerical results and discussion

In order to compare the results with those of two Jaynes–Cummings atoms in Ref. [4], we primarily investigate the concurrence evolution for the resonant case $\omega = \omega_0$.

First, we focus on the concurrence evolution of two qubits with initial state $|\Phi\rangle$. For the RWA model in Ref. [4], Fig. 1 shows that the concurrence periodically vanishes and recovers. The concurrence obtained from the exact model in this paper is shown in Fig. 2.

(A) From Fig. 2(a), we can find that when $\omega_0 = 50g$, the concurrence C_Φ decreases and revives with a smaller amplitude than that in Fig. 2(b), before complete disentanglement of the two atoms. As coupling constant g decreases, the decrease and the revival of concurrence last longer time before complete disentanglement of the two atoms.

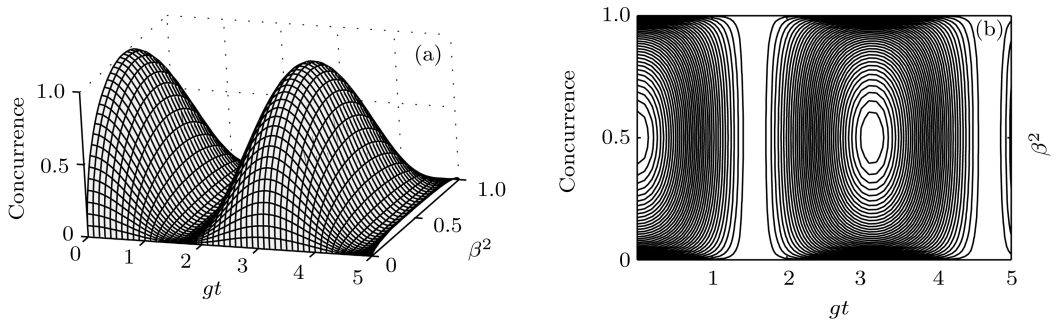


Fig. 1. C_Φ and its contour for resonanct case as a function of gt and β^2 in the RWA.

(B) As $\omega_0 = 10g$, Fig. 2(b) reveals that the concurrence C_Φ decreases to zero with small amplitude oscil-

lation and vanishes for a period of time, then revives with smaller amplitude and vanishes permanently. This characteristic will hold for a much stronger coupling constant.

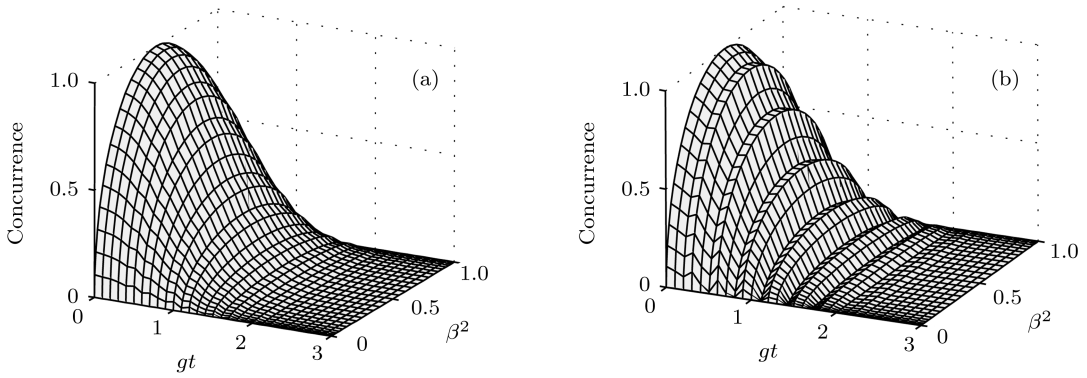


Fig. 2. C_Φ as a function of gt and β^2 for $\delta = 0$ and $\omega_0 = 50g$ (a) and $10g$ (b).

Then, we focus on the decoherence of two qubits with initial state $|\Psi\rangle$. For the RWA model in Ref. [4], Fig. 3 shows that the entanglement represented by C_Ψ can fall abruptly to zero, and will remain zero for a period of time before it recovers. The length of time interval for the zero entanglement is dependent on the degree of initial entanglement. The decoherence behaviour exhibits asymmetric concurrence around $\beta^2 = 1/2$ because two components $|00\rangle$ and $|11\rangle$ have different energies and different decay rates. There are periodical disappearances and revivals of entanglement on a time scale. For the exact model in this paper, the decoherence behaviours of the system can also be categorized into three cases.

(A) As $\omega_0 = 50g$, Fig. 4(a) shows that the entanglement represented by C_Ψ has a similar behaviour to that of C_Φ for $\beta^2 > 1/2$ in Fig. 2(a). In contrast, for $\beta^2 < 1/2$, it decreases smoothly to zero at a finite time and vanishes permanently.

(B) When $\omega_0 = 10g$, Fig. 4(b) reveals that the decoherence behaviour is also symmetrical around $\beta^2 = 1/2$ as mentioned above, but the concurrence C_Ψ decreases exponentially to zero with small amplitude oscillation and vanishes for a period of time, then revives as fractional initial entanglement and vanishes permanently for $\beta^2 > 1/2$. This is because of the strong coupling and the counter–rotating wave interaction between atom and cavity field through the emission and the absorption of virtual photon.

From Figs. 2 and 4, we find that the entanglement decreases to zero finally without adopting the RWA in this paper, and there are no periodical disappearance and recovering of entanglement, like that for the case with adopting the RWA in Ref. [4]. It is the enhancement of spontaneous emission, as an atom resonantly coupled with a cavity, which leads to the disappearance of entanglement.[28] We could conclude that the counter–rotating wave terms which are neglected in Ref. [4] result in the different characteristics.

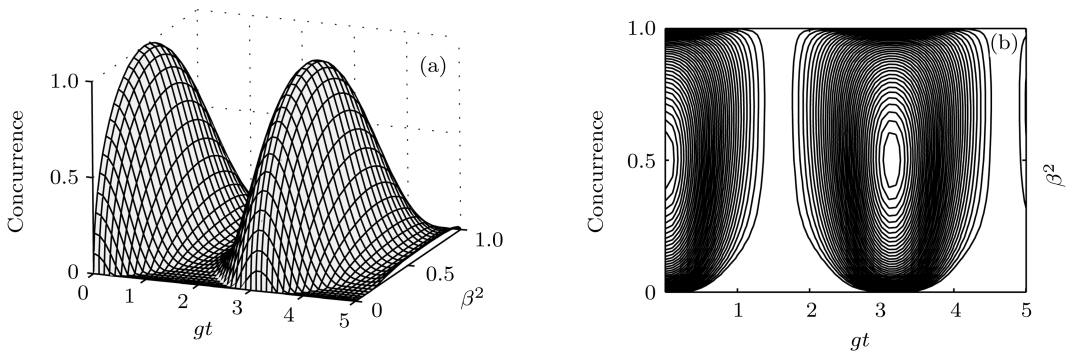


Fig. 3. C_Ψ and its contour as a function of gt and β^2 in the RWA.

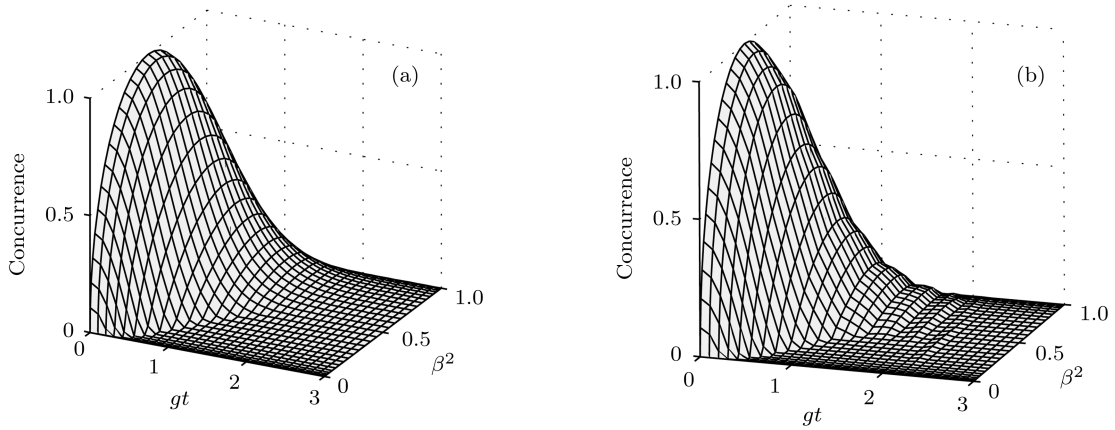


Fig. 4. C_Ψ as a function of gt and β^2 for $\delta = 0$ and $\omega_0 = 50g$ (a) and $10g$ (b).

Finally, we investigate whether there appear the periodical disappearance and the periodical recover of entanglement in the detuning case. Concurrences C_Φ and C_Ψ for the detuning case as a function of gt and β^2 are shown in Figs. 5(a) and 5(b), respectively.

From Fig. 5, we can find that the initial entanglement will recover after a period of time from its disappearance because the spontaneous emission can be greatly inhibited, as an atom non-resonantly coupled with cavity mode.[28] The revival time interval is dependent on detuning δ. The bigger the detuning is, the shorter the time interval will be.

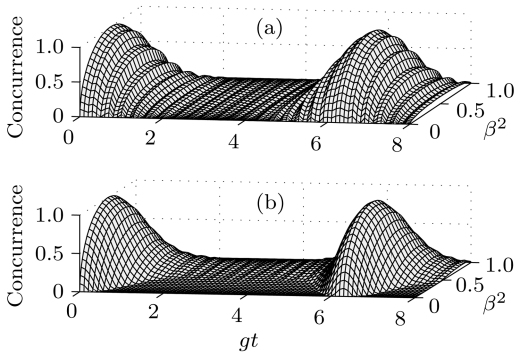


Fig. 5. Concurrences C_Φ (a) and C_Ψ (b) for non-resonant case as a function of gt and β^2 with $\omega_0 = 10g$ and $\delta = 0.1\omega_0$.

4. Conclusion

The concurrence evolution of two initially entangled atoms, coupled with one-mode cavity separately, is discussed exactly. And the results are compared with those in Ref. [4]. The results show that the concurrence evolution behaviour of two qubits depends on the coupling strength and the detuning between atom transition frequency and the cavity mode.

For the resonant case, there is neither periodical disappearance nor periodical revival of entanglement like that for the RWA model (or Jaynes–Cummings model) in Ref. [4]. For the resonant case, the decoherence behaviour of the system is similar to that for the Markovian case when the coupling strength is weak, while there is entanglement revival after complete disentanglement, like that in Ref. [13], when the coupling strength is strong. For the detuning case, the entanglement can periodically recover after complete disentanglement.

The results exhibit that for the resonant case, the entanglement decreases to zero finally, owing to the enhancement of spontaneous emission, but periodically decreases and recovers after complete disentanglement because of the inhibition of spontaneous emission. We can conclude that it is the RWA in Ref. [4] that leads to the recover of entanglement for the resonant case.

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