Sudden death and revival of entanglement of two qubits coupled collectively to a thermal reservoir*

Liao Xiang-Ping(廖湘萍)[a)†], Fang Jian-Shu(方见树)[a)], and Fang Mao-Fa(方卯发)[b)]

[a)] *College of Science, Hunan University of Technology, Zhuzhou* 412008, *China*
[b)] *College of Physics and Information Science, Hunan Normal University, Changsha* 410081, *China*

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In this paper, we investigate the entanglement of two qubits coupled collectively to a common thermal environment and find that the the collective decay can lead to a revival of the entanglement that has already been destroyed. We also show that the ability of the system to revival entanglement relies on the mean photon number of the thermal environment and the degree of entanglement of the initial state.

Keywords: sudden death of entanglement, two qubits, the thermal reservoir
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1. Introduction

Entanglement is a key resource in modern quantum information theory and technology. Entangled states play a central role in quantum key distribution, superdense coding, quantum teleportation and quantum error correction.[1,2] However, realistic quantum systems are never completely isolated from their surroundings. The inevitable interaction between a system and its environment leads to decoherence phenomena and degradation of entanglement.[3]

Once entanglement has been lost, it cannot be restored by local operations.[4] It is therefore important to understand the process of disentanglement in order to control the effects of noise and to preserve entanglement. For these reasons the study of entanglement dynamics of quantum systems in realistic situations has become increasingly important. Recently much attention has been devoted to the process of finite-time disentanglement, also known as 'entanglement sudden death' (ESD), in a bipartite system.[5] This phenomenon consists in the complete disappearance of the bipartite entanglement in a finite time, as opposed to the time-evolution of the coherences of the single parts which vanish only asymptotically. ESD sets a limit on the life-time and usability of entanglement for practical purposes. Hence, lots of efforts have been done in order to understand the conditions under which ESD occurs.[6−12] In a recent paper, Ficek and Tanaś[13] have studied the concept of time delayed creation of entanglement by the dissipative process of spontaneous emission. A threshold effect for the creation of entanglement is found that the initially unentangled qubits can be entangled after a finite time despite the fact that the coherence between the qubits exists for all times. In Ref. [14], Mazzola *et al.* investigated the exact dynamics of two qubits interacting with a common zero-temperature non-Markovian reservior. It is noticed that ESD seems to appear when the state becomes highly mixed, whereas revivals of entanglement are associated to minima of the entropy, where the state becomes purer.

In Ref. [15], the author investigated the entanglement and the nonlocality of two qubits interacting with a thermal reservoir. It is shown that the entanglement and the nonlocality of two qubits can be manipulated by changing the relative phases and the amplitudes of the polarised qubits. Here, we focus on the process of disentanglement, and the ability of the system to revive entanglement. We find that the the collective decay can lead to a revival of the entanglement that has already been detroyed. We also show that the ability of the system to revive entanglement relies on the mean photon number of the thermal environment and the degree of entanglement of the initial states.

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†Corresponding author. E-mail: liaoxp@126.com
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2. The master equation and the entanglement of two qubits

Up to date, much attention has been paid to the environment-induced entanglement.[16−19] Here, we consider a situation in which two qubits collectively interact with a common thermal reservoir. Under the Markovian approximation, the dynamical behaviour of two qubits in this case can be described by the following master equation

$$\frac{\partial\rho}{\partial t} = \frac{(N+1)\gamma}{2}(2J_-\rho J_+ - J_+J_-\rho - \rho J_+J_-) + \frac{N\gamma}{2}(2J_+\rho J_- - J_-J_+\rho - \rho J_-J_+), \quad (1)$$

where γ characterises the coupling strength between two qubits and the thermal reservoir. N is the mean photon number of the thermal environment; $J_\pm$ are the collective atomic operators defined as

$$J_\pm = \sum_{i=1}^{2}\sigma_\pm^{(i)}, \qquad \sigma_+^{(i)} = |1_i\rangle\langle 0_i|, \quad \sigma_-^{(i)} = |0_i\rangle\langle 1_i|, \quad (2)$$

where $|1_i\rangle$ and $|0_i\rangle$ are up and down states of the ith qubit, respectively.

We consider two kinds of initial states in this section. First, we consider the case when the initial state for the two qubits is assumed to be

$$\psi(0) = \sqrt{p}|10\rangle + \sqrt{1-p}|01\rangle, \quad (3)$$

where $0 \le p \le 1$. Then the explicit analytical solution of the master Eq. (1) can be obtained as follows:

$$\rho = \rho_{11}|11\rangle\langle 11| + \rho_{22}|10\rangle\langle 10| + \rho_{33}|01\rangle\langle 01| + \rho_{44}|00\rangle\langle 00| + \rho_{23}|10\rangle\langle 01| + \rho_{32}|01\rangle\langle 10|, \quad (4)$$

where

$$\begin{aligned}
\rho_{11} &= -\frac{A}{N+1} - \frac{r_7 B}{L}(N+1) + \frac{1}{4}\frac{r_4 r_1}{L}(N - \sqrt{N^2+N}) + \frac{1}{4}\frac{r_6 r_5}{L}(N + \sqrt{N^2+N}),\\
\rho_{23} &= \frac{1}{4}\frac{r_7}{L} + \frac{1}{8}\frac{r_6 r_5}{L} + \frac{1}{8}\frac{r_4 r_1}{L},\\
\rho_{44} &= -A\frac{N+1}{N^2} - \frac{r_7 B}{L}\frac{(N+1)}{N^2} - \frac{1}{4}\frac{r_4 r_1}{L}(N+1-\sqrt{N^2+N})\\
&\quad - \frac{1}{4}\frac{r_6 r_5}{L}(N+1+\sqrt{N^2+N}),\\
\rho_{32} &= \rho_{23}^*,
\end{aligned} \quad (5)$$

with

$$\begin{aligned}
L &= 3N^2 + 3N + 1,\\
A &= 8N^4 + 18N^3 + 15N^2 + \frac{11}{2}N + \frac{1}{2} - \sqrt{p-p^2}(32N^4 + 72N^3 + 52N^2 + 13N + 1),\\
B &= 12N^4 + 27N^3 + \frac{41}{2}N^2 + 6N + \frac{1}{2},\\
r_1 &= \exp[-(4N + 2 + 2\sqrt{N^2+N})t],\\
r_4 &= \big(2\sqrt{p-p^2}+1\big)[2N^2 + 2N + 1 + \sqrt{N^2+N}(2N+1)],\\
r_5 &= \exp[-(4N + 2 - 2\sqrt{N^2+N})t],\\
r_6 &= (2\sqrt{p-p^2}+1)[2N^2 + 2N + 1 - \sqrt{N^2+N}(2N+1)],\\
r_7 &= -(2N^2 + 2N + 1) + 2\sqrt{p-p^2}(4N^2 + 4N + 1).
\end{aligned} \quad (6)$$

The formulas of ρ_{22} and ρ_{33} are rather lengthy and we will not adduce them here. Next, we will investigate the entanglement of two qubits interacting with the common thermal environment by using above results.

Several different measures have been proposed to identify the entanglement between two qubits, and we choose the Wootters entanglement measure,[20,21] the concurrence C, defined as

$$C = \max(0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}), \quad (7)$$

where $\lambda_1, \ldots, \lambda_4$ are the eigenvalues of the matrix $\tilde{\boldsymbol{\rho}} = \boldsymbol{\rho}(\boldsymbol{\sigma}_y \otimes \boldsymbol{\sigma}_y)\boldsymbol{\rho}^*(\boldsymbol{\sigma}_y \otimes \boldsymbol{\sigma}_y)$; $\boldsymbol{\sigma}_y$ is the Pauli matrix; $\boldsymbol{\rho}$ is the density matrix representing the quantum state and the matrix elements are taken with respect to the basis $|1\rangle \otimes |1\rangle$, $|1\rangle \otimes |0\rangle$, $|0\rangle \otimes |1\rangle$, $|0\rangle \otimes |0\rangle$. The range of the concurrence is from 0 to 1. For unentangled qubits we have $C = 0$, whereas $C = 1$ for the maximally entangled qubits.

It is straightforward to compute analytically the concurrence C for the density matrix $\boldsymbol{\rho}$

$$C = \max\{0, C_1\}, \tag{8}$$

with

$$C_1 = 2(|\rho_{23}| - \sqrt{\rho_{11}\rho_{44}}). \tag{9}$$

When $C_1 \leq 0$, the two qubits is unentangled. When $1 > C_1 > 0$, the two qubits is entangled.

The entanglement C_1 of two qubits versus γt is shown in Fig. 1 for $p = 0.15$ and different average photon numbers of the thermal reservoir $N = 0.00001$, 0.01, 0.05, 0.1. From Fig. 1, we can see that entanglement can fall abruptly to zero, and remain to be zero for a period of time before the entanglement recovers to a stationary state value. The length of the time interval for the zero entanglement is dependent on the average photon number N. The bigger the average photon number N, the longer the state will stay in the disentangled separable state. This means that under the influence of thermal noise, two-qubit entanglement may terminate abruptly in a finite time. This phenomenon is termed 'sudden death' of entanglement,[22,23] and has been used to elucidate many new characteristics of entanglement evolution in systems of two qubits. Especially when $N = 0.1$, the initial entanglement vanishes for ever and no longer recovers, as shown in Fig. 1(d). This result can be explained as follows: As time $t \longrightarrow \infty$, from Eqs. (4)–(6), the stationary state can be obtained

$$\begin{aligned} \rho_{11} &= -\frac{A}{N+1} - \frac{r_7 B}{L}(N+1), \quad \rho_{23} = \frac{1}{4}\frac{r_7}{L}, \\ \rho_{44} &= -A\frac{N+1}{N^2} - \frac{r_7 B}{L}\frac{(N+1)}{N^2}, \quad \rho_{32} = \rho_{23}^*. \end{aligned} \tag{10}$$

From Eq. (8), the concurrence of the stationary state can be calculated

$$C = 2\max\left(0, \frac{1 - 2\sqrt{p - p^2}(6N^2 + 6N + 1)}{4(3N^2 + 3N + 1)}\right), \tag{11}$$

implying that the stationary state ρ is unentangled for the case $1 \leq 2\sqrt{p - p^2}(6N^2 + 6N + 1)$. When $p = 0.15$ and $N = 0.1$, we can obtain $C = \max(0, -0.07) = 0$. So the initial entanglement vanishes for ever.

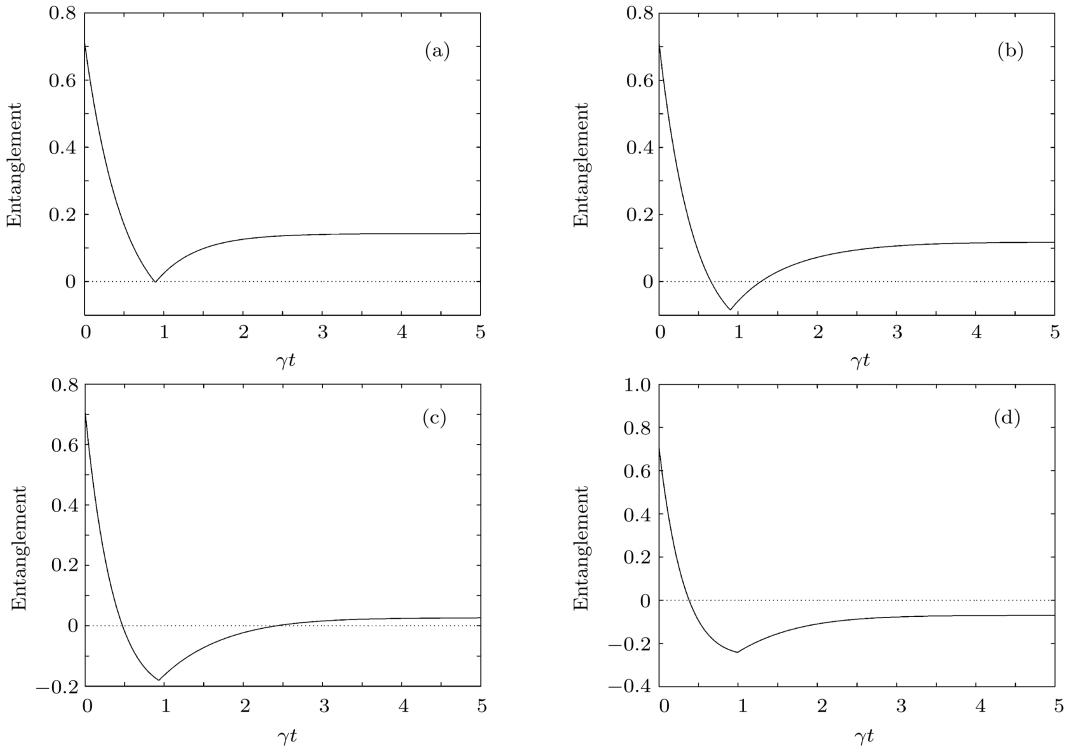


Fig. 1. The entanglement of two qubits versus γt for $p = 0.15$ and different average photon numbers of the thermal reservoir: (a) $N = 0.00001$, (b) $N = 0.01$, (c) $N = 0.05$, (d) $N = 0.1$.

The origin of dark period and the revival of the entanglement can be understood in terms of the populations of ρ_{11}, ρ_{44} and $2|\rho_{23}|$. Figure 2 shows the time evolution of the populations of ρ_{11}, ρ_{44} and $2|\rho_{23}|$ for $p = 0.15$ and the average photon number of the thermal reservoir $N = 0.00001$. As can be seen from the graphs, when $N = 0.00001$, i.e., N tends to zero (namely two qubits collectively interact with a zero-temperature thermal reservoir), the population of ρ_{11} tends to zero. According to Eq. (8), the entanglement becomes $2|\rho_{23}|$. When $2|\rho_{23}|$ collapses at some time, the entanglement collapses at the same time. From Eqs. (5), (6) and (8), as the average photon number $N \longrightarrow 0$, the initial entanglement decreases with time because of the thermal noise, and disappears at time

$$t_{\rm d} \approx \frac{1}{\gamma} \log \left(\frac{1 + 2\sqrt{p - p^2}}{1 - 2\sqrt{p - p^2}} \right)^{\frac{1}{2}}. \tag{12}$$

When $N = 0.00001$ and $p = 0.15$, $2|\rho_{23}|$ collapses at time $t_{\rm d} = 0.89/\gamma$. So, the entanglement collapses at the same time (see Figs. 2 and 1(a)).

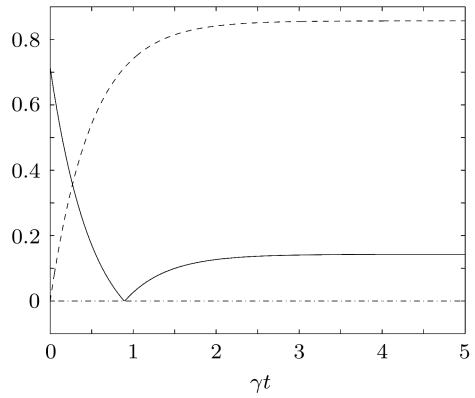


Fig. 2. The time evolution of ρ_{11} (the dashed-dotted line), ρ_{44} (the dashed line) and $2|\rho_{23}|$ (the solid line) for $p = 0.15$ and the average photon number of the thermal reservoir $N = 0.00001$.

Figure 3 shows the time evolution of the entanglement C_1, the populations of $2\sqrt{\rho_{11}\rho_{44}}$ and $2|\rho_{23}|$ for $p = 0.15$ and the average photon number of the thermal reservoir $N = 0.05$. From Fig. 3, we can see that when $2|\rho_{23}|$ intersects $2\sqrt{\rho_{11}\rho_{44}}$ the first time, the entanglement vanishes. When $2|\rho_{23}|$ intersects $2\sqrt{\rho_{11}\rho_{44}}$ the second time, the entanglement emerges. The appearance of the interval (dark periods) in which the entanglement vanishes is due to that $2|\rho_{23}|$ becomes smaller than $2\sqrt{\rho_{11}\rho_{44}}$.

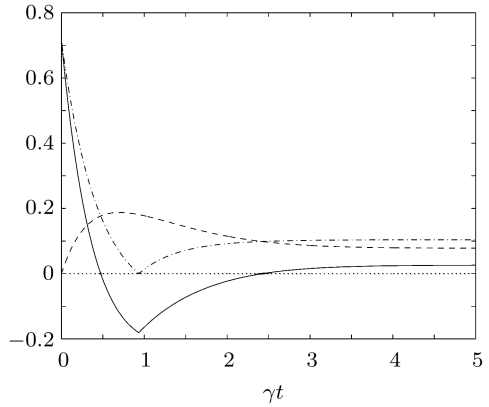


Fig. 3. The time evolution of the entanglement C_1 (the solid line), $2\sqrt{\rho_{11}\rho_{44}}$ (the dashed line) and $2|\rho_{23}|$ (the dot-dashed line) for $p = 0.15$ and the average photon number of the thermal reservoir $N = 0.05$.

Figure 4 shows the time evolution of the entanglement C_1, the populations of $2\sqrt{\rho_{11}\rho_{44}}$ and $2|\rho_{23}|$ for $p = 0.15$ and the average photon number of the thermal reservoir $N = 0.1$. In this case, we can see that when $2|\rho_{23}|$ intersects $2\sqrt{\rho_{11}\rho_{44}}$, the initial entanglement vanishes for ever, and no longer recovers. The reason is that when $2|\rho_{23}|$ intersects $2\sqrt{\rho_{11}\rho_{44}}$, the former becomes smaller than the latter in the time evolution. The initial entanglement can never recover even though the coherence $2|\rho_{23}|$ tends to a stationaty value.

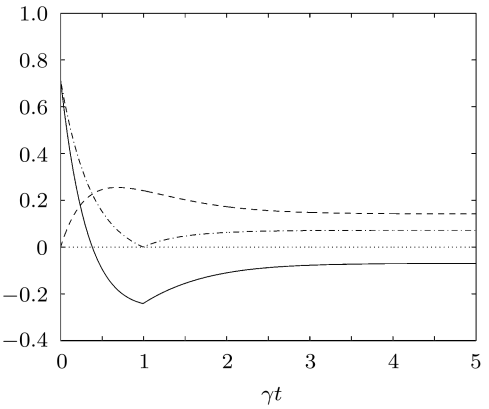


Fig. 4. The time evolution of the entanglement C_1 (the solid line), $2\sqrt{\rho_{11}\rho_{44}}$ (the dashed line) and $2|\rho_{23}|$ (the dot-dashed line) for $p = 0.15$ and the average photon number of the thermal reservoir $N = 0.1$.

The entanglement C_1 of two qubits versus γt is shown in Fig. 5 for $p = 0.05$ and different average photon numbers of the thermal reservoir $N = 0.00001$, 0.01, 0.05, 0.1, 0.2. As in the case of $p = 0.15$, shown in Fig. 1, the entanglement can fall abruptly to zero, and remain to be zero for a period of time before entanglement recovers to a stationary state value. The

length of the time interval for the zero entanglement is dependent on the average photon number N. The bigger the average photon number N, the longer the state will stay in the disentangled separable state. However, a comparison of Figs. 5(a) and 1(a), Figs. 5(b) and 1(b), Figs. 5(c) and 1(c) shows that the length of the time interval for the zero entanglement is also dependent on the degree of entanglement of the initial state. The bigger the initial degree of entanglement, the longer the states will stay in the disentangled separable state. Especially when $p = 0.15$(the initial degree of entanglement is 0.7141), $N = 0.1$, the initial entanglement vanishes for ever and no longer recovers, as shown in Fig. 1(d). While for $p = 0.05$ (the initial degree of entanglement is 0.4359), $N = 0.1$, the zero entanglement can recovers to a stationary state value (see Fig. 5(d)). However, as the average photon number N increases, the initial entanglement vanishes for ever and no longer recovers, as shown in Fig. 5(e).

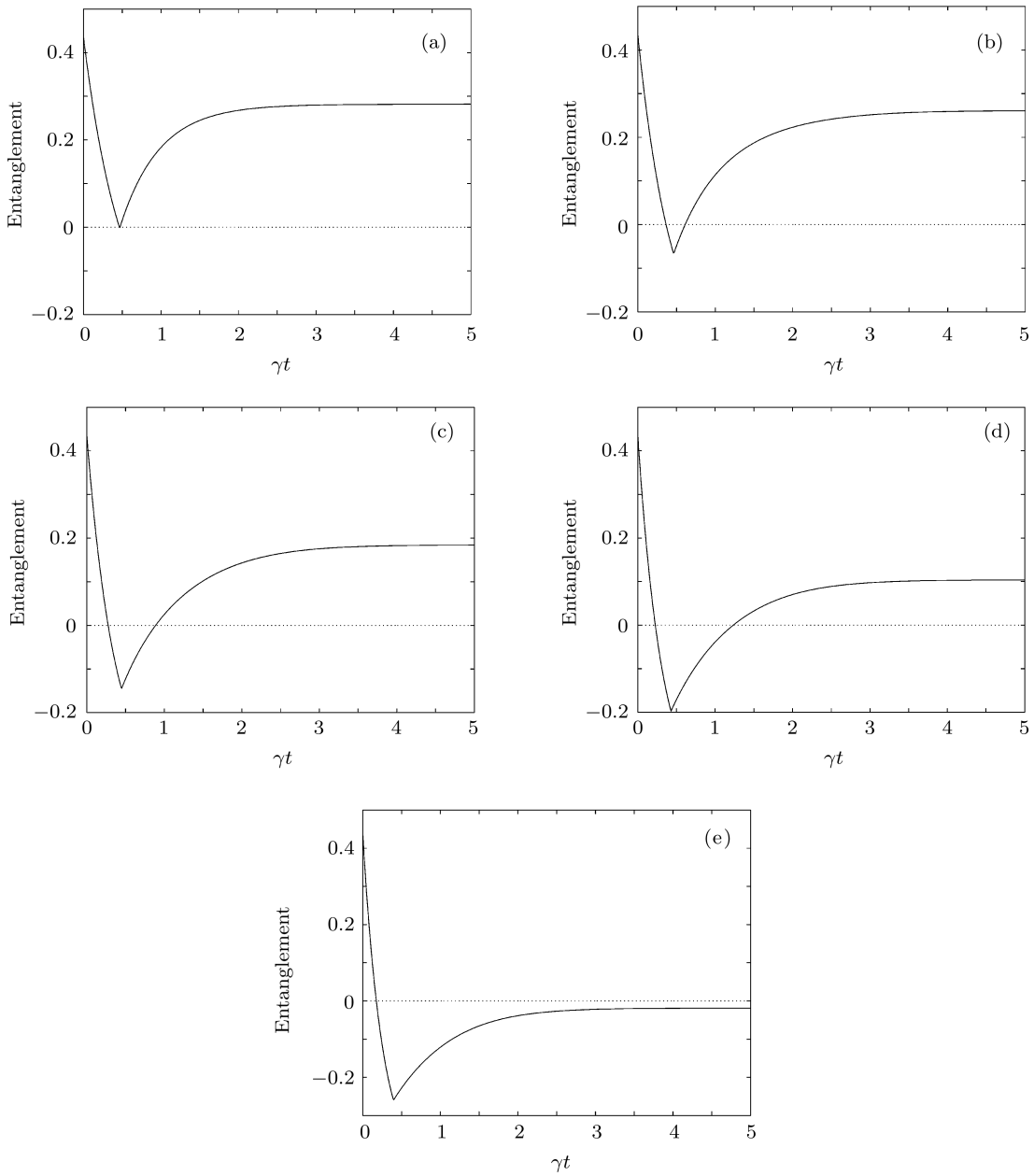


Fig. 5. The entanglement of two qubits versus γt for $p = 0.05$ and different average photon numbers of the thermal reservoir: (a) $N = 0.00001$, (b) $N = 0.01$, (c) $N = 0.05$, (d) $N = 0.1$, (e) $N = 0.2$.

Next, we consider the case when the initial state for two qubits is assumed to be

$$\psi(0) = \sqrt{p}|10\rangle - \sqrt{1-p}|01\rangle, \tag{13}$$

where $0 \leq p \leq 1$. After some calculation, we obtain the analytical solution of the master Eq. (1). Then the dynamical behaviour of the entanglement between the two qubits can be obtained.

Figure 6 shows the time evolution of the entanglement C_1 for $p = 0.15$ and $p = 0.5$ and different average photon numbers of the thermal reservoir $N = 0.00001$, 0.1, 1. In this case, the entanglement cannot fall abruptly to zero. The steady value of entanglement is larger than their initial value (see Figs. 6(a)–6(c)). Moreover, the smaller the average photon number N, the bigger the steady value of the entanglement. This indicates the possibility of obtaining entangled states with a larger amount of entanglement starting from entangled states with a smaller amount of entanglement. It is noted that the initial state with $p = 0.5$ always maintains a maximal entanglement regardless of time and temperature, as shown in Fig. 6(d). It seems that the two quibits in this initial state do not interact with the thermal field all the time. The reason is that the (Bell) state $|\psi^-\rangle = (|10\rangle - |01\rangle)/\sqrt{2}$ (corresponding to the initial condition $p = 0.5$) is in fact a 'dark' state of the system, i.e., $J_-|\psi^-\rangle = J_+|\psi^-\rangle = 0$, implying that this state is unaffected by coupling to the thermal reservoir. This result is the same as that in Ref. [15].

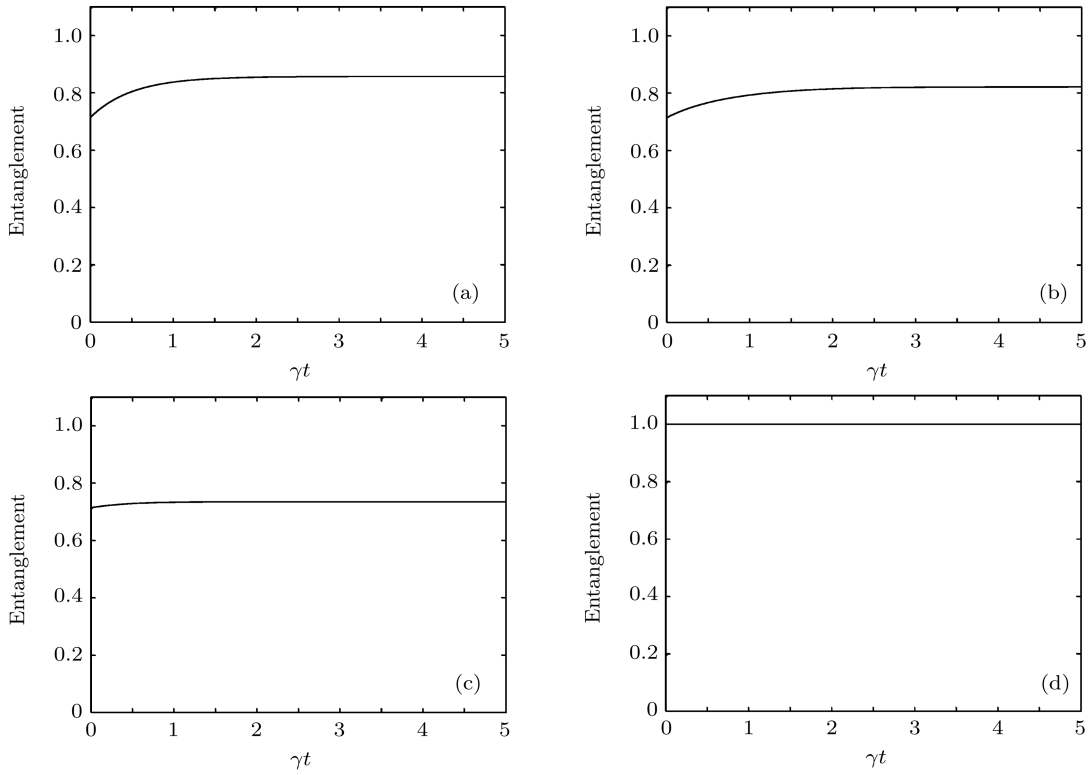


Fig. 6. The entanglement of two qubits versus γt for (a) $p = 0.15, N = 0.00001$, (b) $p = 0.15, N = 0.1$, (c) $p = 0.15, N = 1$, (d) $p = 0.5, N = 1$.

3. Conclusion

In this paper, we have investigated the entanglement of two qubits coupled collectively to a common thermal environment and found that the collective decay can lead to a revival of the entanglement that has already been detroyed. We also show that the ability of the system to revival of entanglement relies on the mean photon number of the thermal environment and the degree of entanglement of the initial state. A deeper understanding of entanglement decay processes is of interest in quantum computation and in any branch of quantum information science where preservation of entanglement is essential for the action of desired operations and devices.

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