Novel pinning control strategies for synchronisation of complex networks with nonlinear coupling dynamics*

Liu Zhao-Bing(刘兆冰), Zhang Hua-Guang(张化光)†, and Sun Qiu-Ye(孙秋野)

College of Information Science and Engineering, Northeastern University, Shenyang 110819, *China*

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This paper considers the global stability of controlling an uncertain complex network to a homogeneous trajectory of the uncoupled system by a local pinning control strategy. Several sufficient conditions are derived to guarantee the network synchronisation by investigating the relationship among pinning synchronisation, network topology, and coupling strength. Also, some fundamental and yet challenging problems in the pinning control of complex networks are discussed: (1) what nodes should be selected as pinned candidates? (2) How many nodes are needed to be pinned for a fixed coupling strength? Furthermore, an adaptive pinning control scheme is developed. In order to achieve synchronisation of an uncertain complex network, the adaptive tuning strategy of either the coupling strength or the control gain is utilised. As an illustrative example, a network with the Lorenz system as node self-dynamics is simulated to verify the efficacy of theoretical results.

Keywords: uncertain complex network, pinning control, synchronisation, adaptive tuning

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1. Introduction

Since 1998, the new discoveries of small-world[1] and scale-free connectivity patterns[2] in many natural and artificial complex networking systems have attracted very wide attention, yielding fruitful results which enrich and deepen the understanding of real-world complex networks, and put an important step forward from the random graph theory.[3] The control and stabilisation of large-scale dynamic systems has been one of the main issues with wide interest over the past decades,[4] and the latest generalisation of nonlinear small-gain theorems has been stated to find their applications in analysis and control synthesis for large-scale network systems.[5,6] It is closer to the interest of this paper that the authors have stabilised a complex bi-directed or directed network to its homogeneous static state with a small fraction of vertices placing local feedback controllers,[7−18] i.e., the so-called pinning control algorithm,[19,20] which was first proposed to control the multi-mode laser systems. It was concluded that the pinning strategies should be more selective according to the detailed complex network topology, and for a class of scale-free networks, the degree-based pinning strategy, i.e., pinning the vertex set with the largest connection degree, yields better stabilisability than randomly placing pinning controllers in the network.[7] More recently, the concept of Lyapunov stability for complex dynamical networks has been proposed and applied to the pinning control algorithm, and the network stability (controllability) is converted into the measure of the negative definiteness of one simple matrix characterising the network topology,[9,14,21−34] which can be measured by the eigenvalues of such matrices.[10] Also, the authors used Lyapunov function to study pinning control in complex networks and gave conditions based on the algebraic properties of the coupling matrix guaranteeing stability.[10,13]

However, to the best of our knowledge, how to choose pinned nodes for general complex dynamical networks still remains to be a challenging problem. This paper aims at developing effective pinned-node selection schemes and practical pinning control schemes for the synchronisation of uncertain complex networks with general topology. Considering the effect of nonlinear coupling dynamics, this paper deeply investigates the relationship among pinning synchronisation, network topology, coupling strength, and selection of pinned nodes. Several sufficient conditions are derived to guarantee to stabilise the proposed uncertain complex network to a homogenous solution

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†Corresponding author. E-mail: zhanghuaguang@mail.neu.edu.cn

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with pinning controllers. Moreover, an adaptive pinning control scheme is developed to achieve the global network synchronisation by using the adaptive tuning strategy of either the coupling strength or the control gains.

The rest of this paper is organised as follows. In Section 2, some preliminaries are presented, and the model is described. In Section 3, the main theorems and corollaries for pinning synchronisation of the uncertain complex network are derived. A representative example and simulations are presented in Section 4. Finally, some conclusions are drawn in Section 5.

2. Preliminaries and model description

First, we introduce the notations employed throughout the paper. $\boldsymbol{W}^{\mathrm{T}}$ defines the transpose of a matrix $\boldsymbol{W}$. We denote by $\boldsymbol{W} > 0$ (≥ 0) that $\boldsymbol{W}$ is positive (semi-positive) definite. $\lambda_{\max}$ ($\boldsymbol{W}$) ($\lambda_{\min}$ ($\boldsymbol{W}$)) is the maximum (minimum) eigenvalue of $\boldsymbol{W}$. $\boldsymbol{I}_m$ is the identity matrix with dimension m. $\|\cdot\|$ refers to Euclidean vector norm or the induced matrix 2-norm. $|\cdot|$ denotes the absolute value. The symbol $\otimes$ is the Kroneck product.

Consider a complex dynamical network consisting of N identical nodes with nonlinear diffusive coupling dynamics, with each node being an n-dinamical system, described by

$$\begin{aligned}&\dot{\boldsymbol{x}}^i(t)\\ &= \boldsymbol{g}(\boldsymbol{x}^i(t),t) + \boldsymbol{h}^i(x^1,x^2,\ldots,x^N)\\ &\quad + c\sum_{j=1,j\neq i}^{N} l_{ij}\boldsymbol{\Gamma}(\boldsymbol{x}^j(t)-\boldsymbol{x}^i(t)), \quad i=1,\ldots,N, \end{aligned} \tag{1}$$

where $\boldsymbol{x}^i(t) = [x_1^i(t),\ldots,x_n^i(t)]^{\mathrm{T}} \in \mathbb{R}^n$ denotes the state variable vector of the ith individual vertex in the coupled system, $t \in \mathbb{R}^+ = [0,\infty)$ denotes the continuous time, $\boldsymbol{g}: \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ is a continuous map representing the self-dynamics of node i, $\boldsymbol{h}^i$ $(x^1,x^2,\ldots,x^N)$ denotes the unknown nonlinear couplings of node i; on the other hand, the last part denotes the known linear couplings, where the positive scalar c denotes the coupling strength, a positive diagonal matrix $\boldsymbol{\Gamma} \in \mathbb{R}^{n\times n}$ denotes the inner coupling matrix, and $\boldsymbol{L} = (l_{ij})_{N\times N}$ is the coupling configuration matrix representing the topological structure of the network, where l_{ij} is defined as follows: if there exists a connection between node i and node j, then $l_{ij} = l_{ji} = 1$; otherwise, $l_{ij} = l_{ji} = 0 (j \neq i)$; and the diagonal elements of matrix $\boldsymbol{L}$ are defined by $l_{ii} = -\sum_{j=1,j\neq i}^{N} l_{ij}$, which ensures the diffusion that $\sum_{j=1}^{N} l_{ij} = 0$. Equivalently, network (1) can be rewritten in a simpler form as follows:

$$\begin{aligned}\dot{\boldsymbol{x}}^i(t) &= \boldsymbol{g}(\boldsymbol{x}^i(t),t) + \boldsymbol{h}^i(x^1,x^2,\ldots,x^N)\\ &\quad + c\sum_{j=1}^{N} l_{ij}\boldsymbol{\Gamma}\boldsymbol{x}^j(t), \quad i=1,\ldots,N. \end{aligned} \tag{2}$$

3. Pinning synchronisation of uncertain complex dynamical networks

In this section, we study the pinning control problem of an uncertain complex dynamical network described by Eq. (2) to a homogeneous orbit satisfying:

$$\dot{\boldsymbol{s}}(t) = \boldsymbol{g}(\boldsymbol{s}(t),t), \tag{3}$$

where $\boldsymbol{s}(t)$ can be an equilibrium, limit cycle, or even a chaotic attractor of the uncoupled system. For this purpose, we apply the pinning control strategy with local feedback controllers to a pinned vertex set, with a very small fraction. Suppose that the nodes $i_1, i_2, \ldots, i_p$ are chosen, where $p = \llcorner \delta N \lrcorner$ represents the integer part of the real number δN. Without loss of generality, rearrange the order of the nodes in the network, and let the first p nodes be controlled, then the pinning controlled network will be described as follows:

$$\begin{aligned}\dot{\boldsymbol{x}}^i(t) &= \boldsymbol{g}(\boldsymbol{x}^i(t),t) + \boldsymbol{h}^i(x^1,x^2,\ldots,x^N)\\ &\quad + c\sum_{j=1}^{N} l_{ij}\boldsymbol{\Gamma}\boldsymbol{x}^j(t) + \boldsymbol{u}^i(t), \quad i=1,\ldots,p,\\ \dot{\boldsymbol{x}}^i(t) &= \boldsymbol{g}(\boldsymbol{x}^i(t),t) + \boldsymbol{h}^i(x^1,x^2,\ldots,x^N)\\ &\quad + c\sum_{j=1}^{N} l_{ij}\boldsymbol{\Gamma}\boldsymbol{x}^j(t), \quad i=p+1,\ldots,N, \end{aligned} \tag{4}$$

where

$$\boldsymbol{u}^i(t) = -c\epsilon^i\boldsymbol{\Gamma}(\boldsymbol{x}^i(t)-\boldsymbol{s}(t)) \in \mathbb{R}^n, \quad i=1,\ldots,p \tag{5}$$

are n-dimensional linear feedback controllers with all the control gains $\epsilon^i > 0$.

The objective of control here is to find some appropriate controllers (5) such that the solutions of the controlled network (4) synchronise with the solution of Eq. (3), in the sense that

$$\lim_{t\to\infty} \|\boldsymbol{x}^i(t) - \boldsymbol{s}(t)\| = 0, \quad i=1,\ldots,N. \tag{6}$$

It is worth noting that the uncertain diffusive coupling $\boldsymbol{h}^i$ will vanish when all the nodes are in the stationary manifold $x^1 = x^2 = \cdots = x^N$. According to this, $\boldsymbol{h}^i$ can be written by $\boldsymbol{h}^i = \boldsymbol{Y}^i\ (\boldsymbol{G} \otimes \boldsymbol{I}_n)\boldsymbol{X}$, in which a suitable matrix $\boldsymbol{Y}^i \in \mathbb{R}^{n\times n(N-1)}$ depends on $\boldsymbol{X} = \left[x^{1\mathrm{T}}, x^{2\mathrm{T}}, \ldots, x^{N\mathrm{T}}\right]^{\mathrm{T}}$, and the matrix $(\boldsymbol{G} \otimes \boldsymbol{I}_n) \in \mathbb{R}^{n(N-1)\times nN}$. From the diffusive property, $\boldsymbol{h}^i = 0$, when $\boldsymbol{x}^i(t) = \boldsymbol{s}(t)$, it is obvious that $\boldsymbol{G}[1, 1, \ldots, 1]^{\mathrm{T}} = 0$. Let the error signal be $\boldsymbol{e}^i(t) = \boldsymbol{x}^i(t) - \boldsymbol{s}(t)$ $(i = 1, 2, \ldots, N)$. Define $\boldsymbol{E} = \left[e^{1\mathrm{T}}, e^{2\mathrm{T}}, \ldots, e^{N\mathrm{T}}\right]^{\mathrm{T}}$, $\boldsymbol{H} = \left[h^{1\mathrm{T}}, h^{2\mathrm{T}}, \ldots, h^{N\mathrm{T}}\right]^{\mathrm{T}}$, and $\boldsymbol{Y} = \left[Y^{1\mathrm{T}}, Y^{2\mathrm{T}}, \ldots, Y^{N\mathrm{T}}\right]^{\mathrm{T}}$, then

$$\begin{aligned}\boldsymbol{H}^{\mathrm{T}}\boldsymbol{H} &= \boldsymbol{X}^{\mathrm{T}}(\boldsymbol{G}^{\mathrm{T}} \otimes \boldsymbol{I}_n)\boldsymbol{Y}^{\mathrm{T}}\boldsymbol{Y}(\boldsymbol{G} \otimes \boldsymbol{I}_n)\boldsymbol{X} \\ &= \boldsymbol{E}^{\mathrm{T}}(\boldsymbol{G}^{\mathrm{T}} \otimes \boldsymbol{I}_n)\boldsymbol{Y}^{\mathrm{T}}\boldsymbol{Y}(\boldsymbol{G} \otimes \boldsymbol{I}_n)\boldsymbol{E}, \quad (7)\end{aligned}$$

where $\boldsymbol{I}_n$ is a unit diagonal matrix with n diagonal elements being unity.

Next, the pinning synchronisation of network (4) is investigated. Subtracting Eq. (3) from Eq. (5) yields the following error dynamical network:

$$\begin{aligned}\dot{\boldsymbol{e}}^i(t) &= \boldsymbol{g}(\boldsymbol{x}^i(t), t) - \boldsymbol{g}(\boldsymbol{s}(t), t) + \boldsymbol{h}^i(x^1, x^2, \ldots, x^N) \\ &\quad + c\sum_{j=1}^{N} l_{ij}\boldsymbol{\Gamma}\boldsymbol{e}^j(t) + \boldsymbol{u}^i(t), \quad i = 1, \ldots, p, \\ \dot{\boldsymbol{e}}^i(t) &= \boldsymbol{g}(\boldsymbol{x}^i(t), t) - \boldsymbol{g}(\boldsymbol{s}(t), t) + \boldsymbol{h}^i(x^1, x^2, \ldots, x^N) \\ &\quad + c\sum_{j=1}^{N} l_{ij}\boldsymbol{\Gamma}\boldsymbol{e}^j(t), \quad i = p+1, \ldots, N, \quad (8)\end{aligned}$$

where $\boldsymbol{e}^i(t) = \boldsymbol{x}^i(t) - \boldsymbol{s}(t),\ i = 1, 2, \ldots, N$.

To derive the stability criteria for the pinning controlled network, the following assumptions and lemma are needed.

Assumption 1 Given a matrix satisfying $\boldsymbol{Q} = \boldsymbol{Q}^{\mathrm{T}} > 0$, there exists a scalar $\theta < 0$ such that

$$\begin{aligned}&(\boldsymbol{x} - \boldsymbol{y})^{\mathrm{T}}\boldsymbol{Q}(\boldsymbol{g}(\boldsymbol{x}, t) - \boldsymbol{g}(\boldsymbol{y}, t)) \\ &\leq -\theta(\boldsymbol{x} - \boldsymbol{y})^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{\Gamma}(\boldsymbol{x} - \boldsymbol{y}) \quad (9)\end{aligned}$$

holds for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and $t \in \mathbb{R}^+$.

Assumption 2 Based on Eq. (7), there exist two known symmetric positive semidefinite matrices $\boldsymbol{M}_1 \in \mathbb{R}^{N\times N}$ and $\boldsymbol{M}_2 \in \mathbb{R}^{n\times n}$ such that

$$\boldsymbol{H}^{\mathrm{T}}\boldsymbol{H} \leq \boldsymbol{E}^{\mathrm{T}}(\boldsymbol{M}_1 \otimes \boldsymbol{M}_2)\boldsymbol{E}. \quad (10)$$

Lemma 1[8] If $\boldsymbol{L}$ is irreducible, $l_{ij} = l_{ji} \geq 0$ for $i \neq j$, and $\sum_{j=1}^{N} l_{ij} = 0$, for all $i = 1, 2, \ldots, N$, then all eigenvalues of the matrix

$$\begin{pmatrix} l_{11} - \alpha & l_{12} & \ldots & l_{1N} \\ l_{21} & l_{22} & \ldots & l_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ l_{N1} & l_{N2} & \ldots & l_{NN} \end{pmatrix}$$

are negative for any positive constant α.

3.1. Pinning control for uncertain complex networks to a homogeneous orbit

In this subsection, a general pinning synchronisation criterion is obtained.

Theorem 1 Suppose that Assumptions 1 and 2 hold. The controlled undirected uncertain complex network (4) is globally synchronised if the following conditions are satisfied:

$$\boldsymbol{Q}\boldsymbol{\Gamma} + \boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{Q} \geq a_1\boldsymbol{M}_2, \quad (11a)$$

$$\boldsymbol{M}_2 \geq a_2\boldsymbol{Q}^2, \quad (11b)$$

$$a_2(a_1(-\theta\boldsymbol{I}_N + c\boldsymbol{L} - c\boldsymbol{\Phi}) + \boldsymbol{M}_1) + \boldsymbol{I}_N \leq 0, \quad (11c)$$

where a_1 and a_2 are positive scalars; $\boldsymbol{\Phi} = \mathrm{diag}(\underbrace{\epsilon^1, \ldots, \epsilon^p}_{p}, \underbrace{0, \ldots, 0}_{N-p})$; θ is the passivity degree in the sense that the nodes need energy from outside to become stable with respect to the objective orbit when $\theta < 0$, whereas the nodes are already stable when $\theta > 0$.

Proof Consider the following Lyapunov function for the whole network (4):

$$V(t) = 2\sum_{i=1}^{N} \boldsymbol{e}^{i\mathrm{T}}(t)\boldsymbol{Q}\boldsymbol{e}^i(t). \quad (12)$$

Its time derivative along the trajectories of Eq. (8) is given by

$$\begin{aligned}&\dot{V}(t) \\ &= 2\sum_{i=1}^{N} \boldsymbol{e}^{i\mathrm{T}}(t)\boldsymbol{Q}\dot{\boldsymbol{e}}^i(t)\end{aligned}$$

$$
\begin{aligned}
&= 2\sum_{i=1}^{N} \boldsymbol{e}^{i\mathrm{T}}(t)\boldsymbol{Q}\left[\boldsymbol{g}(\boldsymbol{x}^i(t),t)-\boldsymbol{g}(\boldsymbol{s}(t),t)+\boldsymbol{h}^i(x^1,x^2,\ldots,x^N)+c\sum_{j=1}^{N} l_{ij}\boldsymbol{\Gamma}\boldsymbol{e}^j(t)\right]-2c\sum_{i=1}^{p}\epsilon^i\boldsymbol{e}^{i\mathrm{T}}(t)\boldsymbol{Q}\boldsymbol{\Gamma}\boldsymbol{e}^i(t)\\
&\leq 2\sum_{i=1}^{N} \boldsymbol{e}^{i\mathrm{T}}(t)\boldsymbol{Q}\left[-\theta\boldsymbol{\Gamma}\boldsymbol{e}^i(t)+\boldsymbol{h}^i(x^1,x^2,\ldots,x^N)+c\sum_{j=1}^{N} l_{ij}\boldsymbol{\Gamma}\boldsymbol{e}^j(t)\right]-2c\sum_{i=1}^{p}\epsilon^i\boldsymbol{e}^{i\mathrm{T}}(t)\boldsymbol{Q}\boldsymbol{\Gamma}\boldsymbol{e}^i(t)\\
&\leq \boldsymbol{e}^{\mathrm{T}}(t)\left[(-\theta\boldsymbol{I}_N+c\boldsymbol{L}-c\boldsymbol{\Phi})\otimes(\boldsymbol{Q}\boldsymbol{\Gamma}+\boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{Q})+\boldsymbol{M}_1\otimes\boldsymbol{M}_2\right]\boldsymbol{e}(t)+2\boldsymbol{e}^{\mathrm{T}}(t)(\boldsymbol{I}_N\otimes\boldsymbol{Q})\boldsymbol{H}-\boldsymbol{H}^{\mathrm{T}}\boldsymbol{H}\\
&\leq \boldsymbol{e}^{\mathrm{T}}((a_1(-\theta\boldsymbol{I}_N+c\boldsymbol{L}-c\boldsymbol{\Phi})+\boldsymbol{M}_1)\otimes\boldsymbol{M}_2)\boldsymbol{e}(t)+2\boldsymbol{e}^{\mathrm{T}}(t)(\boldsymbol{I}_N\otimes\boldsymbol{Q})\boldsymbol{H}-\boldsymbol{H}^{\mathrm{T}}\boldsymbol{H}\\
&= [\,\boldsymbol{e}^{\mathrm{T}}(t)\ \boldsymbol{H}^{\mathrm{T}}\,]\boldsymbol{\Xi}\begin{bmatrix}\boldsymbol{e}(t)\\ \boldsymbol{H}\end{bmatrix},
\end{aligned}
\tag{13}
$$

where $\boldsymbol{e}(t)=[e^{1\mathrm{T}}(t),e^{2\mathrm{T}}(t),\ldots,e^{N\mathrm{T}}(t)]^{\mathrm{T}}$, and $\boldsymbol{\Xi}=\begin{pmatrix}(a_1(-\theta\boldsymbol{I}_N+c\boldsymbol{L}-c\boldsymbol{\Phi})+\boldsymbol{M}_1)\otimes\boldsymbol{M}_2 & \boldsymbol{I}_N\otimes\boldsymbol{Q}\\ \boldsymbol{I}_N\otimes\boldsymbol{Q} & -\boldsymbol{I}_{nN}\end{pmatrix}$.

From inequalities (11b) and (11c), we can easily obtain the following inequality:

$$
(-a_2(a_1(-\theta\boldsymbol{I}_N+c\boldsymbol{L}-c\boldsymbol{\Phi})+\boldsymbol{M}_1)-\boldsymbol{I}_N)\otimes\left(\frac{1}{a_2}\boldsymbol{M}_2-\boldsymbol{Q}^2\right)\geq 0, \tag{14}
$$

which is equivalent to

$$
\begin{aligned}
&(-a_2(a_1(-\theta\boldsymbol{I}_N+c\boldsymbol{L}-c\boldsymbol{\Phi})+\boldsymbol{M}_1))\otimes\frac{1}{a_2}\boldsymbol{M}_2+\boldsymbol{I}_N\otimes\boldsymbol{Q}^2\\
\geq\ &(-a_2(a_1(-\theta\boldsymbol{I}_N+c\boldsymbol{L}-c\boldsymbol{\Phi})+\boldsymbol{M}_1))\otimes\boldsymbol{Q}^2+\boldsymbol{I}_N\otimes\frac{1}{a_2}\boldsymbol{M}_2\\
\geq\ &0.
\end{aligned}
\tag{15}
$$

Based on Schur complements, inequality (15) is equivalent to

$$
(-\boldsymbol{\Xi})=\begin{pmatrix}-a_2(a_1(-\theta\boldsymbol{I}_N+c\boldsymbol{L}-c\boldsymbol{\Phi})+\boldsymbol{M}_1)\otimes\frac{1}{a_2}\boldsymbol{M}_2 & -\boldsymbol{I}_N\otimes\boldsymbol{Q}\\ -\boldsymbol{I}_N\otimes\boldsymbol{Q} & \boldsymbol{I}_{nN}\end{pmatrix}\geq 0. \tag{16}
$$

Therefore, $\boldsymbol{\Xi}\leq 0$, and it is easy to find that $\dot{V}(t)$ is negative semi-definite, so the undirected network (4) is globally synchronised under the given linear feedback pinning controllers. The proof is, thus, completed.

3.2. Coupling strength in pinning control

In this subsection, some criteria for choosing an appropriate coupling strength are established.

Corollary 1 If inequalities (11a), (11b), and the following characteristic inequality

$$
c>\frac{-a_1a_2\theta+a_2\lambda_{\max}(\boldsymbol{M}_1)+1}{|\lambda_{\max}(\boldsymbol{L}-\boldsymbol{\Phi})|} \tag{17}
$$

are satisfied, then controlled network (4) can be stabilised to orbit $s(t)$.

Proof By lemma 1, it is not difficult to see that $\boldsymbol{L}-\boldsymbol{\Phi}^*$ is negative definite, where $\boldsymbol{\Phi}^*$ corresponds to only one controller, i.e. $p=1$. Notice that $\lambda_{\max}(\boldsymbol{L}-\boldsymbol{\Phi})\leq\lambda_{\max}(\boldsymbol{L}-\boldsymbol{\Phi}^*)<0$. From inequality (17), one has $c(\boldsymbol{L}-\boldsymbol{\Phi})-\theta\boldsymbol{I}_N+(1/(a_1a_2))(a_2\boldsymbol{M}_1+\boldsymbol{I}_N)\leq c\lambda_{\max}(\boldsymbol{L}-\boldsymbol{\Phi})-\theta+(1/(a_1a_2))(a_2\lambda_{\max}(\boldsymbol{M}_1)+1)<0$, so inequality (11c) is satisfied. The proof is completed.

Remark 1 An important issue is how large the coupling strength should be chosen to guarantee the network synchronisation. When a complex network is pinned by a single controller,[8] i.e. $p=1$, it requires a very large coupling strength. However, in practice, it cannot be realised if the coupling strength is too large. It is worth noting that the theoretical coupling strength given in inequality (17) is too conservative, usually much larger than the needed value. Clearly, it is desirable to make the coupling strength as small as possible. Here, the adaptive technique is adopted to achieve this goal.

The selected pinning controllers in Eq. (5), associated with the adaptive coupling law, lead to

$$\dot{\boldsymbol{x}}^i(t) = \boldsymbol{g}(\boldsymbol{x}^i(t),t) + \boldsymbol{h}^i(x^1,x^2,\ldots,x^N) + c\sum_{j=1}^{N} l_{ij}\boldsymbol{\Gamma}\boldsymbol{x}^j(t) - c(t)\epsilon^i\boldsymbol{\Gamma}(\boldsymbol{x}^i(t)-\boldsymbol{s}(t)),\quad i=1,\ldots,p,$$

$$\dot{\boldsymbol{x}}^i(t) = \boldsymbol{g}(\boldsymbol{x}^i(t),t) + \boldsymbol{h}^i(x^1,x^2,\ldots,x^N) + c(t)\sum_{j=1}^{N} l_{ij}\boldsymbol{\Gamma}\boldsymbol{x}^j(t),\ i=p+1,\ldots,N,$$

$$\dot{c}(t) = \eta\sum_{i=1}^{N}(\boldsymbol{x}^i-\boldsymbol{s}(t))^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{\Gamma}(\boldsymbol{x}^i-\boldsymbol{s}(t)), \tag{18}$$

where $\boldsymbol{Q}$ is a symmetric and positive definite matrix, η is a small positive constant.

Theorem 2 If inequalities (11a), (11b), and the inequality

$$a_2(a_1(-\theta\boldsymbol{I}_N + c(t)(\boldsymbol{L}-\boldsymbol{\Phi}+\kappa\boldsymbol{I}_N)-\kappa\bar{c}) + \boldsymbol{M}_1) + \boldsymbol{I}_N \le 0 \tag{19}$$

are satisfied, then the adaptive controlled undirected network (18) is globally synchronised for a small constant $\eta > 0$, where κ and $\bar{c}$ are positive constants to be determined below.

Proof Consider the Lyapunov function candidate

$$V(t) = \sum_{i=1}^{N} \boldsymbol{e}^{i\,\mathrm{T}}(t)\boldsymbol{Q}\boldsymbol{e}^i(t) + \frac{\kappa}{\eta}(c(t)-\bar{c})^2, \tag{20}$$

then the derivative of $V(t)$ along the trajectories of (18) will be

$$\begin{aligned}
\dot{V}(t) &= 2\sum_{i=1}^{N} \boldsymbol{e}^{i\,\mathrm{T}}(t)\boldsymbol{Q}\dot{\boldsymbol{e}}^i(t) + 2\kappa(c(t)-\bar{c})\sum_{i=1}^{N} \boldsymbol{e}^{i\,\mathrm{T}}(t)\boldsymbol{Q}\boldsymbol{\Gamma}\boldsymbol{e}^i(t)\\
&= 2\sum_{i=1}^{N} \boldsymbol{e}^{i\,\mathrm{T}}(t)\boldsymbol{Q}\left[\boldsymbol{g}(\boldsymbol{x}^i(t),t) - \boldsymbol{g}(\boldsymbol{s}(t),t) + \boldsymbol{h}^i(x^1,x^2,\ldots,x^N) + c(t)\sum_{j=1}^{N} l_{ij}\boldsymbol{\Gamma}\boldsymbol{e}^j(t)\right]\\
&\quad - 2c(t)\sum_{i=1}^{p}\epsilon^i\boldsymbol{e}^{i\,\mathrm{T}}(t)\boldsymbol{Q}\boldsymbol{\Gamma}\boldsymbol{e}^i(t) + 2\kappa(c(t)-\bar{c})\sum_{i=1}^{N} \boldsymbol{e}^{i\,\mathrm{T}}(t)\boldsymbol{Q}\boldsymbol{\Gamma}\boldsymbol{e}^i(t)\\
&\le \boldsymbol{e}^{\mathrm{T}}(t)\left[(-\theta\boldsymbol{I}_N + c(t)(\boldsymbol{L}-\boldsymbol{\Phi}+\kappa\boldsymbol{I}_N)-\kappa\bar{c})\otimes(\boldsymbol{Q}\boldsymbol{\Gamma}+\boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{Q}) + \boldsymbol{M}_1\otimes\boldsymbol{M}_2\right]\boldsymbol{e}(t)\\
&\quad + 2\boldsymbol{e}^{\mathrm{T}}(t)(\boldsymbol{I}_N\otimes\boldsymbol{Q})\boldsymbol{H} - \boldsymbol{H}^{\mathrm{T}}\boldsymbol{H}\\
&\le \boldsymbol{e}^{\mathrm{T}}((a_1(-\theta\boldsymbol{I}_N + c(t)(\boldsymbol{L}-\boldsymbol{\Phi}+\kappa\boldsymbol{I}_N)-\kappa\bar{c}) + \boldsymbol{M}_1)\otimes\boldsymbol{M}_2)\boldsymbol{e}(t) + 2\boldsymbol{e}^{\mathrm{T}}(t)(\boldsymbol{I}_N\otimes\boldsymbol{Q})\boldsymbol{H} - \boldsymbol{H}^{\mathrm{T}}\boldsymbol{H}\\
&= [\,\boldsymbol{e}^{\mathrm{T}}(t)\ \boldsymbol{H}^{\mathrm{T}}\,]\boldsymbol{\Xi}\begin{bmatrix}\boldsymbol{e}(t)\\ \boldsymbol{H}\end{bmatrix},
\end{aligned} \tag{21}$$

where

$$\boldsymbol{e}(t) = [e^{1\,\mathrm{T}}(t), e^{2\,\mathrm{T}}(t),\ldots,e^{N\,\mathrm{T}}(t)]^{\mathrm{T}},$$

$$\boldsymbol{\Xi} = \begin{pmatrix}(a_1(-\theta\boldsymbol{I}_N + c(t)(\boldsymbol{L}-\boldsymbol{\Phi}+\kappa\boldsymbol{I}_N)-\kappa\bar{c}) + \boldsymbol{M}_1)\otimes\boldsymbol{M}_2 & \boldsymbol{I}_N\otimes\boldsymbol{Q}\\ \boldsymbol{I}_N\otimes\boldsymbol{Q} & -\boldsymbol{I}_{nN}\end{pmatrix}.$$

In a way similar to the proof of Theorem 1, $\boldsymbol{\Xi} \le 0$ can be proved. From Lemma 1, one can see that $\boldsymbol{L}-\boldsymbol{\Phi}+\kappa\boldsymbol{I}_N$ is negative definite when κ is sufficiently small. If one chooses a large initial condition $c(0)$ and let η be small enough, then $c(t)$ increases very slowly. In this case, the synchronisation can be achieved by Corollary 1 when c is very large. After reaching synchronisation, $c(t)$ converges to a constant and is bounded. Since $c(t)$ is bounded, one can always choose a sufficiently large $\bar{c}$ such that $a_2(a_1(\theta\boldsymbol{I}_N + c(t)(\boldsymbol{L}-\boldsymbol{\Phi}+\kappa\boldsymbol{I}_N)-\kappa\bar{c}) + \boldsymbol{M}_1) + \boldsymbol{I}_N < 0$. Then, in the whole time evolution process, $\dot{V}(t)$ is always negative definite. Consequently, the network is synchronised, so the third equation in Eq. (18) will vanish, leading to $c(t) \to$ constant. This completes the proof.

Remark 2 Independent of the network structure and the fraction of the nodes, the condition in Theorem 2 is very mild. Also, the coupling strength can be self-adaptively determined according to Eq. (18) to realise the network synchronisation under any fixed network structure and pinning controllers in the form of Eq. (5).

3.3. Selective pinning scheme and control gains in pinning control

In this subsection, the pinning scheme is designed when the network structure and the coupling strength are fixed.

Let $\boldsymbol{\Omega} = a_2(a_1(-\theta \boldsymbol{I}_N + c\boldsymbol{L} - c\boldsymbol{\Phi}) + \boldsymbol{M}_1) + \boldsymbol{I}_N = (\Omega_{ij})_{N\times N}$ and $k_i = \sum_{j=1,j\neq i}^{N} l_{ij}$, where $\boldsymbol{M}_1 = (m_{ij})_{N\times N}$ and k_i is the degree of node i respectively.

It is not difficult to observe that $\Omega_{ij} = a_1a_2cl_{ij} + a_2m_{ij}(i \neq j), \Omega_{ii} = a_2(a_1(-\theta - c(k_i+\epsilon^i)+m_{ii})+1(1 \leq i \leq p)$, and $\Omega_{ii} = a_2(a_1(-\theta - ck_i) + m_{ii}) + 1(p+1 \leq i \leq N)$.

Corollary 2 If inequalities (11a) and (11b) are guaranteed, then to satisfy condition (19), it is necessary that

$$\begin{aligned} \Omega_{ii} &= a_2(a_1(-\theta - c(k_i+\epsilon^i) + m_{ii}) + 1 < 0, \\ &\quad 1 \leq i \leq p, \\ \Omega_{ii} &= a_2(a_1(-\theta - ck_i) + m_{ii}) + 1 < 0, \\ &\quad p+1 \leq i \leq N. \end{aligned} \tag{22}$$

The proof of this corollary is simple and omitted here. Up to this point, it is still not very clear about how to determine the control gain. Next, a theorem is given to address this concern.

Let $\boldsymbol{\Omega} = \begin{pmatrix} \boldsymbol{A} - \tilde{\boldsymbol{D}} & \boldsymbol{B} \\ \boldsymbol{B}^{\mathrm{T}} & \tilde{\boldsymbol{C}} \end{pmatrix}$, where $\boldsymbol{A}$ and $\boldsymbol{B}$ are matrices with appropriate dimensions, $\tilde{\boldsymbol{D}} = \mathrm{diag}(\epsilon^1, \ldots, \epsilon^p)$, and $\tilde{\boldsymbol{C}}$ is obtained by removing the $1, 2, \ldots, p$ row-column pairs of matrix $\boldsymbol{\Omega}$.

Theorem 3 If inequalities (11a) and (11b) are satisfied and the control gains ϵ^i can be sufficiently large, then $\boldsymbol{\Omega} < 0$ is equivalent to

$$\tilde{\boldsymbol{C}} < 0. \tag{23}$$

For simplicity, one can select $\tilde{\boldsymbol{D}} > \lambda_{\max}(\boldsymbol{A} - \boldsymbol{B}\tilde{\boldsymbol{C}}^{-1}\boldsymbol{B}^{\mathrm{T}})\boldsymbol{I}_p$.

Proof It is easy to see that if $\boldsymbol{\Omega} < 0$ then $\tilde{\boldsymbol{C}} < 0$. So, one only needs to prove that if $\tilde{\boldsymbol{C}} < 0$ then $\boldsymbol{\Omega} < 0$. Choose $\tilde{\boldsymbol{D}} > \boldsymbol{A} - \boldsymbol{B}\tilde{\boldsymbol{C}}^{-1}\boldsymbol{B}^{\mathrm{T}}$. Then, according to Schur complements, one indeed has $\boldsymbol{\Omega} < 0$. The proof is completed.

From the definition of $\tilde{\boldsymbol{C}}$, one has $\tilde{\boldsymbol{C}} = a_2(a_1(-\theta \boldsymbol{I}_{N-p} + c\tilde{\boldsymbol{L}}) + \tilde{\boldsymbol{M}}_1) + \boldsymbol{I}_{N-p}$, where $\tilde{l}_{i-p,j-p} = l_{i,j}$, and $\tilde{m}_{i-p,j-p} = m_{i,j}$, for $i, j = p+1, \ldots, N$.

Corollary 3 If inequalities (11a), (11b), $\tilde{\boldsymbol{D}} > \lambda_{\max}(\boldsymbol{A} - \boldsymbol{B}\tilde{\boldsymbol{C}}^{-1}\boldsymbol{B}^{\mathrm{T}})\boldsymbol{I}_p$ are satisfied, and the following condition holds:

$$a_2(a_1(-\theta + c\lambda_{\max}(\tilde{\boldsymbol{L}})) + \lambda_{\max}(\tilde{\boldsymbol{M}}_1)) + 1 < 0, \tag{24}$$

then controlled undirected network (4) is globally synchronised.

Remark 3 From Corollary 2, it is not difficult to see that if there exist some nodes with degrees $k_i \leq (1/(a_1c))(1/a_2 + m_{ii}) - \theta/c$, then the network cannot achieve the synchronisation without controlling these nodes by Theorem 1 or Theorem 3. This means that when the coupling strength is very small, the nodes with lower degrees should be controlled first.

Remark 4 In practice, it is not easy to select a minimum number of pinned nodes to achieve a network synchronisation. In Ref. [14], a lower bound for the minimum number of pinning controllers for a complex network with nonlinear coupling term is given as follows.

If inequalities (11a) and (11b) are satisfied and $a_2(a_1(-\theta \boldsymbol{I}_N + c\boldsymbol{L}) + \boldsymbol{M}_1) + \boldsymbol{I}_N$ has l nonnegative eigenvalues, then the rank of matrix $\boldsymbol{\Phi}$ cannot be less than l when the closed-loop characteristic inequality (11c) holds strictly.

The theoretical analysis of this condition is not difficult. However, it raises the question whether these l controllers exist and how they can be found. Due to the nonlinear dynamics of the entire network, the number of the controlled nodes may be larger than l. Noticeably, different from the results in Ref. [14] where the choice of the nodes that should be controlled is made in a trial-and-error way, in this paper, as pointed out in Remark 3, the nodes with lower degrees should be controlled first when the coupling strength is very small.

Remark 5 Note that Theorem 3 provides some guidance in the choice of the control gains, and Corollary 3 gives a relationship among network synchronisation, fixed network topology with nonlinear coupling dynamics, coupling strength and pinning scheme. However, gain matrix $\tilde{\boldsymbol{D}} > \lambda_{\max}(\boldsymbol{A} - \boldsymbol{B}\tilde{\boldsymbol{C}}^{-1}\boldsymbol{B}^{\mathrm{T}})\boldsymbol{I}_p$ in inequality (23) may be much larger than the needed value, so the adaptive control approach is used here to tune the control gain to achieve the network synchronisation.

The pinning controllers selected by inequality (23) yield the following controlled network:

$$\dot{\boldsymbol{x}}^i(t) = \boldsymbol{g}(\boldsymbol{x}^i(t),t) + \boldsymbol{h}^i(x^1,x^2,\ldots,x^N) + c\sum_{j=1}^{N} l_{ij}\boldsymbol{\Gamma}\boldsymbol{x}^j(t) - c\epsilon^i(t)\boldsymbol{\Gamma}(\boldsymbol{x}^i(t)-\boldsymbol{s}(t)),\ i=1,\ldots,p,$$
$$\dot{\boldsymbol{x}}^i(t) = \boldsymbol{g}(\boldsymbol{x}^i(t),t) + \boldsymbol{h}^i(x^1,x^2,\ldots,x^N) + c\sum_{j=1}^{N} l_{ij}\boldsymbol{\Gamma}\boldsymbol{x}^j(t),\ i=p+1,\ldots,N,$$
$$\dot{\epsilon}^i(t) = \mu^i(\boldsymbol{x}_i-\boldsymbol{s}(t))^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{\Gamma}(\boldsymbol{x}_i-\boldsymbol{s}(t)),\ i=1,2,\ldots,p, \tag{25}$$

where μ^i are positive constants.

Corollary 4 If inequalities (11a), (11b) and the condition in inequality (23) are satisfied, then the undirected network (25) is globally synchronised under the adaptive scheme (25).

Proof Consider the Lyapunov function candidate

$$V(t) = \sum_{i=1}^{N} \boldsymbol{e}^{i\mathrm{T}}(t)\boldsymbol{Q}\boldsymbol{e}^i(t) + \sum_{i=1}^{p}\frac{c}{\mu^i}(\epsilon^i(t)-\epsilon)^2, \tag{26}$$

where ϵ is a positive constant to be determined below.

Taking the derivative of $V(t)$ along the trajectories of Eq. (25) gives

$$\begin{aligned}
\dot{V}(t) &= 2\sum_{i=1}^{N}\boldsymbol{e}^{i\mathrm{T}}(t)\boldsymbol{Q}\dot{\boldsymbol{e}}^i(t) + 2c\sum_{i=1}^{p}(\epsilon^i(t)-\epsilon)\boldsymbol{e}^{i\mathrm{T}}(t)\boldsymbol{Q}\boldsymbol{\Gamma}\boldsymbol{e}^i(t)\\
&= 2\sum_{i=1}^{N}\boldsymbol{e}^{i\mathrm{T}}(t)\boldsymbol{Q}\left[\boldsymbol{g}(\boldsymbol{x}^i(t),t) - \boldsymbol{g}(\boldsymbol{s}(t),t) + \boldsymbol{h}^i(x^1,x^2,\ldots,x^N) + c\sum_{j=1}^{N} l_{ij}\boldsymbol{\Gamma}\boldsymbol{e}^j(t)\right]\\
&\quad - 2c\sum_{i=1}^{p}\epsilon^i(t)\boldsymbol{e}^{i\mathrm{T}}(t)\boldsymbol{Q}\boldsymbol{\Gamma}\boldsymbol{e}^i(t) + 2c\sum_{i=1}^{p}(\epsilon^i(t)-\epsilon)\boldsymbol{e}^{i\mathrm{T}}(t)\boldsymbol{Q}\boldsymbol{\Gamma}\boldsymbol{e}^i(t)\\
&\le \boldsymbol{e}^{\mathrm{T}}(t)\left[(-\theta\boldsymbol{I}_N + c(\boldsymbol{L}-\epsilon\tilde{\boldsymbol{I}}_N))\otimes(\boldsymbol{Q}\boldsymbol{\Gamma}+\boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{Q}) + \boldsymbol{M}_1\otimes\boldsymbol{M}_2\right]\boldsymbol{e}(t) + 2\boldsymbol{e}^{\mathrm{T}}(t)(\boldsymbol{I}_N\otimes\boldsymbol{Q})\boldsymbol{H} - \boldsymbol{H}^{\mathrm{T}}\boldsymbol{H}\\
&\le \boldsymbol{e}^{\mathrm{T}}\left[(a_1(-\theta\boldsymbol{I}_N + c(\boldsymbol{L}-\epsilon\tilde{\boldsymbol{I}}_N)) + \boldsymbol{M}_1)\otimes\boldsymbol{M}_2\right]\boldsymbol{e}(t) + 2\boldsymbol{e}^{\mathrm{T}}(t)(\boldsymbol{I}_N\otimes\boldsymbol{Q})\boldsymbol{H} - \boldsymbol{H}^{\mathrm{T}}\boldsymbol{H}\\
&= [\,\boldsymbol{e}^{\mathrm{T}}(t)\ \boldsymbol{H}^{\mathrm{T}}\,]\Xi\begin{bmatrix}\boldsymbol{e}(t)\\ \boldsymbol{H}\end{bmatrix},
\end{aligned} \tag{27}$$

where

$$\boldsymbol{e}(t) = [e^{1\mathrm{T}}(t), e^{2\mathrm{T}}(t),\ldots,e^{N\mathrm{T}}(t)]^{\mathrm{T}},\quad \tilde{\boldsymbol{I}}_N = \mathrm{diag}(\underbrace{1,\ldots,1}_{p},\underbrace{0,\ldots,0}_{N-p}),$$

$$\Xi = \begin{pmatrix}(a_1(-\theta\boldsymbol{I}_N + c(\boldsymbol{L}-\epsilon\tilde{\boldsymbol{I}}_N)) + \boldsymbol{M}_1)\otimes\boldsymbol{M}_2 & \boldsymbol{I}_N\otimes\boldsymbol{Q}\\ \boldsymbol{I}_N\otimes\boldsymbol{Q} & -\boldsymbol{I}_{nN}\end{pmatrix}.$$

From theorem 3 it follows that $\tilde{\boldsymbol{C}} < 0$, and by choosing $\epsilon > \lambda_{\max}(\boldsymbol{A} - \boldsymbol{B}\tilde{\boldsymbol{C}}^{-1}\boldsymbol{B}^{\mathrm{T}})$, network (25) is globally synchronised. This completes the proof.

4. Representative example

In this section, a simulation example is given to verify the criteria established above. Here, consider a complex network with both linear and nonlinear coupling dynamics which consists of 40 identical Lorenz systems,[35–37] described by Eq. (1), in which $\boldsymbol{\Gamma} = \mathrm{diag}(1,1,1)$, and

$$\boldsymbol{g}(\boldsymbol{x}^i(t),t) = \begin{bmatrix}10(x_2^i(t)-x_1^i(t))\\ 28x_1^i(t) - x_2^i(t) - x_1^i(t)x_3^i(t)\\ x_1^i(t)x_2^i(t) - (8/3)x_3^i(t)\end{bmatrix}. \tag{28}$$

The degree of each node (from node 1 to node 40) is presented in Table 1.

Table 1. Degree of each node.

parameters	values									
node	1	2	3	4	5	6	7	8	9	10
degree	11	12	13	14	15	16	17	18	19	20
node	11	12	13	14	15	16	17	18	19	20
degree	21	22	23	23	24	24	24	24	24	24
node	21	22	23	24	25	26	27	28	29	30
degree	24	24	24	24	24	24	24	24	15	15
node	31	32	33	34	35	36	37	38	39	40
degree	15	15	14	13	12	11	10	6	4	2

The linear coupling strength $c = 2$. Assume that all the nonlinear coupling $\boldsymbol{h}_i$ are in the following forms:

$$\boldsymbol{h}^i = \psi^i \boldsymbol{B}^i, i = 1, 2, \ldots, 40, \tag{29}$$

where $\boldsymbol{B}^i$ belongs to a convex set

$$\mathcal{C}_0 \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

and ψ^i is assumed to be in the following forms:

$$\psi^j = \sqrt{|\, x_1^{j+1} - x_1^{j+2} \,||\, x_2^{j+3} - x_2^{j+2} \,|},$$
$$\psi^{j+1} = \sqrt{|\, x_2^{j+3} - x_2^{j+2} \,||\, x_3^{j} - x_3^{j+3} \,|},$$
$$\psi^{j+2} = \sqrt{|\, x_3^{j} - x_3^{j+3} \,||\, x_1^{j+3} - x_1^{j} \,|},$$
$$\psi^{j+3} = \sqrt{|\, x_1^{j+3} - x_1^{j} \,||\, x_2^{j+1} - x_2^{j} \,|},$$
$$j = 1, 5, 9, \ldots, 4k-3, \text{for } k = 1, 2, \ldots, 10. \tag{30}$$

Here, it should be pointed out that some information involved in the nonlinear coupling dynamics does not belong to the known information set based on the coupling matrix $\boldsymbol{L}$.

The objective stabilisation state of the network is $\bar{\boldsymbol{X}} = (\bar{x^1}^{\mathrm{T}}, \bar{x^2}^{\mathrm{T}}, \ldots, \bar{x^{40}}^{\mathrm{T}})^{\mathrm{T}} = (s^{\mathrm{T}}, s^{\mathrm{T}}, \ldots, s^{\mathrm{T}})^{\mathrm{T}} = \bar{\boldsymbol{S}}$, where the isolated node s is a Lorenz system and denoted as

$$\dot{\boldsymbol{s}} = \boldsymbol{g}(\boldsymbol{s}). \tag{31}$$

$\boldsymbol{h}^i$ and $\boldsymbol{e}^i$ will vanish when the network is stabilised, and $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$, which satisfy the restrictive condition (10) of $\boldsymbol{h}^i$, are not too hard to obtain as $\boldsymbol{M}_1 = \mathrm{diag}(1, 1, \ldots, 1, 1) \in \mathbb{R}^{40\times 40}$ and $\boldsymbol{M}_2 = \mathrm{diag}(1, 1, 1)$. From the numerical simulation, it is found that there exist some constants $S_1 = 20$, $S_2 = 26$, and $S_3 = 47$, such that the chaotic attractor (s_1, s_2, s_3) of the Lorenz system satisfies $|s_1| < S_1$, $|s_2| < S_2$, and $|s_3| < S_3$. By Eq. (31) and Assumption 1, one has

$$\begin{aligned}
&(\boldsymbol{x}^i - \boldsymbol{s})^{\mathrm{T}}(\boldsymbol{g}(\boldsymbol{x}^i, t) - \boldsymbol{g}(\boldsymbol{s}, t)) \\
=\ & \boldsymbol{e}^{i\,\mathrm{T}}(10e_2^i - 10e_1^i, 28e_1^i - e_2^i - e_1^i e_3^i - s_1 e_3^i - s_3 e_1^i, -\frac{8}{3}e_3^i + e_1^i e_2^i + s_1 e_2^i + s_2 e_1^i)^{\mathrm{T}} \\
\le\ & -10{e_1^i}^2 - {e_2^i}^2 - \frac{8}{3}{e_3^i}^2 + (38 + S_3)|e_1^i e_2^i| + S_2|e_1^i e_3^i| \\
\le\ & \left(-10 + \rho\frac{38 + S_3}{2} + \varrho\frac{S_2}{2}\right){e_1^i}^2 + \left(-1 + \frac{38 + S_3}{2\rho}\right){e_2^i}^2 + \left(-\frac{8}{3} + \frac{S_2}{2\varrho}\right){e_3^i}^2 \\
\le\ & -\theta({e_1^i}^2 + {e_2^i}^2 + {e_3^i}^2).
\end{aligned} \tag{32}$$

Choosing $\rho = 1.0560$ and $\varrho = 0.3113$, one obtains the passivity degree value of each node $\theta = -39.0955$ over the entire domain $\mathbb{R}^3$.

4.1. Pinning strategy for network synchronisation

Let $\boldsymbol{Q}$ be $\boldsymbol{I}_3$, and the scalars a_1 and a_2 are chosen as 2 and 0.5 to satisfy inequalities (11a) and (11b). Then, the main characteristic matrix $-\theta\boldsymbol{I}_N + c\boldsymbol{L} + (1/2)\boldsymbol{M}_1 + \boldsymbol{I}_N$ without control according to inequality (11c) has three positive eigenvalues. From Remark 4, the number of controlled nodes cannot be less than three to stabilise the network to the orbit s. On the other hand, by calculation, we know that when nodes with degree $k \le 20$ exist, the network cannot achieve the synchronisation without controlling these nodes. Therefore, in terms of Table 1, we put the pinning controllers in the nodes 1–10, 29–40. Through simulation, it is found that the number of controllers can be reduced, yet in this situation, the control gains must be increased. For simplicity, the control gain of each node is set to be 19.5, which guarantees that all the eigenvalues of the closed-loop characteristic matrix $\boldsymbol{\Omega} = -\theta\boldsymbol{I}_N + c\boldsymbol{L} - c\boldsymbol{\Phi} + (1/2)\boldsymbol{M}_1 + \boldsymbol{I}_N$ become non-positive; particularly, inequality (11c) is satisfied. The simulation results are shown in Figs. 1 and 2.

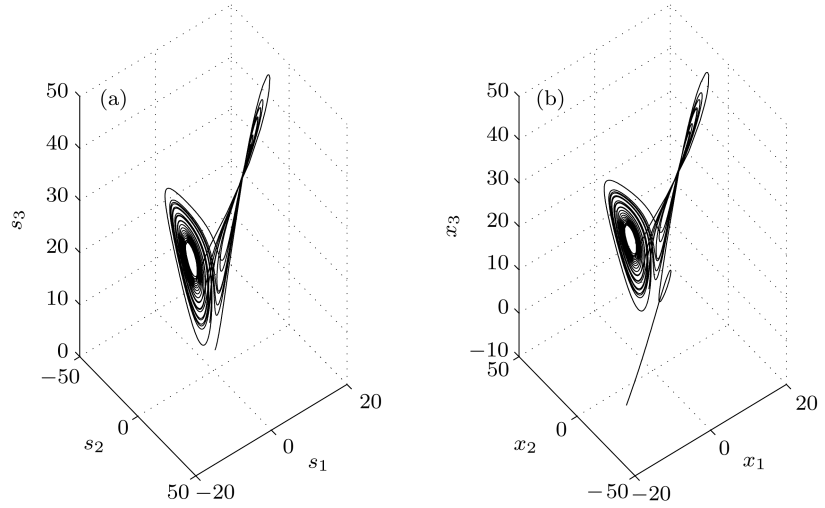


Fig. 1. (a) Dynamics of uncoupled Lorenz system and (b) dynamics of a random network node.

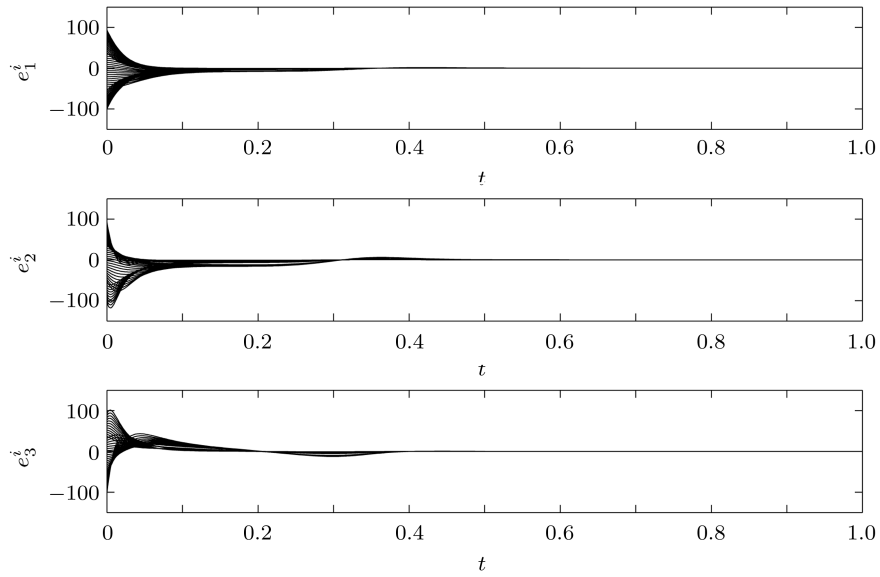


Fig. 2. Synchronisation errors $e_j^i, i = 1, 2, \ldots, 40, j = 1, 2, 3$, between the state of the Lorenz system and the state of the simulated network.

4.2. Adaptive tuning of coupling strength for network synchronisation

In this simulation, the coupling strength is designed by Theorem 2 based on the adaptive tuning strategy for the proposed uncertain complex network, where $c(0) = 0.5$, and the control gain of each pinned node is set to be 9.5.

The states of error $\boldsymbol{e}^i$ and the coupling strength $c(t)$ are illustrated in Figs. 3 and 4, respectively, which represent that the controlled network (18) is globally synchronised when the coupling strength is stable at 28.0948.

It is noted that according to Corollary 2, when $c(0) = 0.5$ and $\epsilon^i = 9.5, i = 1, \ldots, 22$, condition (22) is not satisfied. Therefore, c must be increased by adaptive tuning strategy. We can see that condition (22) is guaranteed when coupling strength c is stable at 28.0948, which gives a full explanation for the above simulation results.

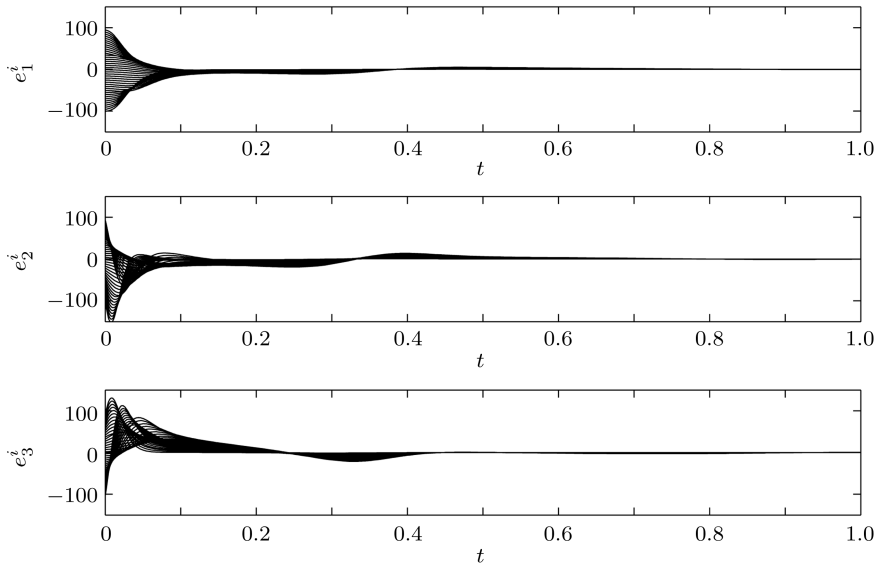


Fig. 3. Synchronisation errors $e_j^i, i = 1, 2, \ldots, 40, j = 1, 2, 3$, between the state of the Lorenz system and the state of the simulated network.

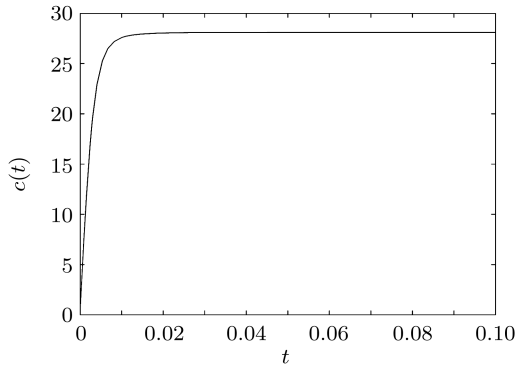


Fig. 4. Adaptive coupling strength $c(t)$ for network synchronisation.

4.3. Adaptive tuning of control gains for network synchronisation

In this subsection, we apply pinning control actions to the selected 22 nodes of the uncertain complex network (25) and tune feedback gains with Eq. (25). The evolutions of the synchronisation error $\boldsymbol{e}^i$ and the pinning feedback gains $\epsilon^i, i = 1, \ldots, 20$ are shown in Figs. 5 and 6, respectively. It can be seen that the proposed uncertain network with 40 nodes is globally synchronised when the control gains become stable.

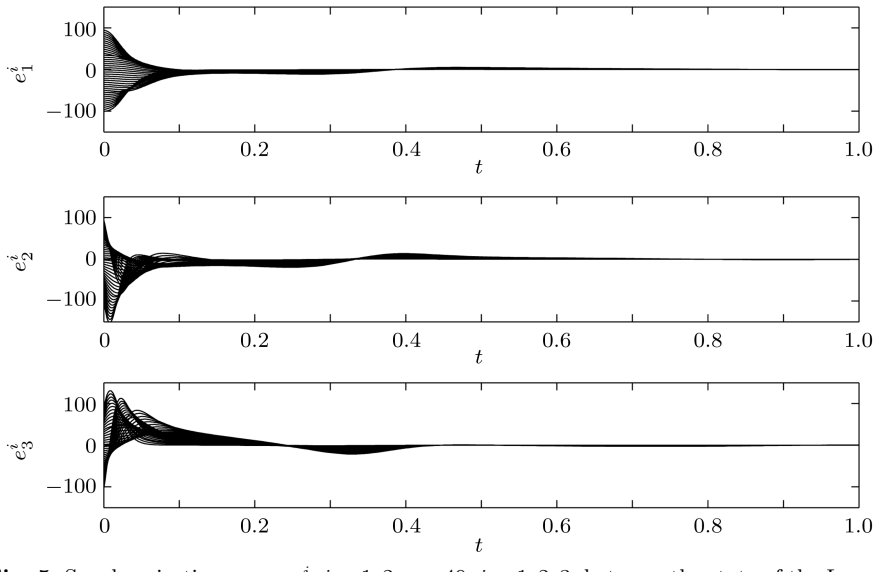


Fig. 5. Synchronisation errors $e_j^i, i = 1, 2, \ldots, 40, j = 1, 2, 3$, between the state of the Lorenz system and the state of the simulated network.

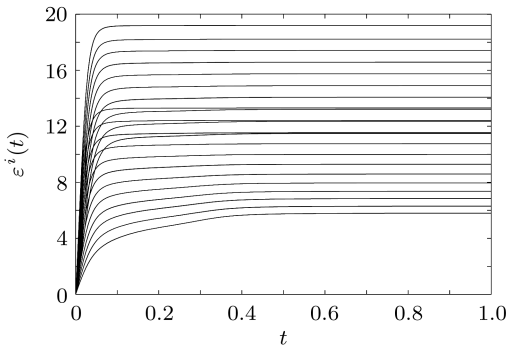


Fig. 6. Adaptive pinning feedback gains $\epsilon^i(t), i = 1, \ldots, 22$, for network synchronisation.

5. Conclusion

To pin a complex network to a homogeneous trajectory of the uncoupled system, the underlying network topology, the coupling strength, and selection of pinned nodes play a key role in stabilising networks. In this paper, we arrive at several conditions to guarantee the global stability of pinning control of a complex network with linear and nonlinear coupling dynamics. In addition, it is worth noting that we deeply discuss the relationship among pinning synchronisation, network topology, coupling strength and selection of pinned nodes. It is found that when the coupling strength is very small, the nodes with low degrees should be selected and pinned. Furthermore, an adaptive pinning control scheme is developed to achieve the synchronisation of the uncertain complex network by using the adaptive tuning of either the coupling strength or the control gains. Finally, an illustrative example is given to verify the effectiveness of the theoretical results.

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