Synchronization between fractional-order chaotic systems and integer orders chaotic systems (fractional-order chaotic systems)*

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Based on the idea of tracking control and stability theory of fractional-order systems, a controller is designed to synchronize the fractional-order chaotic system with chaotic systems of integer orders, and synchronize the different fractional-order chaotic systems. The proposed synchronization approach in this paper shows that the synchronization between fractional-order chaotic systems and chaotic systems of integer orders can be achieved, and the synchronization between different fractional-order chaotic systems can also be realized. Numerical experiments show that the present method works very well.

Keywords: fractional-order chaotic systems, chaotic systems of integer orders, different fractional-order chaotic systems, synchronization

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1. Introduction

Chaotic phenomena can be found in many scientific and engineering fields such as biological systems, electronic circuits, power converters, chemical systems, and so on. Chaotic systems have irregular, complex and unpredictable dynamic behaviours. Since the pioneering work on controlling chaos introduced by Ott, Grebogi and Yorke[1] the synchronizing of the chaotic systems has gained a lot of attention among scientists from a variety of research fields in recent years.[2−4] The chaos synchronization is essential in some applications such as communication and secure communication.[5,6] More recently, many researchers surprisingly found that many nonlinear fractional differential systems also displayed complex bifurcation and chaos phenomena.[7−13] Therefore, many authors begin to investigate the chaotic synchronization for fractional-order dynamical systems.[14−19] The study on chaos synchronization of the fractional-order systems just begins to attract some attention due to its potential applications in secure communications and control processing.[20−25]

However, many existing chaotic synchronization methods mainly focus on the study of fractional-order chaotic systems or chaotic systems of integer orders. So far, there has been little information available in the literature about the synchronization between fractional-order chaotic system and chaotic system of integer orders, and about the synchronization for fractional-order chaotic systems in which the fractional-order value of the drive system is unequal to the fractional-order value of the response system.

Motivated by all the above work, in this paper, we give a synchronization method for fractional-order chaotic systems, which is different from the previous ones. The proposed synchronization technique is based on the ideas of tracking control and stability theory of fractional-order systems, which is simple and theoretically rigorous. The derived method in the present paper shows that the synchronization between fractional-order chaotic systems and chaotic systems of integer orders can be achieved, and the synchronization between different fractional-order chaotic systems can also be realized. The first advantage of our method is needless to absorb all the nonlinear terms of response system. This is different from some previous methods;[7,26−28] The second advantage of our method is that the fractional-order value of drive system is unequal to the fractional-order value of response system. This advantage is different from the

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former synchronization approaches.[13,14,21−25] Two examples are considered and their numerical simulations are performed. The first example is for the synchronization between a fractional-order Lorenz chaotic system and integer orders Chen chaotic system. The second example is for the synchronization between a fractional-order Lü chaotic system and a new fractional-order chaotic system. Numerical simulations accord with the theoretical analysis.

2. Problem formulation

Consider the following n-dimensional fractional-order chaotic system (1) or n-dimensional chaotic system of integer orders (2) as a drive system

$$\mathrm{d}^{\boldsymbol{q}_\mathrm{d}}\boldsymbol{x}/\mathrm{d}t^{\boldsymbol{q}_\mathrm{d}} = \boldsymbol{f}(\boldsymbol{x}), \tag{1}$$

$$\mathrm{d}\boldsymbol{x}/\mathrm{d}t = \boldsymbol{h}(\boldsymbol{x}), \tag{2}$$

where $\boldsymbol{x} \in R^n$, fractional order $\boldsymbol{q}_\mathrm{d} = (q_{\mathrm{d}1}, q_{\mathrm{d}2}, \ldots, q_{\mathrm{d}n})^\mathrm{T}$ $(0 < q_{\mathrm{d}i} < 1)$ may be unequal. $\boldsymbol{f} : R^n \to R^n$, and $\boldsymbol{h} : R^n \to R^n$ are differentiable functions.

Now, consider the following n-dimensional fractional-order chaotic system

$$\mathrm{d}^{\boldsymbol{q}_\mathrm{r}}\boldsymbol{y}/\mathrm{d}t^{\boldsymbol{q}_\mathrm{r}} = \boldsymbol{g}(\boldsymbol{y}), \tag{3}$$

and construct the following fractional order response system:

$$\mathrm{d}^{\boldsymbol{q}_\mathrm{r}}\boldsymbol{y}/\mathrm{d}t^{\boldsymbol{q}_\mathrm{r}} = \boldsymbol{g}(\boldsymbol{y}) + \boldsymbol{u}(\boldsymbol{y}, \boldsymbol{x}), \tag{4}$$

where $\boldsymbol{q}_\mathrm{r} = (q_{r1}, q_{r2}, \ldots, q_{rn})^\mathrm{T}$ $(0 < q_{ri} < 1)$ may be unequal (response system (4) is chaotic system of integer orders if $q_{ri} = 1(i = 1, 2, \ldots, n)$). Fractional order $\boldsymbol{q}_\mathrm{d} = (q_{\mathrm{d}1}, q_{\mathrm{d}2}, \ldots, q_{\mathrm{d}n})^\mathrm{T}$ may be different with fractional order $\boldsymbol{q}_\mathrm{r} = (q_{r1}, q_{r2}, \ldots, q_{rn})^\mathrm{T}$. $\boldsymbol{u}(\boldsymbol{y}, \boldsymbol{x})$ is the controller to be designed. Our goal in this paper is to design controller $\boldsymbol{u}(\boldsymbol{y}, \boldsymbol{x})$ such that

$$\lim_{t\to+\infty} \|\boldsymbol{e}\| = \lim_{t\to+\infty} \|\boldsymbol{y} - \boldsymbol{x}\| = 0, \tag{5}$$

then system(4) and system (1) or system(4) and system (2) are said to be synchronized, where $\|\bullet\|$ is the Euclidean norm.

Remark 1 If $\boldsymbol{x}$ in response system (4) belongs to chaotic system (2), then the synchronization between fractional-order chaotic systems and chaotic systems of integer orders can be achieved.

Remark 2 If $\boldsymbol{x}$ in response system (4) belongs to chaotic system (1), then the synchronization between different fractional-order chaotic systems can be achieved.

Based on the idea of tracking control, in order to achieve the goal of $\lim_{t\to+\infty} \|\boldsymbol{e}\| = \lim_{t\to+\infty} \|\boldsymbol{y} - \boldsymbol{x}\| = 0$, we can let the $\boldsymbol{x}$ of drive system (1) or the $\boldsymbol{x}$ of drive system (2) be a reference signal. The chaotic synchronization between drive system (1) or drive system (2) and response fractional-order chaotic system (4) belongs to the problem of tracking control, i.e. the output signal $\boldsymbol{y}(t)$ of system (4) follows the reference signal $\boldsymbol{x}(t)$ ultimately.

According to the idea of tracking control, let the output signal $\boldsymbol{x}(t)$ of drive system (1) or (2) be a reference signal. Define a compensation controller $\boldsymbol{\theta}(\boldsymbol{x})$ for response system (4) as

$$\boldsymbol{\theta}(\boldsymbol{x}) = \mathrm{d}^{\boldsymbol{q}_\mathrm{r}}\boldsymbol{x}/\mathrm{d}t^{\boldsymbol{q}_\mathrm{r}} - \boldsymbol{g}(\boldsymbol{x})$$

and let the controller be

$$\boldsymbol{u}(\boldsymbol{y}, \boldsymbol{x}) = \boldsymbol{\theta}(\boldsymbol{x}) + \boldsymbol{\tau}(\boldsymbol{y}, \boldsymbol{x}),$$

where $\boldsymbol{\tau}(\boldsymbol{y}, \boldsymbol{x})$ is a feedback controller. So, response system (4) can be rewritten as

$$\mathrm{d}^{\boldsymbol{q}_\mathrm{r}}\boldsymbol{y}/\mathrm{d}t^{\boldsymbol{q}_\mathrm{r}} = \boldsymbol{g}(\boldsymbol{y}) + \boldsymbol{\theta}(\boldsymbol{x}) + \boldsymbol{\tau}(\boldsymbol{y}, \boldsymbol{x}). \tag{6}$$

Let $\boldsymbol{e} = \boldsymbol{y} - \boldsymbol{x}$, response system (4) or response system (6) can be rewritten as

$$\mathrm{d}^{\boldsymbol{q}_\mathrm{r}}\boldsymbol{e}/\mathrm{d}t^{\boldsymbol{q}_\mathrm{r}} = \boldsymbol{g}(\boldsymbol{y}) - \boldsymbol{g}(\boldsymbol{x}) + \boldsymbol{\tau}(\boldsymbol{y}, \boldsymbol{x}). \tag{7}$$

Define $\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{e}) = \boldsymbol{g}(\boldsymbol{y}) - \boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x} + \boldsymbol{e}) - \boldsymbol{g}(\boldsymbol{x})$, and $\boldsymbol{\Omega}(\boldsymbol{x}, \boldsymbol{e}) = \boldsymbol{\tau}(\boldsymbol{y}, \boldsymbol{x}) = \boldsymbol{\tau}(\boldsymbol{x} + \boldsymbol{e}, \boldsymbol{x})$. We can obtain

$$\mathrm{d}^{\boldsymbol{q}_\mathrm{r}}\boldsymbol{e}/\mathrm{d}t^{\boldsymbol{q}_\mathrm{r}} = \boldsymbol{G}(\boldsymbol{x}, \boldsymbol{e}) + \boldsymbol{\Omega}(\boldsymbol{x}, \boldsymbol{e}). \tag{8}$$

In general, we can assume

$$\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{e}) = \begin{pmatrix} \boldsymbol{A}_1\boldsymbol{e}_1 + \boldsymbol{F}_1(\boldsymbol{x}, \boldsymbol{e}_1, \boldsymbol{e}_2) \\ \boldsymbol{A}_2\boldsymbol{e}_2 + \boldsymbol{F}_{21}(\boldsymbol{x}, \boldsymbol{e}_1, \boldsymbol{e}_2) \\ \qquad + \boldsymbol{F}_{22}(\boldsymbol{x}, \boldsymbol{e}_1, \boldsymbol{e}_2) \end{pmatrix}, \tag{9}$$

$$\lim_{\boldsymbol{e}_1\to\boldsymbol{0}} \boldsymbol{F}_{21}(\boldsymbol{x}, \boldsymbol{e}_1, \boldsymbol{e}_2) = \boldsymbol{0}, \tag{10}$$

where $\boldsymbol{F}_1(\boldsymbol{x}, \boldsymbol{e}_1, \boldsymbol{e}_2) \in R^m (1 \le m \le n)$, $\boldsymbol{F}_{21}(\boldsymbol{x}, \boldsymbol{e}_1, \boldsymbol{e}_2) \in R^{n-m}$, and $\boldsymbol{F}_{22}(\boldsymbol{x}, \boldsymbol{e}_1, \boldsymbol{e}_2) \in R^{n-m}$; $\boldsymbol{A}_1 \in R^{m\times m}$ and $\boldsymbol{A}_2 \in R^{(n-m)\times(n-m)}$ are constant real matrixes; $\boldsymbol{e}_1 = (e_{t_1}, \ldots, e_{t_m})^\mathrm{T}$; $\boldsymbol{e}_2 = (e_{t_{m+1}}, \ldots, e_{t_n})^\mathrm{T}$.

Now, we can define

$$\boldsymbol{\Omega}(\boldsymbol{x}, \boldsymbol{e}) = \begin{pmatrix} \boldsymbol{\Omega}_1(\boldsymbol{x}, \boldsymbol{e}_1, \boldsymbol{e}_2) \\ \boldsymbol{\Omega}_2(\boldsymbol{x}, \boldsymbol{e}_1, \boldsymbol{e}_2) \end{pmatrix}. \tag{11}$$

So, response system (6) or (8) can be rewritten as

$$\begin{cases} \mathrm{d}^{\boldsymbol{q}_1}\boldsymbol{e}_1/\mathrm{d}t^{\boldsymbol{q}_1} = \boldsymbol{A}_1\boldsymbol{e}_1 + \boldsymbol{F}_1(\boldsymbol{x},\boldsymbol{e}_1,\boldsymbol{e}_2) + \boldsymbol{\Omega}_1(\boldsymbol{x},\boldsymbol{e}_1,\boldsymbol{e}_2), \\ \mathrm{d}^{\boldsymbol{q}_2}\boldsymbol{e}_2/\mathrm{d}t^{\boldsymbol{q}_2} = \boldsymbol{A}_2\boldsymbol{e}_2 + \boldsymbol{F}_{21}(\boldsymbol{x},\boldsymbol{e}_1,\boldsymbol{e}_2) + \boldsymbol{F}_{22}(\boldsymbol{x},\boldsymbol{e}_1,\boldsymbol{e}_2) + \boldsymbol{\Omega}_2(\boldsymbol{x},\boldsymbol{e}_1,\boldsymbol{e}_2), \end{cases} \tag{12}$$

where $\boldsymbol{q}_1 = (q_{r1},\dots,q_{rm})^{\mathrm{T}}$, and $\boldsymbol{q}_2 = (q_{r(m+1)},\dots,q_{rn})^{\mathrm{T}}$.

Now, Theorem 1 is given based on the above conditions in order to achieve the goal of Eq. (5).

Theorem 1 If controller $\boldsymbol{u} = \boldsymbol{\theta}(\boldsymbol{x}) + \boldsymbol{\tau}(\boldsymbol{y},\boldsymbol{x})$ in response system (6) is

$$\boldsymbol{\tau}(\boldsymbol{y},\boldsymbol{x}) = \boldsymbol{\Omega}(\boldsymbol{x},\boldsymbol{e}) = \begin{pmatrix} \boldsymbol{\Omega}_1(\boldsymbol{x},\boldsymbol{e}_1,\boldsymbol{e}_2) \\ \boldsymbol{\Omega}_2(\boldsymbol{x},\boldsymbol{e}_1,\boldsymbol{e}_2) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Lambda}_1\boldsymbol{e}_1 - \boldsymbol{F}_1(\boldsymbol{x},\boldsymbol{e}_1,\boldsymbol{e}_2) \\ \boldsymbol{\Lambda}_2\boldsymbol{e}_2 - \boldsymbol{F}_{22}(\boldsymbol{x},\boldsymbol{e}_1,\boldsymbol{e}_2) \end{pmatrix}, \tag{13}$$

where $\boldsymbol{\Lambda}_1 \in R^{m\times m}$ and $\boldsymbol{\Lambda}_2 \in R^{(n-m)\times(n-m)}$ are constant real matrixes. If all the eigenvalues of $\boldsymbol{A}_1 + \boldsymbol{\Lambda}_1$ satisfy $|\arg\lambda_i| > 0.5\pi q_1$ ($q_1 = \max(q_{r1},\dots,q_{rm})$, $i = 1,2,\dots,m$), and those of $\boldsymbol{A}_2 + \boldsymbol{\Lambda}_2$ satisfy $|\arg\lambda_i| > 0.5\pi q_2$ ($q_2 = \max(q_{r(m+1)},\dots,q_{rn})$, $i = 1,2,\dots,n-m$), then the synchronization between response (4) and drive system (1) or (2) can be achieved.

Proof

According to Eqs. (12) and (13), we can yield

$$\begin{cases} \mathrm{d}^{q_1}\boldsymbol{e}_1/\mathrm{d}t^{\boldsymbol{q}_1} = \boldsymbol{A}_1\boldsymbol{e}_1 + \boldsymbol{\Lambda}_1\boldsymbol{e}_1, \\ \mathrm{d}^{\boldsymbol{q}_2}\boldsymbol{e}_2/\mathrm{d}t^{\boldsymbol{q}_2} = \boldsymbol{A}_2\boldsymbol{e}_2 + \boldsymbol{\Lambda}_2\boldsymbol{e}_2 + \boldsymbol{F}_{21}(\boldsymbol{x},\boldsymbol{e}_1,\boldsymbol{e}_2). \end{cases} \tag{14}$$

Because all the eigenvalues of $\boldsymbol{A}_1 + \boldsymbol{\Lambda}_1$ satisfy $|\arg\lambda_i| > 0.5\pi q_1$ ($q_1 = \max(q_{r1},\dots,q_{rm})$, $i = 1,2,\dots,m$), according to the stability theory of fractional-order systems,[29,30] the equilibrium point $\boldsymbol{e}_1 = \boldsymbol{0}$ in Eq. (14) is asymptotically stable. So, $\lim\limits_{t\to+\infty}\boldsymbol{e}_1 = \boldsymbol{0}$.

According to $\lim\limits_{\boldsymbol{e}_1\to\boldsymbol{0}}\boldsymbol{F}_{21}(\boldsymbol{x},\boldsymbol{e}_1,\boldsymbol{e}_2) = \boldsymbol{0}$, as time t approaches to infinity, the second equation of Eq. (14) can be changed into

$$\mathrm{d}^{\boldsymbol{q}_2}\boldsymbol{e}_2/\mathrm{d}t^{\boldsymbol{q}_2} = \boldsymbol{A}_2\boldsymbol{e}_2 + \boldsymbol{\Lambda}_2\boldsymbol{e}_2.$$

Because all the eigenvalues of $\boldsymbol{A}_2 + \boldsymbol{\Lambda}_2$ satisfy $|\arg\lambda_i| > 0.5\pi q_2$ ($q_2 = \max(q_{r(m+1)},\dots,q_{rn})$, $i = 1,2,\dots,n-m$), according to the stability theory of fractional-order systems,[29,30] the equilibrium point $\boldsymbol{e}_2 = \boldsymbol{0}$ in Eq. (14) is asymptotically stable. So, $\lim\limits_{t\to+\infty}\boldsymbol{e}_2 = \boldsymbol{0}$.

The above results indicate that $\lim\limits_{t\to+\infty}\|\boldsymbol{e}\| = \lim\limits_{t\to+\infty}\|\boldsymbol{y}-\boldsymbol{x}\| = 0$. So, the synchronization between response (4) and drive system (1) or (2) can be achieved. The proof is completed. ■

Remark 3 Due to the fact that $\boldsymbol{x}$ may be an arbitrary given reference signal, the controlling of the equilibrium or an unstable periodic orbit of fractional-order chaotic system belongs to this class of problem if we take the equilibrium or an unstable periodic orbit as a reference signal.

3. Applications

To illustrate the effectiveness of the proposed synchronization scheme, two examples are considered and their numerical simulations are performed.

3.1. Synchronization between fractional-order Lorenz chaotic system and Chen chaotic system

Consider the synchronization of a fractional-order Lorenz chaotic system[17] with a response Chen chaotic system.[31] The Chen system, introduced by Chen and Ueta in 1999, is similar but not topologically equivalent to the Lorenz system, and it is described by

$$\begin{cases} \mathrm{d}y_1/\mathrm{d}t = a(y_2 - y_1), \\ \mathrm{d}y_2/\mathrm{d}t = (c-a)y_1 - y_1y_3 + cy_2, \\ \mathrm{d}y_3/\mathrm{d}t = y_1y_2 - by_3. \end{cases}$$

The Chen system exhibits a chaotic behaviour at the parameters (a, b, c) = (35, 3, 28), and the chaotic attractor is shown in Fig. 1.

The fractional-order Lorenz chaotic system[17] is

$$\begin{cases} \mathrm{d}^{q_{\mathrm{d}1}}x_1/\mathrm{d}t^{q_{\mathrm{d}1}} = 10(x_2 - x_1), \\ \mathrm{d}^{q_{\mathrm{d}2}}x_2/\mathrm{d}t^{q_{\mathrm{d}2}} = 28x_1 - x_1x_3 - x_2, \\ \mathrm{d}^{q_{\mathrm{d}3}}x_3/\mathrm{d}t^{q_{\mathrm{d}3}} = x_1x_2 - 8x_3/3. \end{cases} \tag{15}$$

The fractional-order Lorenz chaotic system[17] exhibits a chaotic behaviour. For $q_{\mathrm{d}1} = q_{\mathrm{d}2} = q_{\mathrm{d}3} = 0.995$, the chaotic attractor is shown in Fig. 2.

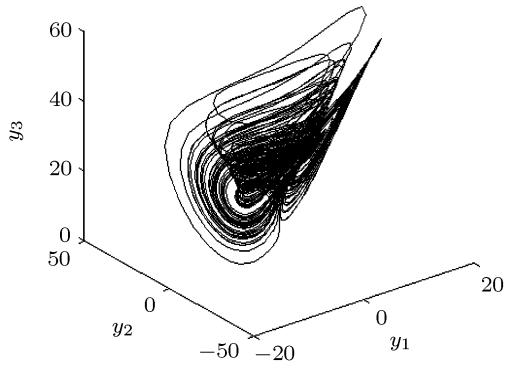


Fig. 1. Chaotic attractor of Chen system.

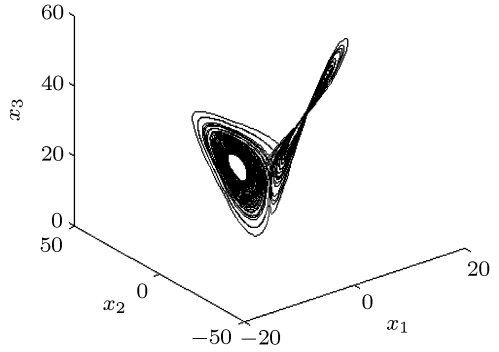


Fig. 2. Chaotic attractor of fractional-order Lorenz chaotic system

Take the fractional-order Lorenz chaotic system with fractional-order $q_{\mathrm{d}1} = q_{\mathrm{d}2} = q_{\mathrm{d}3}$= 0.995 as a drive system, and the Chen system as a response system. The response system is

$$\begin{pmatrix} \mathrm{d}y_1/\mathrm{d}t \\ \mathrm{d}y_2/\mathrm{d}t \\ \mathrm{d}y_3/\mathrm{d}t \end{pmatrix} = \begin{pmatrix} a(y_2 - y_1) \\ (c-a)y_1 - y_1y_3 + cy_2 \\ y_1y_2 - by_3 \end{pmatrix} + \boldsymbol{\theta}(\boldsymbol{x}) + \boldsymbol{\tau}(\boldsymbol{y}, \boldsymbol{x}). \quad (16)$$

According to the above-mentioned, we can obtain compensation controller $\boldsymbol{\theta}(\boldsymbol{x})$ for response system (16) as follows:

$$\boldsymbol{\theta}(\boldsymbol{x}) = \mathrm{d}\boldsymbol{x}/\mathrm{d}t - \boldsymbol{g}(\boldsymbol{x}) = \begin{pmatrix} \mathrm{d}x_1/\mathrm{d}t \\ \mathrm{d}x_2/\mathrm{d}t \\ \mathrm{d}x_3/\mathrm{d}t \end{pmatrix} - \begin{pmatrix} a(x_2 - x_1) \\ (c-a)x_1 - x_1x_3 + cx_2 \\ x_1x_2 - bx_3 \end{pmatrix},$$

and the vector function $\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{e}) = \boldsymbol{g}(\boldsymbol{x} + \boldsymbol{e}) - \boldsymbol{g}(\boldsymbol{x})$ is

$$\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{e}) = \begin{pmatrix} a(e_2 - e_1) \\ (c-a)e_1 - e_1e_3 - e_1x_3 - e_3x_1 + ce_2 \\ e_1e_2 + e_1x_2 + e_2x_1 - be_3 \end{pmatrix}.$$

Hence, we can choose that

$$\boldsymbol{e}_1 = e_1 \quad , \boldsymbol{e}_2 = (e_2, e_3)^{\mathrm{T}},$$

$$\boldsymbol{A}_1 = -a, \quad \boldsymbol{A}_2 = \begin{pmatrix} c & 0 \\ 0 & -b \end{pmatrix},$$

$$\boldsymbol{F}_1(\boldsymbol{x}, \boldsymbol{e}_1, \boldsymbol{e}_2) = ae_2,$$

$$\boldsymbol{F}_{21}(\boldsymbol{x}, \boldsymbol{e}_1, \boldsymbol{e}_2) = \begin{pmatrix} (c-a)e_1 - e_1e_3 - e_1x_3 \\ e_1e_2 + e_1x_2 \end{pmatrix},$$

and

$$\boldsymbol{F}_{22}(\boldsymbol{x}, \boldsymbol{e}_1, \boldsymbol{e}_2) = \begin{pmatrix} -e_3x_1 \\ e_2x_1 \end{pmatrix}.$$

Obviously $\lim\limits_{\boldsymbol{e}_1 \to \boldsymbol{0}} \boldsymbol{F}_{21}(\boldsymbol{x}, \boldsymbol{e}_1, \boldsymbol{e}_2) = \boldsymbol{0}$.

According to Theorem 1, we can yield the feedback controller $\boldsymbol{\tau}(\boldsymbol{y}, \boldsymbol{x}) = \boldsymbol{\Omega}(\boldsymbol{x}, \boldsymbol{e})$ easily, that is,

$$\boldsymbol{\Omega}_1(\boldsymbol{x}, \boldsymbol{e}_1, \boldsymbol{e}_2) = \boldsymbol{\Lambda}_1 e_1 - ae_2,$$

$$\boldsymbol{\Omega}_2(\boldsymbol{x}, \boldsymbol{e}_1, \boldsymbol{e}_2) = \boldsymbol{\Lambda}_2 \boldsymbol{e}_2 - \begin{pmatrix} -x_1e_3 \\ x_1e_2 \end{pmatrix}.$$

Therefore, choose suitable $\boldsymbol{\Lambda}_1 \in R^1$, and $\boldsymbol{\Lambda}_2 \in R^{2\times 2}$, which satisfy $|\arg \lambda_i| > 0.5\pi q_1 (q_1 = q_{r1} = 1)$ and $|\arg \lambda_i| > 0.5\pi q_2$ $(q_2 = q_{r2} = q_{r3} = 1)$, then the synchronization between response system (16) and drive system (15) can be achieved.

For example, Choose $\boldsymbol{\Lambda}_1 = 34$, and $\boldsymbol{\Lambda}_2 = \begin{pmatrix} -29 & -3 \\ 3 & 2 \end{pmatrix}$. The eigenvalue λ_i of $\boldsymbol{A}_1 + \boldsymbol{\Lambda}_1$, and eigenvalue λ_i of $\boldsymbol{A}_2 + \boldsymbol{\Lambda}_2$ are $\lambda_1 = -1$, and $\lambda_{2,3} = -1 \pm 3\mathrm{j}$, respectively. The corresponding numerical result is shown in Fig. 3, where $\varepsilon = \left(\sum_{i=1}^{3} e_i^2\right)^{1/2}$.

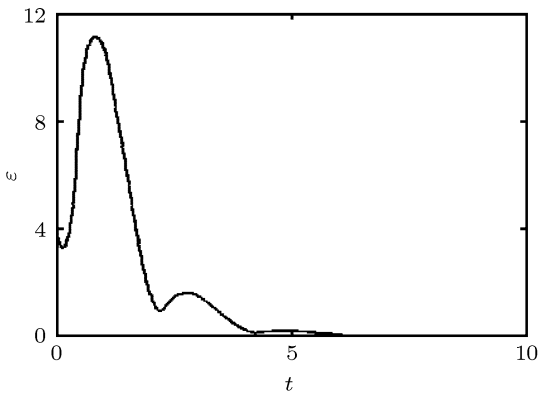


Fig. 3. Synchronization between response system (16) and drive system (15).

3.2. Synchronization between two different fractional-order chaotic systems

Now, consider the synchronization between the 3D fractional-order Lü chaotic system and a new 3D fractional-order chaotic system.

The Lü system, found by Lü and Chen, is known as a bridge between the Lorenz and the Chen systems and represents the transition from one to the other.[32] The Lü system is described by

$$\begin{cases} \mathrm{d}x_1/\mathrm{d}t = a(x_2 - x_1), \\ \mathrm{d}x_2/\mathrm{d}t = -x_1x_3 + cx_2, \\ \mathrm{d}x_3/\mathrm{d}t = x_1x_2 - bx_3. \end{cases}$$

Lü system exhibits a chaotic behaviour at the parameters (a, b, c) = (35, 3, 28). The 3D fractional-order Lü chaotic system[13] is

$$\begin{cases} \mathrm{d}^{q_{\mathrm{d}1}}x_1/\mathrm{d}t^{q_{\mathrm{d}1}} = a(x_2 - x_1), \\ \mathrm{d}^{q_{\mathrm{d}2}}x_2/\mathrm{d}t^{q_{\mathrm{d}2}} = -x_1x_3 + cx_2, \\ \mathrm{d}^{q_{\mathrm{d}3}}x_3/\mathrm{d}t^{q_{\mathrm{d}3}} = x_1x_2 - bx_3. \end{cases} \tag{17}$$

The fractional-order Lü system (17) shows to be chaotic[13] for fractional-order $q > 0.82$. The chaotic attractors for $q_{\mathrm{d}1} = 0.85$, and $q_{\mathrm{d}2} = 0.9$, and $q_{\mathrm{d}3} = 0.9$ are shown in Fig. 4.

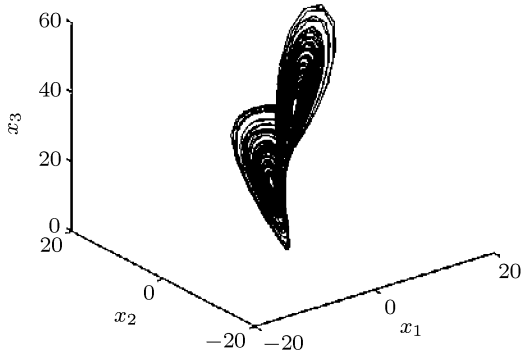


Fig. 4. Chaotic attractor of fractional-order Lü system.

A new 3D fractional-order chaotic system[33] is reported by Sheu *et al.* It is described by

$$\begin{cases} \mathrm{d}^{q_{r1}}y_1/\mathrm{d}t^{q_{r1}} = 5y_1 - y_2y_3, \\ \mathrm{d}^{q_{r2}}y_2/\mathrm{d}t^{q_{r2}} = -10y_2 + y_1y_3, \\ \mathrm{d}^{q_{r3}}y_3/\mathrm{d}t^{q_{r3}} = y_1y_2/3 - 3.8y_3. \end{cases} \tag{18}$$

The chaotic attractors for $q_{r1} = 0.8$, and $q_{r2} = 0.9$, and $q_{r3} = 0.95$ are shown in Fig. 5.

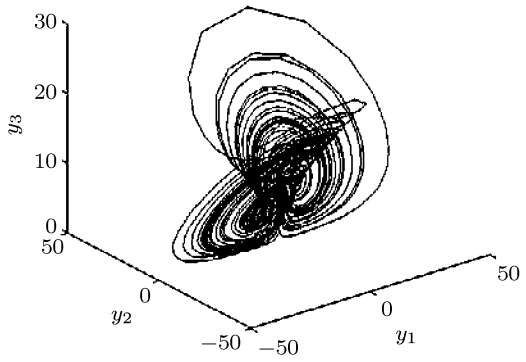


Fig. 5. Chaotic attractor of a new fractional-order system.

Now, we take fractional-order Lü chaotic system (17) as a drive system, and the new fractional-order chaotic system (18) as a response system. The response system is

$$\begin{pmatrix} \mathrm{d}^{q_{r1}}y_1/\mathrm{d}t^{q_{r1}} \\ \mathrm{d}^{q_{r2}}y_2/\mathrm{d}t^{q_{r2}} \\ \mathrm{d}^{q_{r3}}y_3/\mathrm{d}t^{q_{r3}} \end{pmatrix} = \begin{pmatrix} 5y_1 - y_2y_3 \\ -10y_2 + y_1y_3 \\ y_1y_2/3 - 3.8y_3 \end{pmatrix} + \boldsymbol{\theta}(\boldsymbol{x}) + \tau(\boldsymbol{y}, \boldsymbol{x}). \tag{19}$$

According to the above-mentioned, we can choose that

$$\boldsymbol{e}_1 = e_1 \quad , \boldsymbol{e}_2 = (e_2, e_3)^{\mathrm{T}},$$

$$\boldsymbol{A}_1 = 5, \quad \boldsymbol{A}_2 = \begin{pmatrix} -10 & 0 \\ 0 & -3.8 \end{pmatrix},$$

$$\boldsymbol{F}_1(\boldsymbol{x}, \boldsymbol{e}_1, \boldsymbol{e}_2) = -x_3e_2 - x_2e_3 - e_2e_3,$$

$$\boldsymbol{F}_{21}(\boldsymbol{x}, \boldsymbol{e}_1, \boldsymbol{e}_2) = \begin{pmatrix} e_1e_3 + e_1x_3 \\ (e_1e_2 + e_1x_2)/3 \end{pmatrix},$$

and

$$\boldsymbol{F}_{22}(\boldsymbol{x}, \boldsymbol{e}_1, \boldsymbol{e}_2) = \begin{pmatrix} e_3x_1 \\ e_2x_1/3 \end{pmatrix}.$$

Obviously $\lim\limits_{\boldsymbol{e}_1 \to \boldsymbol{0}} \boldsymbol{F}_{21}(\boldsymbol{x}, \boldsymbol{e}_1, \boldsymbol{e}_2) = \boldsymbol{0}$.

Therefore, choose suitable $\boldsymbol{\Lambda}_1 \in R^1$, and $\boldsymbol{\Lambda}_2 \in R^{2\times 2}$, which satisfy $|\arg \lambda_i| > 0.5\pi q_1$ ($q_1 = q_{r1}$, $i = 1$) and $|\arg \lambda_i| > 0.5\pi q_2$ ($q_2 = \max(q_{r2}, q_{r3})$, $i = 1, 2$), then the synchronization between response system (19) and drive system (17) can be achieved.

For example, Choose $\boldsymbol{\Lambda}_1 = -10$, and $\boldsymbol{\Lambda}_2 = \begin{pmatrix} -10 & 1 \\ -1 & -16.2 \end{pmatrix}$. The eigenvalue λ_i of $\boldsymbol{A}_1 + \boldsymbol{\Lambda}_1$, and eigenvalue λ_i of $\boldsymbol{A}_2 + \boldsymbol{\Lambda}_2$ are $\lambda_1 = -5$, and $\lambda_{2,3} = -20 \pm \mathrm{j}$, respectively. The corresponding numerical result is shown in Fig. 6, where $\varepsilon = (\sum_{i=1}^{3} e_i^2)^{1/2}$.

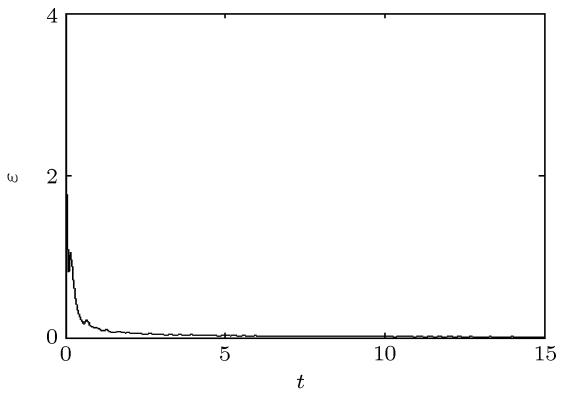


Fig. 6. Synchronization between response system (19) and drive system (17).

4. Conclusions

In this paper, the synchronization between fractional-order chaotic systems and chaotic systems of integer orders is achieved, and the synchronization between different fractional-order chaotic systems is also realized. The proposed synchronization approach is simple and theoretically rigorous. Finally, the synchronization between fractional-order Lorenz chaotic system and integer order Chen chaotic system, and the synchronization between fractional-order Lü chaotic system and a new fractional-order chaotic system, are used to illustrate the effectiveness of the proposed scheme.

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