An improved impulsive control approach to robust lag synchronization between two different chaotic systems*

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In this paper, a novel robust impulsive lag synchronization scheme for different chaotic systems with parametric uncertainties is proposed. Based on the theory of impulsive functional differential equations and a new differential inequality, some new and less conservative sufficient conditions are established to guarantee that the error dynamics can converge to a predetermined region. Finally, some numerical simulations for the Lorenz system and Chen system are given to demonstrate the effectiveness and feasibility of the proposed method. Compared with the existing results based on so-called dual-stage impulsive control, the derived results reduce the complexity of impulsive controller, moreover, a larger stable region can be obtained under the same parameters, which can be shown in the numerical simulations finally.

Keywords: chaotic systems, robust lag synchronization, impulsive control, parametric uncertainty

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1. Introduction

Since the pioneering work of Pecora and Carroll,[1] many different methods have been used theoretically and experimentally to synchronize the chaotic systems, such as linear and nonlinear feedback control,[2−4] backstepping control,[5−7] sliding mode control,[8−10] adaptive control,[11−14] impulsive control,[15−20] active control,[21,22] nonlinear observer approach,[23−25] etc. Among these methods, impulsive control is an efficient method to deal with the dynamical systems which cannot be controlled by continuous control methods. In addition, in the synchronization process the response system receives the information from the drive system only at the discrete time instants, which dramatically reduces the amount of synchronization information transmitted from the drive system to the response system and has been applied to a large number of chaotic-based communication systems which exhibit good performance for the security purposes.[26]

Most synchronization methods are focused on synchronizing two identical chaotic systems. However, experimental and even more real systems are often not fully identical. In particular, there are mismatches in parameters of the systems. It is thus important and also interesting to investigate synchronization behaviour between two different chaotic systems. Thus, some studies on synchronization between two different chaotic systems have been performed.[17−19]

In Refs. [17] and [18], the robust impulsive synchronizations with error bound for different uncertain chaotic systems are proposed. However, the impulsive controller takes effect only when the error satisfies $\|\boldsymbol{e}\| > \xi$ (constant $\xi > 0$ refers to the synchronization error bound), and once the error magnitude $\|\boldsymbol{e}\|$ is less than or equal to ξ, the criteria in Refs. [17] and [18] will become unsupported. To overcome this kind of instability, Refs. [19] and [20] propose the so-called dual-stage impulsive control method (corresponding to the switch of impulsive distances δ_1 and δ_2), which can keep the error within a predetermined region. It is worth pointing out that, the switch time is very often short in comparison with the whole synchronization process. This is to say, the dual-stage impulsive method mainly uses the second impulsive distance, δ_2. In fact, to a large extent the switch increases the complexity of the controller in practical application. In this paper, we further study the robust lag synchronization scheme described in Ref. [19], and by establishing a new differential inequality, a novel and efficient impulsive control method is proposed. Compared with the existing results in Refs. [19] and [20], the results in this paper are less conservative and more practical. Some numerical simulations show the effec-

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tiveness and feasibility of the proposed method finally.

Notations: Throughout the paper, $\mathbb{R}$, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the real number, the n-dimensional Euclidean space and the set of all $n\times m$ real matrices, respectively. $\mathbb{R}^+ \doteq [0,\infty)$. $\mathbb{N} \doteq \{1,2,\ldots\}$. $\boldsymbol{I}$ denotes the identity matrix with an appropriate dimension. $\|\cdot\|$ refers to Euclidean vector norm or the induced matrix 2-norm. $\lambda_{\max}(\boldsymbol{A})(\lambda_{\min}(\boldsymbol{A}))$ represents the maximum (minimum) eigenvalue of matrix $\boldsymbol{A}$.

2. Problem formulation and some preliminaries

Considering channel time-delay η, we give the drive system at time $t-\eta$ as follows:

$$\dot{\boldsymbol{x}}(t-\eta) = (\boldsymbol{A}+\Delta\boldsymbol{A}(t-\eta))\boldsymbol{x}(t-\eta)+f(\boldsymbol{x}(t-\eta)), \quad (1)$$

where $\boldsymbol{x}(\cdot)\in\mathbb{R}^n$ is the state vector, $\boldsymbol{A}\in\mathbb{R}^{n\times n}$ is a known constant matrix, and continuous nonlinear function $f(\boldsymbol{x})\in\mathbb{R}^n$ satisfies the Lipschitz condition with Lipschitz constant $\sigma>0$, i.e.

$$\|f(\boldsymbol{x})-f(\tilde{\boldsymbol{x}})\|\le\sigma\|\boldsymbol{x}-\tilde{\boldsymbol{x}}\|, \quad (2)$$

where $\boldsymbol{x}\in\mathbb{R}^n$ and $\tilde{\boldsymbol{x}}\in\mathbb{R}^n$.

Assume that the parametric uncertainty satisfies the following assumptions:

Assumption 1 Parametric uncertainty $\Delta\boldsymbol{A}(t)$ is in the form of

$$\Delta\boldsymbol{A}(t)=\boldsymbol{E}\boldsymbol{F}(t)\boldsymbol{H}, \quad (3)$$

where $\boldsymbol{E}$ and $\boldsymbol{H}$ are the known real constant matrices with appropriate dimensions, and the uncertain matrix $\boldsymbol{F}(t)\in\mathbb{R}^{n\times n}$ satisfies

$$\|\boldsymbol{F}(t)\|\le 1. \quad (4)$$

Assumption 2[19] Parametric uncertainty $\Delta\boldsymbol{A}(t)$ does not destroy the chaotic behaviour of the chaotic system (1).

We consider the response system as follows:

$$\dot{\tilde{\boldsymbol{x}}}(t)=(\tilde{\boldsymbol{A}}+\Delta\tilde{\boldsymbol{A}}(t))\tilde{\boldsymbol{x}}(t)+g(\tilde{\boldsymbol{x}}(t)), \quad (5)$$

where $\tilde{\boldsymbol{x}}(\cdot)\in\mathbb{R}^n$ is the state vector, $\tilde{\boldsymbol{A}}\in\mathbb{R}^{n\times n}$ is a known constant matrix, and $g(\tilde{\boldsymbol{x}})\in\mathbb{R}^n$ is a continuous nonlinear function. The parametric uncertainty $\Delta\tilde{\boldsymbol{A}}(t)$ satisfies the same assumptions as that of $\Delta\boldsymbol{A}(t)$, i.e. $\Delta\tilde{\boldsymbol{A}}(t)=\tilde{\boldsymbol{E}}\tilde{\boldsymbol{F}}(t)\tilde{\boldsymbol{H}}$ and $\|\tilde{\boldsymbol{F}}(t)\|\le 1$.

Due to the boundedness of the chaotic signals, there exists a constant $\varsigma>0$ such that $\|\tilde{\boldsymbol{x}}\|\le\varsigma$.

Assumption 3[19] For the nonlinear functions $f(\cdot)$ and $g(\cdot)$, there exists a positive constant μ

$$\|f(\tilde{\boldsymbol{x}})-g(\tilde{\boldsymbol{x}})\|\le\mu\|\tilde{\boldsymbol{x}}\|. \quad (6)$$

Suppose that a discrete instant set $\{t_k\}$ satisfies $0<t_1<t_2<\ldots<t_k<t_{k+1}<\ldots$, $\lim_{k\to\infty}t_k=\infty$, $k\in\mathbb{N}$, $0\le t_0<t_1$. At discrete time t_k, the state variables of the drive system are transmitted to the response system as the control input so that the state variables of the response system are suddenly changed at these instants. Therefore, the response system can be written in the following form:

$$\begin{cases}\dot{\tilde{\boldsymbol{x}}}(t)=(\tilde{\boldsymbol{A}}+\Delta\tilde{\boldsymbol{A}}(t))\tilde{\boldsymbol{x}}(t)+g(\tilde{\boldsymbol{x}}(t)),\ t\ne t_k,\ k\in\mathbb{N},\\ \Delta\tilde{\boldsymbol{x}}(t_k)=\tilde{\boldsymbol{x}}(t_k^+)-\tilde{\boldsymbol{x}}(t_k^-)=-\boldsymbol{B}_k\boldsymbol{e}(t_k),\ t=t_k,\\ \tilde{\boldsymbol{x}}(t_0^+)=\tilde{\boldsymbol{x}}_0,\end{cases} \quad (7)$$

where $\boldsymbol{e}(t)=\boldsymbol{x}(t-\eta)-\tilde{\boldsymbol{x}}(t)=[x_1(t-\eta)-\tilde{x}_1(t),x_2(t-\eta)-\tilde{x}_2(t),\ldots,x_n(t-\eta)-\tilde{x}_n(t)]^{\mathrm{T}}$ is the synchronization error, $\lim_{t\to t_k^+}\tilde{\boldsymbol{x}}(t)=\tilde{\boldsymbol{x}}(t_k^+)$, $\lim_{t\to t_k^-}\tilde{\boldsymbol{x}}(t)=\tilde{\boldsymbol{x}}(t_k^-)$, $\tilde{\boldsymbol{x}}(t)$ is left-continuous at $t=t_k$ which implies $\tilde{\boldsymbol{x}}(t_k)=\tilde{\boldsymbol{x}}(t_k^-)$, $\boldsymbol{B}_k\in\mathbb{R}^{n\times n}$ are the impulsive control gains. $\Delta\boldsymbol{x}(t_k)=\boldsymbol{x}(t_k^+)-\boldsymbol{x}(t_k^-)=0$. Then from Eqs. (1) and (7), the following error system is obtained:

$$\begin{cases}\dot{\boldsymbol{e}}(t)=(\boldsymbol{A}+\Delta\boldsymbol{A}(t-\eta))\boldsymbol{e}(t)+h(t)\\ \qquad\quad +(\boldsymbol{A}-\tilde{\boldsymbol{A}}+\Delta\boldsymbol{A}(t-\eta)-\Delta\tilde{\boldsymbol{A}}(t))\tilde{\boldsymbol{x}}(t)\\ \qquad\quad +f(\tilde{\boldsymbol{x}}(t))-g(\tilde{\boldsymbol{x}}(t)),\ t\ne t_k,\ k\in\mathbb{N},\\ \Delta\boldsymbol{e}|_{t=t_k}=\boldsymbol{e}(t_k^+)-\boldsymbol{e}(t_k)=\boldsymbol{B}_k\boldsymbol{e}(t_k),\ t=t_k,\\ \boldsymbol{e}(t_0^+)=\boldsymbol{e}_0,\end{cases} \quad (8)$$

where $h(t)\doteq f(\boldsymbol{x}(t-\eta))-f(\tilde{\boldsymbol{x}}(t))$.

Note that origin $\boldsymbol{e}(t)=0$ is not an equilibrium point of error system (8), which implies that it is impossible to achieve an accurate (complete) synchronization between systems (1) and (7). Therefore, the following definition of synchronization with error bound is employed.

Definition 1 The synchronization between systems (1) and (7) is said to be achieved with error bound $\xi>0$ if, for arbitrary initial conditions $\boldsymbol{x}_0$ and $\tilde{\boldsymbol{x}}_0$, the trivial solution of error system (8) asymptotically converges to a predetermined neighbourhood of the origin for any admissible parametric uncertainty that satisfies Assumptions 1 and 2.

Here our goal is to set control gains $\boldsymbol{B}_k$ and impulsive distances $\delta_k=t_k-t_{k-1}$ ($k\in\mathbb{N}$) such that the error magnitude $\|\boldsymbol{e}\|$ converges to a neighbourhood of the origin with given radius ξ (i.e. region $\Upsilon\doteq\{\boldsymbol{e}\in\mathbb{R}^n:\|\boldsymbol{e}\|\le\xi\}$) if the error starts with

$\|\boldsymbol{e}\| > \xi$, which implies that impulsive controlled response system (7) is synchronized with drive system (1) for arbitrary initial conditions.

In Refs. [19] and [20], to keep the synchronization error within region $\Upsilon_1 \doteq \{\boldsymbol{e} \in \mathbb{R}^n : \|\boldsymbol{e}\| \leq \xi_1\}$ all along, the so-called dual-stage impulsive control method is proposed. If error $\boldsymbol{e}(t)$ starts with $\|\boldsymbol{e}\| > \xi_1$, one chooses impulsive distance δ_1 such that $\boldsymbol{e}(t)$ enters Υ_1. If error $\boldsymbol{e}(t)$ enters Υ_1, one chooses impulsive distance δ_2 such that $\boldsymbol{e}(t)$ enters $\Upsilon_2 \doteq \{\boldsymbol{e} \in \mathbb{R}^n : \|\boldsymbol{e}\| \leq \xi_2\}$. It is worth noting that the time for entering Υ_1 (corresponding to T_1 in Refs. [19] and [20]) is very short in comparison with the whole synchronization process. In other words, the main impulsive distance, which is used for almost the whole synchronization process, is δ_2. On the other hand, the switching impulsive distances will produce the complexity of impulsive controller, therefore, the single impulsive controller with δ_2 maybe more reasonable in the sense of the implementation simplicity. In this paper, under the same synchronization scheme and the same parameters as those in Ref. [19], we will use single impulsive controller to achieve the synchronization with error bound in which it will be proven that the derived impulsive distance is larger than δ_2 in the simulation, and thus the derived result is less conservative and also avoids unnecessary expense of control energy.

Before giving our main results, we present the following dilemma which will be employed for future derivations.

Lemma 1[27] For any continuous function $u(t) : \mathbb{R}^+ \to \mathbb{R}^+$, if there exist positive constants α and β such that

$$\dot{u}(t) \leq \alpha u(t) + \beta\sqrt{u(t)},\ t \geq t^* \geq 0, \tag{9}$$

then $u(t)$ can be bounded by

$$\sqrt{u(t)} \leq \mathrm{e}^{\frac{\alpha}{2}(t-t^*)}\sqrt{u(t^*)} + \frac{\beta}{\alpha}\left(\mathrm{e}^{\frac{\alpha}{2}(t-t^*)} - 1\right). \tag{10}$$

By using the Grownwell–Bellman inequality[27] and some simple mathematical deduction, Lemma 1 can be obtained easily, and the details are omitted for avoiding unnecessary presentation.

For simplicity, in the following sections, $\boldsymbol{x}(t-\eta)$, $\tilde{\boldsymbol{x}}(t)$, $h(t)$, $\Delta\boldsymbol{A}(t-\eta)$ and $\Delta\tilde{\boldsymbol{A}}(t)$ are denoted by $\boldsymbol{x}$, $\tilde{\boldsymbol{x}}$, h, $\Delta\boldsymbol{A}$ and $\Delta\tilde{\boldsymbol{A}}$ respectively.

3. Main results

The following theorem will give the sufficient conditions for robust stability of error system (8), which implies that impulsively controlled response system (7) is synchronized with drive system (1) with error bound ξ.

Theorem 1 If the following conditions hold:

$$\lambda_j \exp\left(\frac{\alpha\delta_{j+1}}{2}\right) < 1, j \in \mathbb{N}, \tag{11}$$

$$\frac{\beta}{\alpha}\left(\sum_{i=1}^{k}\left(\exp\left(\frac{\alpha\delta_{k-i+1}}{2}\right)-1\right)\left(\prod_{l=k-i+1}^{k}\lambda_l \exp\left(\frac{\alpha\delta_{l+1}}{2}\right)\right) + \exp\left(\frac{\alpha\delta_{k+1}}{2}\right) - 1\right) < \xi, \tag{12}$$

then response system (7) is synchronized with drive system (1) with error bound ξ, where $\alpha = \lambda_A + 2\sigma + 2\|\boldsymbol{E}\|\|\boldsymbol{H}\|$, $\beta = 2\varsigma\left(\|\boldsymbol{A}-\tilde{\boldsymbol{A}}\| + \|\boldsymbol{E}\|\|\boldsymbol{H}\| + \|\tilde{\boldsymbol{E}}\|\|\tilde{\boldsymbol{H}}\| + \mu\right)$, $\lambda_k = \sqrt{\lambda_{\max}((\boldsymbol{I}+\boldsymbol{B}_k)^{\mathrm{T}}(\boldsymbol{I}+\boldsymbol{B}_k))}$ and $\lambda_A = \lambda_{\max}(\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}})$.

Proof Choose the following Lyapunov function:

$$V(\boldsymbol{e}) = \boldsymbol{e}^{\mathrm{T}}\boldsymbol{e}. \tag{13}$$

For $t \in (t_{k-1}, t_k]$ and $k \in \mathbb{N}$, the time derivative of $V(\boldsymbol{e})$ along the solution of Eq. (8) is

$$\begin{aligned}
\dot{V}(\boldsymbol{e}(t)) &= \left((\boldsymbol{A}+\Delta\boldsymbol{A})\boldsymbol{e} + h + (\boldsymbol{A}-\tilde{\boldsymbol{A}}+\Delta\boldsymbol{A}-\Delta\tilde{\boldsymbol{A}})\tilde{\boldsymbol{x}} + f(\tilde{\boldsymbol{x}}) - g(\tilde{\boldsymbol{x}})\right)^{\mathrm{T}}\boldsymbol{e} \\
&\quad + \boldsymbol{e}^{\mathrm{T}}\left((\boldsymbol{A}+\Delta\boldsymbol{A})\boldsymbol{e} + h + (\boldsymbol{A}-\tilde{\boldsymbol{A}}+\Delta\boldsymbol{A}-\Delta\tilde{\boldsymbol{A}})\tilde{\boldsymbol{x}} + f(\tilde{\boldsymbol{x}}) - g(\tilde{\boldsymbol{x}})\right) \\
&= \boldsymbol{e}^{\mathrm{T}}(\boldsymbol{A}+\boldsymbol{A}^{\mathrm{T}})\boldsymbol{e} + 2\boldsymbol{e}^{\mathrm{T}}h + \boldsymbol{e}^{\mathrm{T}}(\Delta\boldsymbol{A}+\Delta\boldsymbol{A}^{\mathrm{T}})\boldsymbol{e} + 2\boldsymbol{e}^{\mathrm{T}}(\boldsymbol{A}-\tilde{\boldsymbol{A}}+\Delta\boldsymbol{A}-\Delta\tilde{\boldsymbol{A}})\tilde{\boldsymbol{x}} + 2\boldsymbol{e}^{\mathrm{T}}(f(\tilde{\boldsymbol{x}}) - g(\tilde{\boldsymbol{x}})) \\
&\leq (\lambda_A + 2\sigma + 2\|\boldsymbol{E}\|\|\boldsymbol{H}\|)\boldsymbol{e}^{\mathrm{T}}\boldsymbol{e} + \beta\|\boldsymbol{e}\| \\
&= \alpha V(\boldsymbol{e}) + \beta\sqrt{V(\boldsymbol{e})}.
\end{aligned} \tag{14}$$

From Lemma 1, inequality (14) further becomes

$$\begin{aligned}\|\boldsymbol{e}(t)\| = \sqrt{V(\boldsymbol{e})} &\leq \exp\left(\frac{\alpha}{2}(t-t_{k-1})\right)\left\|\boldsymbol{e}(t_{k-1}^{+})\right\| + \frac{\beta}{\alpha}\left(\exp\left(\frac{\alpha}{2}(t-t_{k-1})\right)-1\right)\\ &\leq \exp\left(\frac{\alpha\delta_k}{2}\right)\left\|\boldsymbol{e}(t_{k-1}^{+})\right\| + \frac{\beta}{\alpha}\left(\exp\left(\frac{\alpha\delta_k}{2}\right)-1\right). \end{aligned} \tag{15}$$

On the other hand, from the second equation of Eq. (8), we obtain

$$\left\|\boldsymbol{e}(t_k^{+})\right\| = \|(\boldsymbol{I}+\boldsymbol{B}_k)\boldsymbol{e}(t_k)\| = \sqrt{\boldsymbol{e}^{\mathrm{T}}(t_k)(\boldsymbol{I}+\boldsymbol{B}_k)^{\mathrm{T}}(\boldsymbol{I}+\boldsymbol{B}_k)\boldsymbol{e}(t_k)} \leq \lambda_k\|\boldsymbol{e}(t_k)\|. \tag{16}$$

Thus, let $k=1$ in inequality (15), then we will have

$$\|\boldsymbol{e}(t)\| \leq \exp\left(\frac{\alpha\delta_1}{2}\right)\|\boldsymbol{e}(t_0)\| + \frac{\beta}{\alpha}\left(\exp\left(\frac{\alpha\delta_1}{2}\right)-1\right), \quad t\in(t_0,t_1],$$

which leads to

$$\|\boldsymbol{e}(t_1)\| \leq \exp\left(\frac{\alpha\delta_1}{2}\right)\|\boldsymbol{e}(t_0)\| + \frac{\beta}{\alpha}\left(\exp\left(\frac{\alpha\delta_1}{2}\right)-1\right),$$

and from Eq. (16), we further have

$$\left\|\boldsymbol{e}(t_1^{+})\right\| \leq \lambda_1\|\boldsymbol{e}(t_1)\| \leq \lambda_1\exp\left(\frac{\alpha\delta_1}{2}\right)\|\boldsymbol{e}(t_0)\| + \frac{\lambda_1\beta}{\alpha}\left(\exp\left(\frac{\alpha\delta_1}{2}\right)-1\right).$$

Similarly, for $t\in(t_1,t_2]$,

$$\begin{aligned}\|\boldsymbol{e}(t)\| &\leq \exp\left(\frac{\alpha\delta_2}{2}\right)\left\|\boldsymbol{e}(t_1^{+})\right\| + \frac{\beta}{\alpha}\left(\exp\left(\frac{\alpha\delta_2}{2}\right)-1\right)\\ &\leq \exp\left(\frac{\alpha\delta_2}{2}\right)\left(\lambda_1\exp\left(\frac{\alpha\delta_1}{2}\right)\|\boldsymbol{e}(t_0)\| + \frac{\lambda_1\beta}{\alpha}\left(\exp\left(\frac{\alpha\delta_1}{2}\right)-1\right)\right) + \frac{\beta}{\alpha}\left(\exp\left(\frac{\alpha\delta_2}{2}\right)-1\right)\\ &= \lambda_1\exp\left(\frac{\alpha\delta_2}{2}\right)\exp\left(\frac{\alpha\delta_1}{2}\right)\|\boldsymbol{e}(t_0)\| + \frac{\beta}{\alpha}\left(\lambda_1\exp\left(\frac{\alpha\delta_2}{2}\right)\left(\exp\left(\frac{\alpha\delta_1}{2}\right)-1\right)+\exp\left(\frac{\alpha\delta_2}{2}\right)-1\right).\end{aligned}$$

In general, for any $t\in(t_k,t_{k+1}]$, we can deduce that

$$\begin{aligned}\|\boldsymbol{e}(t)\| \leq& \lambda_1\exp\left(\frac{\alpha\delta_2}{2}\right)\lambda_2\exp\left(\frac{\alpha\delta_3}{2}\right)\cdots\lambda_k\exp\left(\frac{\alpha\delta_{k+1}}{2}\right)\exp\left(\frac{\alpha\delta_1}{2}\right)\|\boldsymbol{e}(t_0)\|\\ &+\frac{\beta}{\alpha}\left(\lambda_1\exp\left(\frac{\alpha\delta_2}{2}\right)\cdots\lambda_k\exp\left(\frac{\alpha\delta_{k+1}}{2}\right)\left(\exp\left(\frac{\alpha\delta_1}{2}\right)-1\right)\right.\\ &+\lambda_2\exp\left(\frac{\alpha\delta_3}{2}\right)\cdots\lambda_k\exp\left(\frac{\alpha\delta_{k+1}}{2}\right)\left(\exp\left(\frac{\alpha\delta_2}{2}\right)-1\right)\\ &\left.+\cdots+\lambda_k\exp\left(\frac{\alpha\delta_{k+1}}{2}\right)\left(\exp\left(\frac{\alpha\delta_k}{2}\right)-1\right)+\exp\left(\frac{\alpha\delta_{k+1}}{2}\right)-1\right).\end{aligned} \tag{17}$$

From inequality (11), there must exist a positive constant $\rho<1$ such that

$$\lambda_j\exp\left(\frac{\alpha\delta_{j+1}}{2}\right)\leq\rho,\ j\in\mathbb{N}, \tag{18}$$

then inequality (17) further becomes

$$\begin{aligned}\|\boldsymbol{e}(t)\| &\leq \rho^k\exp\left(\frac{\alpha\delta_1}{2}\right)\|\boldsymbol{e}(t_0)\| + \frac{\beta}{\alpha}\left(\sum_{i=1}^{k}\left(\exp\left(\frac{\alpha\delta_{k-i+1}}{2}\right)-1\right)\right.\\ &\quad\times\left.\left(\prod_{l=k-i+1}^{k}\lambda_l\exp\left(\frac{\alpha\delta_{l+1}}{2}\right)\right)+\exp\left(\frac{\alpha\delta_{k+1}}{2}\right)-1\right)\\ &< \rho^k\exp\left(\frac{\alpha\delta_1}{2}\right)\|\boldsymbol{e}(t_0)\| + \xi.\end{aligned} \tag{19}$$

Thus $\|\boldsymbol{e}(t)\| \leq \xi$ as $k \to \infty$ (or $t \to \infty$), and we can conclude that synchronization error magnitude $\|\boldsymbol{e}(t)\|$ can asymptotically converge to a value below bound ξ, which implies that response system (7) is synchronized with drive system (1) with error bound ξ. This completes the proof. □

Remark 1 To show the reasonableness of inequality (12), next we will prove the convergence of the following series:

$$\Sigma_k = \sum_{i=1}^{k} \left(\exp\left(\frac{\alpha\delta_{k-i+1}}{2}\right) - 1 \right) \left(\prod_{l=k-i+1}^{k} \lambda_l \exp\left(\frac{\alpha\delta_{l+1}}{2}\right) \right) + \exp\left(\frac{\alpha\delta_{k+1}}{2}\right) - 1. \tag{20}$$

Let $\bar{\delta} = \sup_{k\in\mathbb{N}} \delta_k$. From inequality (18), Eq. (20) becomes

$$\Sigma_k \leq \left(\exp\left(\frac{\alpha\bar{\delta}}{2}\right) - 1 \right) \sum_{i=1}^{k} \rho^i + \exp\left(\frac{\alpha\bar{\delta}}{2}\right) - 1 = \sum_{i=0}^{k} \rho^i \left(\exp\left(\frac{\alpha\bar{\delta}}{2}\right) - 1 \right) \leq \frac{1}{1-\rho} \left(\exp\left(\frac{\alpha\bar{\delta}}{2}\right) - 1 \right), \tag{21}$$

where the last inequality in (21) uses the fact $\rho < 1$. Therefore, we can obtain that the series Σ_k is convergent, which implies the reasonableness of inequality (12).

In practice, for the sake of convenience, gain matrices $\boldsymbol{B}_k$ are always selected as a constant matrix and impulsive distances δ_k are set to be positive constants, then we have the following corollary straightforwardly.

Corollary 1 Assume that $\delta_k = \delta$ and $\boldsymbol{B}_k = \boldsymbol{B}$ ($k \in \mathbb{N}$). If the following conditions hold:

$$\lambda \exp\left(\frac{\alpha\delta}{2}\right) < 1, \tag{22}$$

$$\frac{\beta\left(\exp\left(\frac{\alpha\delta}{2}\right) - 1\right)}{\alpha\left(1 - \lambda\exp\left(\frac{\alpha\delta}{2}\right)\right)} < \xi, \tag{23}$$

then response system (7) is synchronized with drive system (1) with error bound ξ, where

$$\lambda = \sqrt{\lambda_{\max}((\boldsymbol{I}+\boldsymbol{B})^{\mathrm{T}}(\boldsymbol{I}+\boldsymbol{B}))},$$

α and β are defined as those in Theorem 1.

Proof Replace λ_k and δ_k in inequality (11) with λ and δ respectively, then one can have inequality (22). Meanwhile, from inequalities (22) and (23), inequality (17) further becomes

$$\begin{aligned}
\|\boldsymbol{e}(t)\| &\leq \rho^k \exp\left(\frac{\alpha\delta}{2}\right) \|\boldsymbol{e}(t_0)\| + \frac{\beta}{\alpha}\left(\sum_{i=1}^{k} \left(\exp\left(\frac{\alpha\delta}{2}\right) - 1 \right) \left(\prod_{l=k-i+1}^{k} \lambda \exp\left(\frac{\alpha\delta}{2}\right) \right) + \exp\left(\frac{\alpha\delta}{2}\right) - 1 \right) \\
&\leq \rho^k \exp\left(\frac{\alpha\delta}{2}\right) \|\boldsymbol{e}(t_0)\| + \frac{\beta}{\alpha}\left(\left(\exp\left(\frac{\alpha\delta}{2}\right) - 1 \right) \sum_{i=1}^{k} \left(\lambda \exp\left(\frac{\alpha\delta}{2}\right) \right)^i + \exp\left(\frac{\alpha\delta}{2}\right) - 1 \right) \\
&= \rho^k \exp\left(\frac{\alpha\delta}{2}\right) \|\boldsymbol{e}(t_0)\| + \frac{\beta}{\alpha}\left(\exp\left(\frac{\alpha\delta}{2}\right) - 1 \right) \sum_{i=0}^{k} \left(\lambda \exp\left(\frac{\alpha\delta}{2}\right) \right)^i \\
&\leq \rho^k \exp\left(\frac{\alpha\delta}{2}\right) \|\boldsymbol{e}(t_0)\| + \frac{\beta\left(\exp\left(\frac{\alpha\delta}{2}\right) - 1\right)}{\alpha\left(1 - \lambda\exp\left(\frac{\alpha\delta}{2}\right)\right)} \\
&< \rho^k \exp\left(\frac{\alpha\delta}{2}\right) \|\boldsymbol{e}(t_0)\| + \xi.
\end{aligned} \tag{24}$$

Thus response system (7) is synchronized with drive system (1) with error bound ξ. To avoid unnecessary iteration, the details of proof are omitted. This completes the proof. □

4. Simulation results

Assume that Lorenz system[28] drives Chen system.[29] Therefore, the drive and the response systems are as follows:

$$\begin{cases} \dot{x}_1 = a_1(x_2 - x_1), \\ \dot{x}_2 = a_2 x_1 - x_1 x_3 - x_2, \\ \dot{x}_3 = x_1 x_2 - a_3 x_3 \end{cases} \tag{25}$$

and

$$\begin{cases} \dot{\tilde{x}}_1 = b_1(\tilde{x}_2 - \tilde{x}_1), \\ \dot{\tilde{x}}_2 = (b_2 - b_1)\tilde{x}_1 - \tilde{x}_1\tilde{x}_3 + b_2\tilde{x}_2, \\ \dot{\tilde{x}}_3 = \tilde{x}_1\tilde{x}_2 - b_3\tilde{x}_3. \end{cases} \tag{26}$$

From Eqs. (8), (25) and (26), we can have the following error system:

$$\begin{cases} \dot{\boldsymbol{e}} = (\boldsymbol{A} + \Delta\boldsymbol{A})\boldsymbol{e} + h + (\boldsymbol{A} - \tilde{\boldsymbol{A}} + \Delta\boldsymbol{A} - \Delta\tilde{\boldsymbol{A}})\tilde{\boldsymbol{x}} \\ \qquad + f(\tilde{\boldsymbol{x}}) - g(\tilde{\boldsymbol{x}}),\ t \neq t_k,\ k \in \mathbb{N}, \\ \Delta\boldsymbol{e}|_{t=t_k} = \boldsymbol{e}(t_k^+) - \boldsymbol{e}(t_k) = \boldsymbol{B}\boldsymbol{e}(t_k),\ t = t_k, \\ \boldsymbol{e}(t_0^+) = \boldsymbol{e}_0. \end{cases}$$

To compare our results with the existing ones, in the simulation, we use the same parameters as those in Ref. [19]. Let $\boldsymbol{B}_k = \boldsymbol{B} = \mathrm{diag}\{d, d, d\}$, and then $\lambda = |d + 1|$. Let $a_1 = 10$, $a_2 = 28$, $a_3 = 8/3$, $b_1 = 35$, $b_2 = 28$ and $b_3 = 3$, then we will have

$$\boldsymbol{A} = \begin{pmatrix} -a_1 & a_1 & 0 \\ a_2 & -1 & 0 \\ 0 & 0 & -a_3 \end{pmatrix} = \begin{pmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -8/3 \end{pmatrix},$$

$$\tilde{\boldsymbol{A}} = \begin{pmatrix} -b_1 & b_1 & 0 \\ b_2 - b_1 & b_2 & 0 \\ 0 & 0 & -b_3 \end{pmatrix} = \begin{pmatrix} -35 & 35 & 0 \\ -7 & 28 & 0 \\ 0 & 0 & -3 \end{pmatrix},$$

$$\Delta\boldsymbol{A} = -\Delta\tilde{\boldsymbol{A}} = 0.04 \begin{pmatrix} \sin t & 0 & 0 \\ 0 & \cos t & 0 \\ 0 & 0 & \sin t \end{pmatrix},$$

where

$$\boldsymbol{E} = \boldsymbol{H} = \tilde{\boldsymbol{E}} = \tilde{\boldsymbol{H}} = 0.2\boldsymbol{I},$$

$$\boldsymbol{F}(t) = -\tilde{\boldsymbol{F}}(t) = \begin{pmatrix} \sin t & 0 & 0 \\ 0 & \cos t & 0 \\ 0 & 0 & \sin t \end{pmatrix}.$$

From $f(\tilde{\boldsymbol{x}}) = g(\tilde{\boldsymbol{x}}) = (0, -\tilde{x}_1\tilde{x}_3, \tilde{x}_1\tilde{x}_2)^{\mathrm{T}}$, we have $\mu = 0$.

In this simulation, we choose $\sigma = 27.5$ and $\varsigma = 60$. By simple computation, we can have $\lambda_A = 28.0512$, $\|\boldsymbol{A} - \tilde{\boldsymbol{A}}\| = 57.5257$, $\|\boldsymbol{E}\|\,\|\boldsymbol{H}\| = \|\tilde{\boldsymbol{E}}\|\|\tilde{\boldsymbol{H}}\| = 0.04$. Let $\xi = 0.5$, $d = -0.98$, 'unknown' channel time-delay $\eta = 3$ s, the initial conditions of the drive and the response systems will be taken as $\boldsymbol{x}_0 = (4,\ 1,\ 3)^{\mathrm{T}}$ and $\tilde{\boldsymbol{x}}_0 = (3,\ -1,\ 5)^{\mathrm{T}}$, respectively. Hence, from inequalities (22) and (23) in Corollary 1, the estimate of bound of stable region is given by $\delta < 1.4134 \times 10^{-4}$.

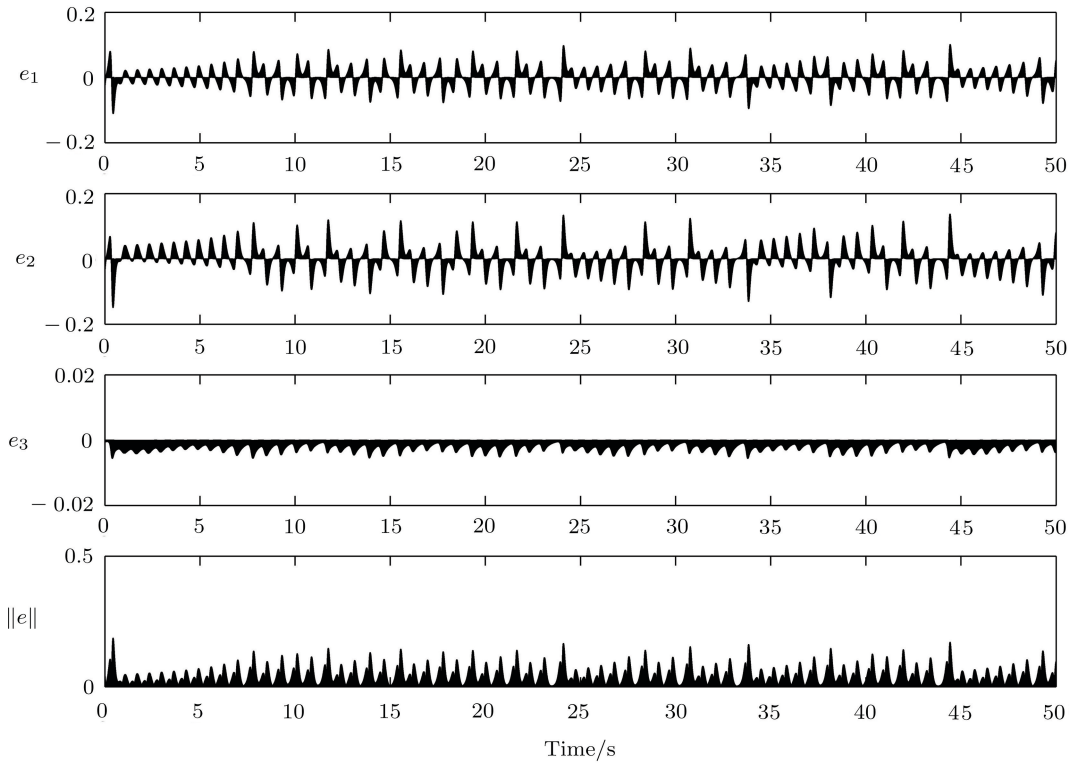


Fig. 1. Synchronization errors and their magnitudes with $\delta = 1.4 \times 10^{-4}$ and $\lambda = 0.02$.

Under the same synchronization scheme and the same system parameters, Ref. [19] uses the so-called dual-stage impulsive control method (i.e., under the fixed control gain, if $\boldsymbol{e}(t)$ starts with $\|\boldsymbol{e}(t)\| > \xi$, one chooses impulsive distance δ_1, and if $\|\boldsymbol{e}(t)\| \leq \xi$, the impulsive distance switches to δ_2), and the impulsive distances obtained are $\delta_1 = 5.5568 \times 10^{-4}$ and $\delta_2 = 1.1412 \times 10^{-4}$ respectively. It is worth noting that under stronger impulsive control gain (corresponding to $\lambda = |d+1| = 0.02$) and higher impulsive control frequency (corresponding to $\delta_1 = 5.5568 \times 10^{-4}$), synchronization error magnitude $\|\boldsymbol{e}(t)\|$ will converge to a value below bound ξ in a very short time. That is to say, to achieve the robust synchronization with error bound ξ, impulsive distance $\delta_2 = 1.1412 \times 10^{-4}$ is necessary for almost the whole synchronization process. In comparison, in this paper, we use the single impulsive controller with larger impulsive distance $\delta = 1.4134 \times 10^{-4}$ (other parameters are unchangeable) in the whole synchronization process, which is less conservative ($\delta > \delta_2$ implies larger stable region) and more practical (the switching impulsive distance in Refs. [19] and [20] will add complexity to impulsive controller inevitably).

Robust impulsive lag synchronization between Lorenz system and Chen system with impulsive distance $\delta = 1.4 \times 10^{-4}$ and gain coefficient $\lambda = |d+1| = 0.02$ is shown in Fig.1. As can be seen, the synchronization has been achieved practically and $\|\boldsymbol{e}\|$ is smaller than $\xi = 0.5$.

5. Conclusions

In this paper, we present an improved impulsive control method to synchronize two different chaotic systems with different parametric uncertainties. Based on an effective differential inequality and the theory of impulsive differential equations, a robust lag synchronization with error bound together with some less conservative conditions is presented to guarantee that the synchronization error dynamics can converge to a predetermined level. Compared with the existing results, ours can obtain a larger stable region, moreover, in the sense of the implementation simplicity, the method in the paper is more practical. Finally, numerical simulations are given to demonstrate the effectiveness of the method.

References

[1] Pecora L M and Carroll T L 1990 *Phys. Rev. Lett.* **64** 821
[2] Ma T D, Zhang H G and Fu J 2008 *Chin. Phys.* B **17** 4407
[3] Meng J and Wang X Y 2008 *Chaos* **18** 023108
[4] Cui B T and Lou X Y 2009 *Chaos, Solitons and Fractals* **39** 288
[5] Vincent U E, Ucar A, Laoye J A and Kareem S O 2008 *Physica* C **468** 374
[6] Lin C M, Peng Y F and Lin M H 2009 *Chaos, Solitons and Fractals* **42** 981
[7] Lü L, Zhang Q L and Guo Z A 2008 *Chin. Phys.* B **17** 498
[8] Vasegh N and Khellat F 2009 *Chaos, Solitons and Fractals* **42** 1054
[9] Zribi M, Smaoui N and Salim H 2009 *Chaos, Solitons and Fractals* **42** 3197
[10] Kong C C and Chen S H 2009 *Chin. Phys.* B **18** 91
[11] Zhang H G, Huang W, Wang Z L and Chai T Y 2006 *Phys. Lett.* A **350** 363
[12] Zhang H G, Xie Y H, Wang Z L and Zheng C D 2007 *IEEE Trans. Neural Network* **18** 1841
[13] Lee T H and Park J H 2009 *Chin. Phys. Lett.* **26** 090507
[14] Zhu C X 2009 *Appl. Math. Comput.* **215** 557
[15] Ma T D, Zhang H G and Wang Z L 2007 *Acta Phys. Sin.* **56** 3796 (in Chinese)
[16] Zhang H G, Ma T D, Fu J and Tong S C 2009 *Chin. Phys.* B **18** 3742
[17] Haeri M and Dehghani M 2008 *Chaos, Solitons and Fractals* **38** 120
[18] Zhang H G, Ma T D, Yu W and Fu J 2008 *Chin. Phys.* B **17** 3616
[19] Zhang H G, Ma T D, Fu J and Tong S C 2009 *Chin. Phys.* B **18** 3751
[20] Zhang H G, Ma T D, Huang G B and Wang Z L 2010 *IEEE Trans. Syst. Man Cybern.* B **40** 831
[21] Njah A N 2009*J. Sound Vib.* **327** 322
[22] Tang R A, Liu Y L and Xue J K 2009 *Phys. Lett.* A **373** 1449
[23] Perruquetti W, Floquet T and Moulay E 2008 *IEEE Trans. Autom. Contr.* **53** 356
[24] Fradkov A L, Andrievsky B and Evans R J 2008 *IEEE Trans. Circuits Syst. I* Reg. Papers **55** 1685
[25] Wang X Y and Meng J 2008 *Chaos* **18** 033102
[26] Yang T 2001 *Impulsive Control Theory* (Berlin: Spinger-Verlag) p48
[27] Khalil H K 1995 *Nonlinear Systems* 2nd Edition (New Jersey: Prentice Hall) p63
[28] Stewart I 2000 *Nature* **406** 948
[29] Chen G R and Ueta T 1999 *Int. J. Bifurc. Chaos* **9** 1465