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Energy and first law of thermodynamics for Born–Infeld–anti-de-Sitter black hole*

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We calculate the local energy and the energy density of the Reisner–Norström–anti-de-Sitter black hole, study the first law of thermodynamics and show the Smarr formula for the Born–Infeld–anti-de-Sitter black hole. Applying the first law of thermodynamics to the black hole region, we analyse the three energy exchange processes between the black hole region and the outer and the inner regions.

Keywords: first law of thermodynamics, energy distribution, energy exchange
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1. Introduction

The energy localization is one of the most important problems in the general relativity theory.[1−6] To calculate the local energy, Einstein first proposed the energy–momentum complex (a pseudo-tensor). After that, the Landau–Lifshitz, Tolman, Papapetrou, Bergmann, Weinberg and Møller and Komar proposed some different energy–momentum complexes (see Refs. [7]and [8]). Using the Landau–Lifshitz energy–momentum complex, the energy and the angular momentum distributions of the Kerr–Newmann black hole were calculated.[9] The energy distributions for the spherically symmetric black holes including the Reisner–Norström–anti-de-Sitter (RN–AdS) black hole were studied by some authors.[10−13] Every energy–momentum complex can be associated with a Hamiltonian boundary term.[14] So, finding the appropriate energy–momentum complex is usually a difficult task.

There is a deep relation between the first law of thermodynamics for black hole, $\mathrm{d}E = \mathrm{d}Q + \mathrm{d}W$, and the gravitational field equations.[15−20] It is found that the local energy obtained from the Einstein energy distribution satisfies the first law of thermodynamics.[21,22] This implies that in the sense of thermodynamics the Einstein energy distribution should be appropriate. The energy of the Schwarzschild black hole is fully localized in the region enclosed by the event horizon. For the Schwarzschild black hole with a global monopole, the energy can be distributed in the whole spacetime.[23] For the Reisner–Norström black hole, the energy in the black hole region is $\frac{1}{2}(r_+ - r_-)$ and the energies localized in the outer and the inner regions are both $\frac{1}{2}r_-$.[22] The Born–Infeld action includes the coupling term of gravity with a cosmological constant and the non-linear electrodynamic term.[24−26] The Born–Infeld–anti-de-Sitter (BI–AdS) black hole may be described by mass, electric charge, cosmological radius (or cosmological constant) and Born–Infeld parameter. The properties of the BI–AdS black hole, including the global and the local stabilities, Hawking–Page phase transition, and phase structure were studied.[27] The thermodynamic properties of the BI–AdS black hole with the cosmological term[28,29] and without the cosmological term[30] were also discussed.

In this paper, we further study the thermodynamic properties of the BI–AdS black hole. First, we calculate the energy and the energy density of the RN–AdS black hole. Then, we show the Smarr formula of the BI–AdS black hole and give the first laws of thermodynamics for the mass and the local energies. In order to obtain a further understanding of the first law of thermodynamics for the black hole region, we analyse the three typical thermodynamic processes.

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2. First law of thermodynamics of BI–AdS black hole

This section is composed of three subsections. In the first subsection, we calculate the energy and the energy density of the RN–AdS black hole. In the second subsection, we give the Smarr mass formula for the BI–AdS black hole, calculate the local energy of the BI–AdS black hole and show the first law of thermodynamics. In the third subsection, we analyse the three energy exchange processes of the black hole region.

2.1. Energy and energy density of RN–AdS black hole region

The spherically symmetric black hole spacetime metric is given by

$$\mathrm{d}s^2 = f\mathrm{d}t^2 - f^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2). \quad (1)$$

For the RN–AdS black hole,

$$f = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{1}{3}\Lambda r^2$$

and the radii of the outer and the inner horizons are determined by

$$1 - \frac{2M}{r_\pm} + \frac{Q^2}{r_\pm^2} - \frac{1}{3}\Lambda r_\pm^2 = 0, \quad (2)$$

where $\Lambda = -3/l^2$, with l being the AdS cosmological radius, M and Q are the mass and electric charge, respectively. Solving Eq. (2) yields the mass and the electric charge of the RN–AdS black hole as

$$M = \frac{r_+ + r_-}{2}\left(1 + \frac{r_+^2 + r_-^2}{l^2}\right), \quad (3)$$

$$Q = \sqrt{r_+ r_-\left(1 + \frac{r_+^2 + r_+ r_- + r_-^2}{l^2}\right)}. \quad (4)$$

The thermal properties such as the Hawking radiation of the RN–AdS black hole were studied.[31,32] We discuss the problem of the energy of the RN–AdS black hole. From the Einstein energy distribution,[11] the energy inside the region of radius r in the spacetime described by metric (1) is obtained as

$$E = E(r) = \frac{1}{2}r(1 - f). \quad (5)$$

Though the energy localized in the region outside the outer horizon depends on metric function f, the energies inside the outer and the inner horizons appear independent of it. For the RN–AdS black hole, Eq. (5) gives the following energy distribution function:

$$E = E(r) = M - \frac{Q^2}{2r} + \frac{1}{6}\Lambda r^3. \quad (6)$$

From Eq. (6), the energies in the regions enclosed by the outer and the inner horizons are obtained as

$$E_\pm = E(r_\pm) = M - \frac{Q^2}{2r_\pm} + \frac{1}{6}\Lambda r_\pm^3. \quad (7)$$

The total black hole spacetime may be divided into three regions: the black hole region ($r_- \le r \le r_+$), the outer and inner regions ($0 < r \le r_-$ and $r \ge r_+$). Defining the energy in the black hole region as the energy difference,[21,22] $E_{\mathrm{BH}} = E_+ - E_-$, then for the RN–AdS black hole one can have

$$E_{\mathrm{BH}} = \frac{1}{2}(r_+ - r_-), \quad (8)$$

where $r_\pm = r_\pm(M, Q, l)$ are the two functions of M, Q and l.

The area of the horizon of black hole is calculated from[33]

$$A = \int \sqrt{g}\mathrm{d}\theta\mathrm{d}\varphi, \quad (9)$$

where g is the determinant of the two-dimensional sphere metric. For metric (1), there is $g_{tt}g_{rr} = -1$. So, Eq. (9) may also be written as $A = \int \sqrt{-g_{(4)}}\mathrm{d}\theta\mathrm{d}\varphi$, with $g_{(4)}$ being the determinant of the 4-spacetime metric. The volume (three-dimensional spatial volume) of the spacetime described by metric (1) is given by[34]

$$V = \int \mathrm{d}^3x\sqrt{-g_{(4)}} = \int r^2\mathrm{d}r \int \sin\theta\mathrm{d}\theta\mathrm{d}\varphi. \quad (10)$$

According to Eq. (10), the volume of the region enclosed by the two-dimensional hyper-surface of radius $r < r_-$ is obtained as $\bar{V}_{\mathrm{in}} = \frac{4}{3}\pi r^3$, which gives the volume of the inner region bounded by the inner horizon, $V_{\mathrm{in}} = \frac{4}{3}\pi r_-^3$, when $r = r_-$. The black hole region bounded by the outer and the inner horizons has a different topological property from those the outer and the inner regions have, which is the lack of the spacetime having the property of $f(r) > 0$. From this logic and considering volume formula (10), we infer that the volume of the black hole region should be given as

$$V_{\mathrm{BH}} = V_{\mathrm{out}} - V_{\mathrm{in}} = \frac{4}{3}\pi(r_+^3 - r_-^3) \quad (11)$$

with $V_{\mathrm{out}} = \frac{4}{3}\pi r_+^3$. Defining the energy density of the black hole region as $\rho_{\mathrm{BH}} = E_{\mathrm{BH}}/V_{\mathrm{BH}}$, then we obtain

$$\rho_{\mathrm{BH}} = \frac{3}{4\pi(r_+^2 + r_+ r_- + r_-^2)}. \quad (12)$$

When $r_- \ll r_+$, the energy density reduces to $\rho_{\rm BH} \simeq 3/4\pi r_+^2$, which is the energy density of the Schwarzschild black hole, the maximum for a given outer event radius; for $r_- \simeq r_+$ there is $\rho_{\rm BH} \simeq 1/4\pi r_+^2$, the minimum for a given outer event radius.

2.2. First laws of thermodynamics for mass and local energies of BI−AdS black hole

The $(n+1)$-dimensional Einstein–Born–Infeld action with a cosmological constant is given by[24,25]

$$S = \int \mathrm{d}^n x\sqrt{-g}\left[\frac{R-2\Lambda}{16\pi G} + 4b^2\left(1+\sqrt{1+\frac{F^2}{2b^2}}\right)\right], \tag{13}$$

where R is the scalar curvature, $F^2 = F^{\mu\nu}F_{\mu\nu}$ with $F_{\mu\nu}$ denoting the electromagnetic fields, Λ is cosmological constant and b (β in Refs. [24] and [25]) is the Born–Infeld parameter having the dimension of mass.

The $(3+1)$-dimensional Born–Infeld–anti-de-Sitter black hole is described by metric (1) with f given by[28]

$$f = 1 - \frac{2M}{r} + \frac{r^2}{l^2} + \frac{2b^2r^2}{3}\left(1-\sqrt{1+\frac{Q^2}{b^2r^4}}\right) + \frac{4Q^2}{3r^2}F\left(\frac{1}{4},\frac{1}{2},\frac{5}{4},-\frac{Q^2}{b^2r^4}\right). \tag{14}$$

When $b\to\infty$, the BI–AdS metric approaches the RN–AdS metric. Assume that the BI–AdS black hole has two horizons. From Eq. (14), one can obtain the mass of the BI–AdS black hole as

$$M(r_\pm,Q,l,b) = \frac{1}{2}r_\pm + \frac{r_\pm^3}{2l^2} + \frac{b^2r_\pm^3}{3}\left(1-\sqrt{1+\frac{Q^2}{b^2r_\pm^4}}\right) + \frac{2Q^2}{3r_\pm}F\left(\frac{1}{4},\frac{1}{2},\frac{5}{4},-\frac{Q^2}{b^2r_\pm^4}\right), \tag{15}$$

which gives the mass expression shown in Ref. [28] for the case of $M(r_+,Q,l,b)$. Considering the mass as a function of r_+ (or r_-), Q, l and b, and then differentiating it yield[35]

$$\mathrm{d}M = T_\pm \mathrm{d}S_\pm + \Phi_\pm \mathrm{d}Q + \Theta_\pm \mathrm{d}l + \Gamma_\pm \mathrm{d}b, \tag{16}$$

where

$$T_\pm = \left(\frac{\partial M}{\partial S_\pm}\right)_{Q,l,b} = \frac{1}{4\pi}\left[\sqrt{\frac{\pi}{S_\pm}} + \frac{3}{l^2}\sqrt{\frac{S_\pm}{\pi}} + 2b^2\sqrt{\frac{S_\pm}{\pi}}\left(1-\sqrt{1+\frac{\pi^2Q^2}{b^2S_\pm^2}}\right)\right], \tag{17}$$

$$\Phi_\pm = \left(\frac{\partial M}{\partial Q}\right)_{r_\pm,l,b} = Q\sqrt{\frac{\pi}{S_\pm}}F\left(\frac{1}{4},\frac{1}{2},\frac{5}{4},-\frac{\pi^2Q^2}{b^2S_\pm^2}\right), \tag{18}$$

$$\Theta_\pm = -\left(\frac{\partial M}{\partial l}\right)_{r_\pm,Q,b} = \frac{1}{l^3}\left(\frac{S_\pm}{\pi}\right)^{3/2}, \tag{19}$$

$$\Gamma_\pm = -\left(\frac{\partial M}{\partial b}\right)_{r_\pm,Q,l} = -\frac{2b}{3}\left(\frac{S_\pm}{\pi}\right)^{3/2}\left(1-\sqrt{1+\frac{\pi^2Q^2}{b^2S_\pm^2}}\right) - \frac{Q^2}{3br_\pm}\sqrt{\frac{\pi}{S_\pm}}F\left(\frac{1}{4},\frac{1}{2},\frac{5}{4},-\frac{\pi^2Q^2}{b^2S_\pm^2}\right). \tag{20}$$

Equation (16) is the first law of thermodynamics associated with the mass of the BI–AdS black hole. With parameters l and b fixed, Eq. (16) reduces to Eq. (3.4) in Ref. [24]. From Eqs. (17)–(20), we obtain the Smarr mass formula for the BI–AdS black hole, expressed as

$$M = 2T_\pm S_\pm + \Phi_\pm Q + \Theta_\pm l + \Gamma_\pm b. \tag{21}$$

From Eqs. (5) and (14) we have the energies in the regions enclosed by the outer and the inner horizons as follows:

$$E_\pm = E(r_\pm) = M - \frac{r_\pm^3}{2l^2} - \frac{b^2r_\pm^3}{3}\left(1-\sqrt{1+\frac{Q^2}{b^2r_\pm^4}}\right) - \frac{2Q^2}{3r_\pm}F\left(\frac{1}{4},\frac{1}{2},\frac{5}{4},-\frac{Q^2}{b^2r_\pm^4}\right). \tag{22}$$

From $P(r) = -\rho(r) = -(1/4\pi r^2)(\mathrm{d}E/\mathrm{d}r)$, with $\rho(r)$ being the energy density, the pressures on the outer and the inner horizons of the BI–AdS black hole are given as

$$P_\pm = P(r_\pm) = \frac{1}{4\pi r_\pm^2}\left[\frac{3r_\pm^2}{2l^2} + b^2r_\pm^2\left(1-\sqrt{1+\frac{Q^2}{b^2r_\pm^4}}\right)\right]. \tag{23}$$

Substituting Eq. (15) into Eq. (22) gives the energies in the regions inclosed by the outer and the inner horizons as follows:

$$E_{\pm} = \frac{1}{2} r_{\pm}. \tag{24}$$

From the temperatures given in Eq. (17), the pressures given in Eq. (23), the entropies of the outer and the inner horizons, $S_{\pm} = \pi r_{\pm}^2$, and the volumes enclosed by them, $V_{\pm} = \frac{4}{3}\pi r_{\pm}^3$, one can have

$$\mathrm{d}E_{\pm} = T_{\pm}\mathrm{d}S_{\pm} - P_{\pm}\mathrm{d}V_{\pm}, \tag{25}$$

which we call the first law of thermodynamics associated with the energies inside the outer and the inner horizons of the BI–AdS black hole.

Combining the two equations in Eq. (25) leads to[21]

$$\mathrm{d}E_{\mathrm{BH}} = \mathrm{d}Q_{\mathrm{BH}} + \mathrm{d}W_{\mathrm{BH}}, \tag{26}$$

where

$$\mathrm{d}Q_{\mathrm{BH}} = T_{+}\mathrm{d}S_{+} - T_{-}\mathrm{d}S_{-} \tag{27}$$

and

$$\mathrm{d}W_{\mathrm{BH}} = -(P_{+}\mathrm{d}V_{+} - P_{-}\mathrm{d}V_{-}) \tag{28}$$

denote the heat flux into hole and the work done by the two horizons in an infinitesimal thermodynamic process, respectively. We call Eq. (26) the first law of thermodynamics associated with the energy of the black hole region of the BI–AdS black hole.

2.3. Energy exchange process of BI–AdS black hole region

The first law of thermodynamics associated with the energy in the region enclosed by the outer horizon describes the energy exchange process of the hole with the outer region. Though the inner region of the black hole is not visible, the energy exchange of the hole through the inner horizon can occur. The first law of thermodynamics for the black hole region of the BI–AdS black hole can reflect how the hole exchanges the energy with both the outer and the inner regions.

For the purpose of illustrating the first law of thermodynamics given by Eq. (26), we consider the RN–AdS black hole. First, we analyse the process in which the heat flowing into the hole through one horizon equals the heat flowing out through the other horizon. For such a process, $\mathrm{d}Q_{\mathrm{BH}} = 0$ yields

$$\left(\frac{\mathrm{d}r_{-}}{\mathrm{d}r_{+}}\right)_{\mathrm{A}} = -\frac{r_{-}}{r_{+}}\frac{l^2 + 2r_{+}^2 + (r_{+}+r_{-})^2}{l^2 + 2r_{-}^2 + (r_{+}+r_{-})^2}, \tag{29}$$

where the subscript A denotes the adiabatic process and the temperatures of the outer and the inner horizons

$$T_{\pm} = \frac{r_{\pm} - r_{\mp}}{4\pi r_{\pm}^2}\left[1 + \frac{(r_{+}+r_{-})^2 + 2r_{\pm}^2}{l^2}\right] \tag{30}$$

have been used. For $l \to \infty$, i.e., the case of the RN black hole, Eq. (29) reduces to

$$\left(\frac{\mathrm{d}r_{-}}{\mathrm{d}r_{+}}\right)_{\mathrm{A}} = -\frac{r_{-}}{r_{+}}.$$

When

$$\frac{\mathrm{d}r_{-}}{\mathrm{d}r_{+}} > \left(\frac{\mathrm{d}r_{-}}{\mathrm{d}r_{+}}\right)_{\mathrm{A}},$$

there is $\mathrm{d}Q_{\mathrm{BH}} > 0$, which indicates that the heat flows into the black hole through the outer or the inner horizon; otherwise, the heat flows out from the hole.

Next, we come to the process in which no work is done by the two horizon. For $b \to \infty$, Eq. (23) gives the pressures of the outer and the inner horizons of the RN–AdS black hole as

$$P_{\pm} = \frac{1}{4\pi r_{\pm}^2}\left(\frac{3r_{\pm}^2}{2l^2} - \frac{Q^2}{2r_{\pm}^2}\right). \tag{31}$$

From $\mathrm{d}W_{\mathrm{BH}} = 0$, we have

$$\left(\frac{\mathrm{d}r_{-}}{\mathrm{d}r_{+}}\right)_{\mathrm{W}} = \frac{Q^2/r_{+}^2 - 3r_{+}^2/l^2}{Q^2/r_{-}^2 - 3r_{-}^2/l^2}, \tag{32}$$

where the subscript W denotes the process of the work being zero, which reduces to

$$\left(\frac{\mathrm{d}r_{-}}{\mathrm{d}r_{+}}\right)_{\mathrm{W}} = \left(\frac{r_{-}}{r_{+}}\right)^2$$

for $l \to \infty$. When $\frac{\mathrm{d}r_{-}}{\mathrm{d}r_{+}} < \left(\frac{\mathrm{d}r_{-}}{\mathrm{d}r_{+}}\right)_{\mathrm{W}}$, $\mathrm{d}W_{\mathrm{BH}} > 0$ holds, namely, the total work done by the outer and the inner horizons is positive.

Assume that the energy of the black hole region keeps unchanged, then there will be

$$\left(\frac{\mathrm{d}r_{-}}{\mathrm{d}r_{+}}\right)_{\mathrm{C}} = 1, \tag{33}$$

where the subscript C denotes the energy conversation in the hole region, which yields

$$\left(\frac{\mathrm{d}Q}{\mathrm{d}M}\right)_{\mathrm{C}} = \frac{r_{+}+r_{-}}{2Q}\frac{l^2 + r_{+}^2 + 4r_{+}r_{-} + r_{-}^2}{l^2 + r_{+}^2 + r_{-}^2 + (r_{+}+r_{-})^2}. \tag{34}$$

In order to make clear the implication of Eq. (34), we consider the case of $l \to \infty$. For the non-extremal case, Eq. (34) reduces to

$$\left(\frac{\mathrm{d}Q}{\mathrm{d}M}\right)_{\mathrm{C}} = \frac{r_+ + r_-}{2\sqrt{r_+ r_-}} > 1, \tag{35}$$

when $l \to \infty$. For the RN black hole, the extreme condition implies the charge–mass ratio $Q/M = 1$. Consequently, assuming that the constraint $(\mathrm{d}Q/\mathrm{d}M)_{\mathrm{C}} \leq 1$ is imposed, the process in which E_{BH} keeps unchanged cannot occur.

3. Discussions

The energy of the black hole region for the space–time described by metric (1) takes the form $E_{\mathrm{BH}} = \frac{1}{2}(r_+ - r_-)$. The first law of thermodynamics associated with the mass may be expressed in terms of the thermodynamic quantities defined on the outer horizon or the inner horizon for the BI–AdS black hole, as is for the case of KN black hole.[36−38] Combining the first laws of thermodynamics associated with the energies inside the outer and the inner horizons of the BI–AdS black hole, we give the first law of thermodynamics for the black hole region.

The energy expression (5), with f given by Eq. (14), can describe the energy distributions of the BI–AdS black hole in the outer and the inner regions. In the black hole region, $f(r) < 0$, the coordinates r and t interchange the roles in denoting space and time. This implies that expression (5) becomes invalid in the black hole region. Nevertheless, we can still take the energy of the black hole region as the difference between E_+ and E_-.

The two horizons of black hole are the two boundaries of the black hole region. So, though the energy exchange process of the hole with the inner region of spacetime is shielded by the outer horizon and thus is invisible, it can occur indeed. The first law of thermodynamics for the black hole region can describe the energy exchange process of the hole with the inner and the outer regions of the spacetime.

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