Thermodynamic geometry of the Garfinkle–Horowitz–Strominger dilaton black hole*

Lan Ming-Jian(兰明建)†, Chen Gang(陈 刚), and Han Yi-Wen(韩亦文)

College of Computer Science, Chongqing Technology and Business University, Chongqing 400067, *China*

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This paper studies the thermodynamic properties of the Garfinkle–Horowitz–Strominger dilaton black hole from the viewpoint of geometry. It calculates the heat capacity and the temperature of the black hole, Weinhold metric and Ruppeiner metric are also obtained respectively. It finds that they are both curved and the scalar curvature of the Weinhold geometry consists with the first-order transition point reproduced from the capacity, while the Ruppeiner one is both in accordance with the first-order and the second-order phase transition points reproduced from the capacity.

Keywords: black hole, thermodynamic geometry, phase transition

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1. Introduction

Several decades ago, Bekenstein proposed that a black hole can be described as a thermodynamic system, he held the opinion that the surface gravity of a black hole corresponds to its temperature whereas the entropy is proportional to the area of its event horizon.[1,2] Two years later, Hawking made a striking discovery that a classical black hole emits thermal spectrum, which leads to a disturbing and difficult problem about information conservation during black hole evaporation.[3] Based on their works, the four laws of black hole thermodynamics have been established, Hawking radiation from all kinds of black holes has been also studied, it is still one of the focuses in theoretical physics today.[4−10]

In 1975, a geometrical concept was introduced into ordinary thermodynamics by Weinhold.[11] According to his viewpoint, a Riemannian metric can be defined as the second derivative of the internal energy, with respect to entropy and other extensive variables of the system. Several years later, Ruppeiner[12] proposed another geometrical way to study the thermodynamics of equilibrium systems. According to his viewpoint, the line element of a Riemannian metric can be expressed as the second derivative of the entropy, with respect to internal energy and other extensive variables of a thermodynamic system. One of the aims of the application of thermodynamic geometry is to describe phase transitions in terms of curvature singularities. For a thermodynamic system, it is quite interesting to investigate the corresponding relationship between the curvature of Weinhold's and Ruppeiner's metric and the phase transitions. In fact this viewpoint has been applied to various black holes.[13−27] Even though it is widely believed that the thermodynamic geometry of a black hole is still a most fascinating and unresolved subject at present.

In this paper, we study the thermodynamic properties from the viewpoint of geometry and calculate the heat capacity and the temperature of the Garfinkle–Horowitz–Strominger (GHS) dilaton black hole. We also calculate the corresponding curvature scalars in Weinhold geometric and Ruppeiner geometric respectively, we find out the relationship between the curvature of Weinhold's and Ruppeiner's metric and the phase transitions. Throughout the paper, the units $G = c = \hbar = 1$ are used.

2. The event horizon of the GHS dilaton black hole

The action describing the Maxwell field with a dilaton field in 3+1 dimensional space–time is given by[28]

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[-R + 2\left(\nabla\varphi\right)^2 + \mathrm{e}^{-2a\varphi}F^2\right], \quad (1)$$

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†Corresponding author. E-mail: lanmingjian@sohu.com

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where φ is a dilaton field, $F^2 = F_{\mu\nu}F^{\mu\nu}$ is the Maxwell field and a is an arbitrary parameter governing the strength of the coupling between them. In addition, φ and $g_{\mu\nu}$ yield the field equations

$$\nabla_\mu \left(\mathrm{e}^{-2a\varphi} F^{\mu\nu} \right) = 0, \tag{2}$$

$$\nabla^2 \varphi + \frac{1}{2} \mathrm{e}^{-2a\varphi} F_{\mu\nu} F^{\mu\nu} = 0, \tag{3}$$

$$\begin{aligned} & R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \\ = \; & 2\nabla_\mu \varphi \nabla_\nu \varphi - g_{\mu\nu} \left(\nabla \varphi \right)^2 + 2\,\mathrm{e}^{-2a\varphi} F_{\mu\rho} F^{\rho}_{\nu} \\ & - \frac{1}{2} g_{\mu\nu}\, \mathrm{e}^{-2a\varphi} F_{\mu\nu} F^{\mu\nu}. \end{aligned} \tag{4}$$

From these equations, the line element of the GHS dilaton black hole can be expressed as

$$\begin{aligned} \mathrm{d}s^2 = & -\left(1 - \frac{r_+}{r}\right) \left(1 - \frac{r_-}{r}\right)^{\frac{1-a^2}{1+a^2}} \mathrm{d}t^2 \\ & + \left(1 - \frac{r_+}{r}\right)^{-1} \left(1 - \frac{r_-}{r}\right)^{\frac{a^2-1}{1+a^2}} \mathrm{d}r^2 \\ & + r^2 \left(1 - \frac{r_-}{r}\right)^{\frac{2a^2}{1+a^2}} \mathrm{d}\Omega^2, \end{aligned} \tag{5}$$

obviously, it is a spherically symmetric and asymptotically flat solution with the Maxwell field

$$F = Q \sin\theta \mathrm{d}\theta \wedge \mathrm{d}\phi, \tag{6}$$

where $\wedge$ denote the wedge product and the dilaton field is given by

$$\mathrm{e}^{-2\varphi} = \left(1 - \frac{r_-}{r}\right)^{\frac{2a}{1+a^2}}, \tag{7}$$

here the asymptotic constant value of the dilaton field is zero. Under a specified condition, i.e. $a = 1$, one can verify that these expressions reduce to the more common forms of the GHS dilaton black hole,[28−30] but we do not discuss the latter in this paper. In the former expressions, r_+ and r_- are the outer and inner event horizons of the GHS dilaton black hole respectively. In order to simplify the calculation, we let

$$\beta = \frac{1 - a^2}{1 + a^2}, \tag{8}$$

so we have the outer and inner event horizons of the GHS dilaton black hole, namely

$$r_+ = \frac{M + \beta M + \sqrt{(1+\beta)\left[(1+\beta)M^2 - 2\beta Q^2\right]}}{1+\beta}, \tag{9}$$

and

$$r_- = \frac{M + \beta M - \sqrt{(1+\beta)\left[(1+\beta)M^2 - 2\beta Q^2\right]}}{\beta(1+\beta)}, \tag{10}$$

where M and Q are the mass and electric charge of the black hole respectively.

3. Heat capacity of the GHS charged dilaton black hole

To describe its thermodynamic geometry, both the mass and electric charge of the GHS dilaton black hole need to be expressed in terms of the outer and inner event horizons as

$$M = \frac{r_+ + \beta r_-}{2}, \tag{11}$$

and

$$Q^2 = \frac{(1+\beta) r_+ r_-}{2}. \tag{12}$$

By the area law, the entropy of the black hole is given as

$$S = \frac{1}{4} k_{\mathrm{B}} A = k_{\mathrm{B}} \pi r_+^2 = r_+^2, \tag{13}$$

where k_{B} is the Boltzmann constant and we let $k_{\mathrm{B}} = 1/\pi$. From Eq. (11)–(13) , we can obtain

$$M = \frac{\sqrt{S}}{2} \left[1 + \frac{2\beta Q^2}{(1+\beta) S} \right]. \tag{14}$$

According to the first law of black hole thermodynamics

$$\mathrm{d}M = T \mathrm{d}S + \Phi \mathrm{d}Q, \tag{15}$$

where T and Φ are the temperature and the electric potential of the GHS dilaton black hole respectively and they can be expressed in terms of the outer and inner event horizons as

$$T = \left(\frac{\partial M}{\partial S} \right)_Q = \frac{r_+ - \beta r_-}{4 r_+^2}, \tag{16}$$

and

$$\Phi = \left(\frac{\partial M}{\partial Q} \right)_S = \beta \sqrt{\frac{2 r_-}{(1+\beta) r_+}}, \tag{17}$$

where $\beta \leq 1$, therefore, the temperature T is always positive except at the extremal limit where $r_+ = \beta r_-$. For the fixed charge, the heat capacity can be expressed in terms of the outer and inner event horizons as

$$C_Q = T \left(\frac{\partial S}{\partial T} \right)_Q = \frac{2 r_+^2 (r_+ - \beta r_-)}{3\beta r_- - r_+}, \tag{18}$$

we find it vanishes at $r_+ = \beta r_-$, which is considered to be the first-order phase transition point. It is generally believed that the Davies' point where the second-order phase transition takes place corresponds to the diverge point of the heat capacity, so it may indicate that the second-order phase transition takes place at $r_+ = 3\beta r_-$.

4. Weinhold geometry of the GHS dilaton black hole

Now, we turn to discuss thermodynamic geometry of the GHS dilaton black hole. According to Weinhold's viewpoint, the metric can be defined as the second derivative of internal energy.[11] In general, that is

$$g_{ij}^{W} = \partial_i \partial_j M\left(S, N^a\right), \tag{19}$$

where M, S and N^a are the mass, the entropy and any other extensive variables of the black hole, respectively. Substituting Eq. (14) into Eq. (19), we can write the Weinhold metric of the GHS dilaton black hole as

$$g_{SS}^{\mathrm{W}} = -\frac{1}{8\sqrt{S^3}} + \frac{3\beta Q^2}{4\left(1+\beta\right)\sqrt{S^5}}, \tag{20}$$

$$g_{QS}^{\mathrm{W}} = g_{SQ}^{\mathrm{W}} = -\frac{\beta Q}{\left(1+\beta\right)\sqrt{S^3}}, \tag{21}$$

$$g_{QQ}^{\mathrm{W}} = \frac{2\beta}{\left(1+\beta\right)\sqrt{S}}, \tag{22}$$

so the corresponding line element of the Weinhold geometry can be expressed as

$$\begin{aligned} \mathrm{d}s_{\mathrm{W}}^2 = \frac{1}{8\sqrt{S^3}}\Bigg[& -\left(1-\frac{6\beta Q^2}{S+\beta S}\right)\mathrm{d}S^2 \\ & -\frac{16\beta Q}{1+\beta}\mathrm{d}Q\,\mathrm{d}S + \frac{16\beta S}{1+\beta}\mathrm{d}Q^2\Bigg]. \end{aligned} \tag{23}$$

Then we can calculate the scalar curvature of the Weinhold geometry. The Ricci curvature tensor can be expressed as

$$R_{\mu\nu} = \Gamma^{\kappa}_{\mu\nu,\kappa} - \Gamma^{\kappa}_{\mu\kappa,\nu} + \Gamma^{\kappa}_{\rho\kappa}\cdot\Gamma^{\rho}_{\mu\nu} - \Gamma^{\kappa}_{\rho\nu}\cdot\Gamma^{\rho}_{\mu\kappa}, \tag{24}$$

where the Christoffel symbols $\Gamma^{\kappa}_{\mu\nu}$ can be worked out by

$$\Gamma^{\kappa}_{\mu\nu} = \frac{1}{2}g^{\alpha\kappa}\left(\frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}\right). \tag{25}$$

From the Ricci curvature tensor, we can obtain the Riemannian curvature scalar by

$$R = g^{\mu\nu}R_{\mu\nu}. \tag{26}$$

Then, we obtain the curvature scalar of the GHS dilaton black hole by the complicated computation, it can be expressed in terms of the outer and inner event horizons as

$$R_{\mathrm{W}} = -\frac{2r_+}{\left(r_+ - \beta r_-\right)^2}. \tag{27}$$

This curvature scalar is always negative for any value of the horizon radius and no singular point comes into being except the extremal limit at $r_+ = \beta r_-$. It means that the Weinhold metric of the GHS dilaton black hole with the mass as thermodynamic potential presents a curvature singularity only in the limit of an extremal black hole. The result is exactly consistent with the first-order transition point reproduced from the capacity. However, it implies no information about the second-order phase transition.

5. Ruppeiner geometry of the GHS dilaton black hole

The obtained Weinhold geometry of the GHS dilaton black hole consists of the first-order transition point reproduced from the capacity. However, it fails to give the information about the second-order phase transition point, so we turn to do some research on the Ruppeiner curvature of the black hole. First, we can try to obtain the Ruppeiner metric by a usual method. The Ruppeiner metric and the Weinhold metric are related to each other with the following relation:[12]

$$\mathrm{d}s_{\mathrm{R}}^2 = \frac{1}{T}\mathrm{d}s_{\mathrm{W}}^2. \tag{28}$$

By introducing a new coordinate transformation, i.e. $u = Q/\sqrt{S}$, we can obtain the following Weinhold metric after diagonalization:

$$\begin{aligned} & \mathrm{d}s_{\mathrm{W}}^2 \\ & = \frac{1}{8\sqrt{S^3}}\left[-\left(1-\frac{2\beta u^2}{1+\beta}\right)\mathrm{d}S^2 + \frac{16\beta S^2}{1+\beta}\mathrm{d}u^2\right]. \end{aligned} \tag{29}$$

Then we can turn the above Weinhold metric into the Ruppeiner metric in the same coordinates, that is

$$\mathrm{d}s_{\mathrm{R}}^2 = -\frac{1}{2S}\mathrm{d}S^2 + \frac{8\beta S}{1+\beta-2\beta u^2}\mathrm{d}u^2. \tag{30}$$

Now we let $\tau = \sqrt{2S}$ and $u = \sqrt{(1+\beta)/2\beta}\,\sin\left(\sigma/\sqrt{2}\right)$, then we obtain the Ruppeiner metric as

$$\mathrm{d}s_{\mathrm{R}}^2 = -\mathrm{d}\tau^2 + \tau^2\mathrm{d}\sigma^2. \tag{31}$$

Obviously, the metric is flat and the corresponding scalar curvature is zero. This result is not in agreement with the phase transition of the thermodynamic system. In fact, the similar results also come out in some pre-existing research,[17] such as the 3+1 dimension Reissner–Nordström black hole. Nevertheless, a simple change of the thermodynamic potential adopted by Shen *et al.*[26] can solve this problem. In fact, this change is based on the viewpoint

that the phase structure of some black holes is analogous to that of the van der Waals–Maxwell liquid–gas system.[31−35] In order to avoid our result that the Ruppeiner geometry carries no information about phase structure of the thermodynamic system, just like the approach which Shen *et al.* have used, we also redefine the internal energy of the black hole, namely the ADM gravitational mass of the black hole subtracted by the energy of electric field outside the black hole horizon. Therefore the appropriate internal energy should be written as

$$U = M - \Phi Q = \frac{r_+ - \beta r_-}{2}. \tag{32}$$

Combining Eqs. (15) and (32), expressed by the internal energy, we can write the first law of thermodynamics as

$$\mathrm{d}U = T\mathrm{d}S - Q\mathrm{d}\Phi. \tag{33}$$

Considering the entropy as a function of the internal energy and electric potential, and combining it with Eqs.(11)–(13) and Eq. (17), we obtain

$$S\left(U,\Phi\right) = \left[\frac{4\beta U}{2\beta - (1+\beta)\,\Phi^2}\right]^2. \tag{34}$$

The thermodynamic metric introduced in Ruppeiner's theory[12−14] is defined by the second derivative of the entropy, that is

$$g^{\mathrm{R}}_{ij} = \partial_i\partial_j S\left(U, x^\alpha\right), \tag{35}$$

where U denotes the internal energy and $x^\alpha\,(\alpha = 1, 2, 3, \ldots, n)$ is the extensive variables. So the Ruppeiner metric of the GHS dilaton black hole can be written down as

$$g^{\mathrm{R}}_{ij} = \partial_i\partial_j S\left(U, \Phi\right). \tag{36}$$

Substituting Eq. (34) into Eq. (36), we can work out the new Ruppeiner metric of the GHS dilaton black hole as

$$g^{\mathrm{R}}_{UU} = \frac{32\beta^2}{\left(2\beta - (1+\beta)\,\Phi^2\right)^2}, \tag{37}$$

$$g^{\mathrm{R}}_{U\Phi} = g^{\mathrm{R}}_{\Phi U} = \frac{128\beta^2\,(1+\beta)}{\left(2\beta - (1+\beta)\,\Phi^2\right)^3}, \tag{38}$$

$$g^{\mathrm{R}}_{\Phi\Phi} = \frac{64\beta^2U^2\,(1+\beta)\left(2\beta + 5\Phi^2 + 5\beta\Phi^2\right)}{\left(2\beta - (1+\beta)\,\Phi^2\right)^4}, \tag{39}$$

substituting them into Eq. (25), we can work out the Christoffel symbols $\Gamma^\kappa_{\mu\nu}$, then substituting the Christoffel symbols into Eq. (24) we obtain the Ricci curvature tensor $R_{\mu\nu}$. Finally, we can obtain the curvature scalar of the GHS dilaton black hole by using Eq. (26), which can be expressed in terms of the outer and inner event horizons as

$$R_{\mathrm{R}} = -\frac{r_+ - \beta r_-}{r_+\left(r_+ - 3\beta r_-\right)^2}. \tag{40}$$

Obviously, this scalar curvature vanishes only at the extremal limit where $r_+ = \beta r_-$. On the other hand, it goes to negative infinity when $r_+ = 3\beta r_-$, which consists with the second-order transition point reproduced from the capacity.

6. Conclusions

We have studied the thermodynamic geometry of the GHS dilaton black hole. The results show that both the Weinhold geometry and Ruppeiner geometry of it are curved, but the latter is based on the definition of the internal energy which is the ADM gravitational mass of the black hole subtracted by the energy of electric field outside the black hole horizon. In addition, we calculate the temperature and the heat capacity of the black hole and obtain the phase transition point. The results indicate that the scalar curvature of the Weinhold geometry is always negative for any value of the horizon radius and it diverges at $r_+ = \beta r_-$, which only consists with the first-order phase transition point reproduced from the capacity, however, it implies no information about the second-order phase transition. While the Ruppeiner one vanishes only at the extremal limit where $r_+ = \beta r_-$ and it goes to negative infinity when $r_+ = 3\beta r_-$. Obviously, it can give both the first-order and the second-order phase transition points reproduced from the capacity.

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