Thermal entanglement in two-qutrit spin-1 anisotropic Heisenberg model with inhomogeneous magnetic field

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The thermal entanglement of a two-qutrit spin-1 anisotropic Heisenberg XXZ chain in an inhomogeneous magnetic field is studied in detail. The effects of the external magnetic field (B), a parameter b which controls the inhomogeneity of B, and the bilinear interaction parameters $J_x = J_y \neq J_z$ on the thermal variation of the negativity are studied in detail. It is found that negativity N decreases when the values of magnetic field, inhomogeneity b and temperature are increasing. In addition, N remains at higher temperatures for higher values of J_z and lower values of B and b.

Keywords: negativity, XXZ model, thermal entanglement, qutrit, spin-1, magnetic field
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1. Introduction

Entanglement is a nonlocal correlation for the quantum systems, such as for the two-qubit or two-qutrit systems, so this non-classical quantum property can be used as a key concept, such as in quantum computation and quantum-information processing, quantum teleportation, quantum cryptography. Thus it has attracted a lot of attention in recent years both theoretically and experimentally.

There are a wide variety of the studies that are carried out on entanglement. The entangled rings were studied in a translationally invariant quantum state and asked to what extent each pair of nearest neighbours can be entangled.[1] The entanglement between any two spins in one-dimensional Heisenberg chain as a function of temperature and external magnetic field was investigated.[2] The entanglement arising naturally from a one-dimensional Ising chain in a magnetic field in an arbitrary direction was studied.[3] The distributed entanglement was considered in a three-qubit system A, B and C which may be entangled.[4] In order to quantify the entanglement, the conditions that every measure of entanglement has to satisfy were given and a whole class of 'good' entanglement measures were constructed.[5] The entanglement and the spin squeezing in the three-qubit transverse Ising model were studied.[6] It was demonstrated that local transformations on a composite quantum systems can be enhanced in the presence of certain entangled states.[7] The problem of environmentally induced decoherence in a near-critical one-dimensional system of $N >> 1$ coupled qubits was studied.[8] An exact formula of the entanglement of formation for all mixed states of two qubits having no more than two nonzero eigenvalues was obtained.[9] The entanglement formation of an arbitrary state of two-qubits was considered.[10] And the effect of the Dzyaloshinskii–Moriya (DM) interaction on entanglement was investigated.[11]

The Heisenberg models are usually classified according to the equality of the bilinear interaction parameter components, i.e. J_x, J_y and J_z. Thus, the entanglement of a two-qubit anisotropic Heisenberg XY chain was investigated in thermal equilibrium at temperature T in the presence of an external magnetic field along the z-axis,[12] with arbitrarily-directed static magnetic field[13] and with a nonuniform magnetic field.[14] The concurrences of a 2×2 and 3×3 square lattices were calculated for the XY model.[15] The entanglement was studied in the XY model in which the so-called W entangled states can be generated for 3 or 4 qubits.[16] The entanglement teleportation via two separate thermally entangled states of a two-qubit Heisenberg XX chain was studied.[17] The natural thermal entanglement of the two-qubit Heisenberg XXX model was investigated and the results showed for the antiferromagnet that the increasing of the coupling strength or the decreasing of the temperature at the critical point increases the entanglement.[18] The thermal entanglement in the Heisenberg XYZ chain was investigated in the presence of an external magnetic field[19] and in the presence of DM interaction.[20] We can also men-

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tion a few studies of the thermal entanglement of the XXZ Heisenberg chain as follows: under the rectangle magnetic field,[21] for pairwise thermal entanglement in three-qubit model,[22] on how to generate maximally entangled states with defects,[23] for two-qubits with DM interaction,[24] on the relations among concurrence C quantum phase transition and correlation length at zero temperature[25] and in the presence of DM anisotropic antisymmetric interaction.[26]

The above mentioned studies are related to the spin-1/2 models, where the concurrence is usually used as a measure of entanglement. Another measure, negativity, is used as a measure of entanglement in spin-1 models but this model, because of its complexity, has received much less attention. The entanglement in an anisotropic spin-1 Heisenberg chain with a uniform magnetic field was investigated by the use of negativity.[27] The thermal entanglement in a two-qutrit spin-1 system with exchange interaction was investigated in terms of the measure of entanglement called 'negativity'.[28] The entanglement in spin-1 Heisenberg chains was studied by using the concept of negativity, and both bilinear and bilinear-biquadratic chains were considered.[29] The negativity and the entanglement witnesses were used to study quantum phase transitions in bilinear-biquadratic and anisotropic XXZ spin-1 models.[30] The thermal entanglement in a two-qutrit system with two spins coupled with the exchange interaction was investigated in terms of negativity.[31] The thermal entanglement in ferrimagnetic chains was investigated in terms of negativity.[32] The entanglement properties in the mixed spin-(1, s) systems were investigated in terms of negativity.[33] The time evolution of the bound entanglement under the bilinear-biquadratic Hamiltonian was studied by using the partial transpose and realignment method.[34] The negativity was exactly calculated in the two-site extended Hubbard model with external magnetic field.[35] The entanglement was investigated numerically in the supermolecular dimer $[Mn_4]_2$ consisting of a pair of single molecular magnets with antiferromagnetic exchange coupling.[36] Finally, the entangled states with positive partial transposes were constructed by using indecomposable positive linear maps between matrix algebras.[37]

The effect of inhomogeneities in the magnetic field on the thermal entanglement of a two spin system was also studied and it was found that a very small inhomogeneity is capable of producing large values of thermal entanglement for the ferromagnetic (FM) case.[38] Similarly, the thermal entanglement in a two-qubit Heisenberg XXZ spin chain was investigated under an inhomogeneous magnetic field, and it was concluded that the critical magnetic field increases with the increase of inhomogeneity but the maximum entanglement that the system can reach becomes smaller.[39] The thermal entanglement of a two-qubit anisotropic Heisenberg XXZ chain with DM interaction in an inhomogeneous magnetic field was studied in detail.[40]

So far the effects of inhomogeneities in the magnetic field on the thermal entanglement for spin-1 models have not been considered as far as we know. Since the addition of an extra term b makes the problem difficult to cope with. In this work, we consider the thermal entanglement of a two-qutrit spin-1 anisotropic Heisenberg XXZ chain in an inhomogeneous magnetic field and analyse the effects of external magnetic field (B), parameter b which controls the inhomogeneity of B and bilinear interaction parameters $J_x = J_y \neq J_z$ on the thermal variation of the negativity.

The rest of the work is set up as follows: The next section is devoted to the detailed formulation of the problem. In the last section we present our findings and give some discussion and comparisons.

2. Formulations

The Hamiltonian for the isotropic two-qutrit system may be given as

$$\mathcal{H} = J\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + (B+b)\sigma_1^z + (B-b)\sigma_2^z, \qquad (1)$$

where J is the isotropic bilinear interaction between the spin pairs; $B \geq 0$ is the external magnetic field, of which the inhomogeneity is controlled by the parameter b; $\boldsymbol{\sigma}_i$ is the spin operator, $i = 1$ or 2; the σ_i^z is the z-th component. This Hamiltonian, when $J_x = J_y = J \neq J_z$ is taken, is called the anisotropic XXZ model and written as

$$\begin{aligned}\mathcal{H} = {} & J\sigma_1^x\sigma_2^x + J\sigma_1^y\sigma_2^y + J_z\sigma_1^z\sigma_2^z \\ & +(B+b)\sigma_1^z + (B-b)\sigma_2^z. \qquad (2)\end{aligned}$$

The final form of the Hamiltonian can be obtained in terms of the lowering operator (σ_i^-) and raising operator (σ_i^+) for the XXZ model as

$$\begin{aligned}\mathcal{H} = {} & 2J(\sigma_1^+\sigma_2^- + \sigma_1^-\sigma_2^+) + J_z\sigma_1^z\sigma_2^z \\ & +(B+b)\sigma_1^z + (B-b)\sigma_2^z. \qquad (3)\end{aligned}$$

In order to obtain the eigenvalues and the eigenvectors for the two-qutrit spin-1 system from Eq. (3),

we use the usual standard basis states, i.e., $|1,1\rangle$, $|1,0\rangle$, $|1,-1\rangle$, $|0,1\rangle$, $|0,0\rangle$, $|0,-1\rangle$, $|-1,1\rangle$, $|-1,0\rangle$ and $|-1,-1\rangle$. Thus the Hamiltonian can be represented in the matrix form as

$$\mathcal{H}=\begin{pmatrix} J_z+2B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B+b & 0 & 2J & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -J_z+2b & 0 & 2J & 0 & 0 & 0 & 0 \\ 0 & 2J & 0 & B-b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2J & 0 & 0 & 0 & 2J & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -(B-b) & 0 & 2J & 0 \\ 0 & 0 & 0 & 0 & 2J & 0 & -J_z-2b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2J & 0 & -(B+b) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & J_z-2B \end{pmatrix}. \quad (4)$$

The eigenvectors of Eq. (4) can be found after some algebra as

$$\begin{aligned}
|\Psi_1\rangle &= |1,1\rangle, \quad |\Psi_2\rangle = \frac{b+\gamma}{\xi}|1,0\rangle + \frac{2J}{\xi}|0,1\rangle, \\
|\Psi_3\rangle &= \frac{2J}{\xi}|1,0\rangle - \frac{b+\gamma}{\xi}|0,1\rangle, \\
|\Psi_4\rangle &= \frac{b+\gamma}{\xi}|0,-1\rangle + \frac{2J}{\xi}|-1,0\rangle, \\
|\Psi_5\rangle &= \frac{2J}{\xi}|0,-1\rangle - \frac{b+\gamma}{\xi}|-1,0\rangle, \\
|\Psi_6\rangle &= \frac{1}{\sqrt{4J^2(\zeta_1^2+\zeta_2^2)+\zeta_1^2\zeta_2^2}}(2J\zeta_1|1,-1\rangle \\
&\quad +\zeta_1\zeta_2|0,0\rangle + 2J\zeta_2|-1,1\rangle), \\
|\Psi_7\rangle &= \frac{1}{\sqrt{4J^2(\eta_1^2+\eta_2^2)+\eta_1^2\eta_2^2}}(2J\eta_1|1,-1\rangle \\
&\quad +\eta_1\eta_2|0,0\rangle + 2J\eta_2|-1,1\rangle), \\
|\Psi_8\rangle &= \frac{4J^2}{\sqrt{16J^4+(4J^2-E_8\lambda_2)^2}}|1,-1\rangle \\
&\quad -\frac{4J^2-E_8\lambda_2}{\sqrt{16J^4+(4J^2-E_8\lambda_2)^2}}|-1,1\rangle, \\
|\Psi_9\rangle &= |-1,-1\rangle, \qquad (5)
\end{aligned}$$

where $\gamma=\sqrt{b^2+4J^2}$, $\xi=\sqrt{(b+\gamma)^2+4J^2}$, $\zeta_1=J_z+2b+E_6$, $\zeta_2=J_z-2b+E_6$, $\eta_1=J_z+2b+E_7$, $\eta_2=J_z-2b+E_7$, and $\lambda_2=J_z-2b+E_8$ with E_6, E_7 and E_8 being the corresponding eigenvalues. The eigenvalues that correspond to the above eigenvectors are given respectively as

$$\begin{aligned}
&E_1=J_z+2B, \quad E_2=B+\gamma, \quad E_3=B-\gamma, \\
&E_4=-B+\gamma, \; E_5=-B-\gamma, \; E_9=J_z-2B. \qquad (6)
\end{aligned}$$

The other three eigenvalues are found from the below equation

$$E_i^3+a_2E_i^2+a_1E_i+a_0=0 \qquad (7)$$

with $a_2=2J_z$, $a_1=J_z^2-4b^2-8J^2$ and $a_0=-8J^2J_z$ by using the mathematical identity,

$$\begin{aligned}
&E_{n+6}=2y\cos\left[\frac{1}{3}\left\{\arccos\left(-\frac{G}{3y^3}\right)+2n\pi\right\}\right]-\frac{a_2}{3}, \\
&n=0,1,2. \qquad (8)
\end{aligned}$$

The density operator at thermal equilibrium $\rho(T)=\frac{1}{Z}\exp(-\beta\mathcal{H})$, where the partition function is given as $Z=\mathrm{Tr}[\exp(-\beta\mathcal{H})]$ and $\beta=1/kT$, with k being the Boltzmann constant and taken to be equal to unity. The density operator, $\rho(T)=\frac{1}{Z}\sum_{i=1}^{9}\exp(-\beta E_i)|\psi_i\rangle\langle\psi_i|$, can be calculated in the standard basis as

$$\begin{aligned}
\rho(T) &= \frac{1}{Z}\Big\{\mathrm{e}^{-\beta(J_z+2B)}|1,1\rangle\langle 1,1| \\
&\quad +\mathrm{e}^{-\beta(J_z-2B)}|-1,-1\rangle\langle -1,-1| \\
&\quad +\frac{\mathrm{e}^{-\beta B}}{\xi^2}[\mathrm{e}^{-\beta\gamma}(b+\gamma)^2+\mathrm{e}^{\beta\gamma}4J^2]|1,0\rangle\langle 1,0| \\
&\quad +\frac{\mathrm{e}^{-\beta B}}{\xi^2}[\mathrm{e}^{-\beta\gamma}4J^2+\mathrm{e}^{\beta\gamma}(b+\gamma)^2]|0,1\rangle\langle 0,1| \\
&\quad +\frac{\mathrm{e}^{\beta B}}{\xi^2}[\mathrm{e}^{-\beta\gamma}(b+\gamma)^2+\mathrm{e}^{\beta\gamma}4J^2]|0,-1\rangle\langle 0,-1| \\
&\quad +\frac{\mathrm{e}^{\beta B}}{\xi^2}[\mathrm{e}^{-\beta\gamma}4J^2+\mathrm{e}^{\beta\gamma}(b+\gamma)^2]|-1,0\rangle\langle -1,0| \\
&\quad -4\sinh(\beta\gamma)\frac{(b+\gamma)J}{\xi^2} \\
&\quad \times[\mathrm{e}^{-\beta B}(|1,0\rangle\langle 0,1|+|0,1\rangle\langle 1,0|) \\
&\quad +\mathrm{e}^{\beta B}(|0,-1\rangle\langle -1,0|+|-1,0\rangle\langle 0,-1|)] \\
&\quad +\mathrm{e}^{-BE_6}\varrho_6+\mathrm{e}^{-BE_7}\varrho_7+\mathrm{e}^{-BE_8}\varrho_8\Big\}, \qquad (9)
\end{aligned}$$

where

$$Z = \mathrm{e}^{-\beta(J_z+2B)} + \mathrm{e}^{-\beta(B+\gamma)} + \mathrm{e}^{-\beta(B-\gamma)} + \mathrm{e}^{-\beta(-B+\gamma)} + \mathrm{e}^{-\beta(-B-\gamma)} + \mathrm{e}^{-\beta E_6} + \mathrm{e}^{-\beta E_7} + \mathrm{e}^{-\beta E_8} + \mathrm{e}^{-\beta(J_z-2B)}, \tag{10}$$

and

$$\begin{aligned}\varrho_6 = {} & c_3^2|1,-1\rangle\langle 1,-1| + c_3c_5|1,-1\rangle\langle 0,0| + c_3c_7|1,-1\rangle\langle -1,1| \\ & + c_5c_3|0,0\rangle\langle 1,-1| + c_5^2|0,0\rangle\langle 0,0| + c_5c_7|0,0\rangle\langle -1,1| \\ & + c_7c_3|-1,1\rangle\langle 1,-1| + c_7c_5|-1,1\rangle\langle 0,0| + c_7^2|-1,1\rangle\langle -1,1|.\end{aligned} \tag{11}$$

Note also that (c_3, c_5, c_7) are replaced with (c_3', c_5', c_7') and $(c_3'', c_5'' = 0, c_7'')$ for ϱ_7 and ϱ_8, respectively. Meanwhile, the coefficients are

$$\begin{aligned} c_3 &= \frac{2J\zeta_1}{\kappa_1}, \quad c_5 = \frac{\zeta_1\zeta_2}{\kappa_1}, \quad c_7 = \frac{2J\zeta_2}{\kappa_1} \quad \text{with} \quad \kappa_1 = \sqrt{4J^2(\zeta_1^2+\zeta_2^2)+\zeta_1^2\zeta_2^2} \\ c_3' &= \frac{2J\eta_1}{\kappa_2}, \quad c_5' = \frac{\eta_1\eta_2}{\kappa_2}, \quad c_7' = \frac{2J\eta_2}{\kappa_2} \quad \text{with} \quad \kappa_2 = \sqrt{4J^2(\eta_1^2+\eta_2^2)+\eta_1^2\eta_2^2} \\ c_3'' &= \frac{4J^2}{\kappa_3}, \quad c_5'' = 0, \quad c_7'' = -\frac{4J^2 - E_8\lambda_2}{\kappa_3} \quad \text{with} \quad \kappa_3 = \sqrt{16J^4 + (4J^2 - E_8\lambda_2)^2}. \end{aligned} \tag{12}$$

The negativity as a measure of entanglement[41] is defined as

$$N = \frac{\| \boldsymbol{\rho}^{\mathrm{T_A}} \| - 1}{2}, \tag{13}$$

which corresponds to the absolute value of the sum of the negative values of $\boldsymbol{\rho}^{\mathrm{T_A}}$, where $\| \boldsymbol{\rho}^{\mathrm{T_A}} \| = \sqrt{\mathrm{Tr}[(\boldsymbol{\rho}^{\mathrm{T_A}})^\dagger \boldsymbol{\rho}^{\mathrm{T_A}}]}$ denotes the trace norm of the partially transposed density matrix $\boldsymbol{\rho}^{\mathrm{T_A}}$. Thus the partially transposed matrix $\boldsymbol{\rho}^{\mathrm{T_A}}$ of the original density matrix $\boldsymbol{\rho}^{\mathrm{T}}$ is found in the form

$$\boldsymbol{\rho}^{\mathrm{T_A}} = \begin{pmatrix} \rho_{11} & 0 & 0 & 0 & \rho_{15} & 0 & 0 & 0 & \rho_{19} \\ 0 & \rho_{22} & 0 & 0 & 0 & \rho_{26} & 0 & 0 & 0 \\ 0 & 0 & \rho_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_{44} & 0 & 0 & 0 & \rho_{48} & 0 \\ \rho_{51} & 0 & 0 & 0 & \rho_{55} & 0 & 0 & 0 & \rho_{59} \\ 0 & \rho_{62} & 0 & 0 & 0 & \rho_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \rho_{77} & 0 & 0 \\ 0 & 0 & 0 & \rho_{84} & 0 & 0 & 0 & \rho_{88} & 0 \\ \rho_{91} & 0 & 0 & 0 & \rho_{95} & 0 & 0 & 0 & \rho_{99} \end{pmatrix}, \tag{14}$$

which is a real and symmetric matrix whose elements are calculated as

$$\begin{aligned} \rho_{11} &= \frac{1}{Z}\mathrm{e}^{-\beta(J_z+2B)}, \\ \rho_{15} &= \rho_{51} = \frac{1}{Z}\left[-4\sinh(\beta\gamma)\frac{(b+\gamma)J}{\xi^2}\mathrm{e}^{-\beta B}\right], \\ \rho_{19} &= \rho_{91} = \frac{1}{Z}[c_3c_7\mathrm{e}^{-\beta E_6} + c_3'c_7'\mathrm{e}^{-\beta E_7} + c_3''c_7''\mathrm{e}^{-\beta E_8}], \\ \rho_{22} &= \frac{1}{Z}\left[\frac{\mathrm{e}^{-\beta B}}{\xi^2}\{\mathrm{e}^{-\beta\gamma}(b+\gamma)^2 + \mathrm{e}^{\beta\gamma}4J^2\}\right], \\ \rho_{26} &= \rho_{62} = \frac{1}{Z}[c_3c_5\mathrm{e}^{-\beta E_6} + c_3'c_5'\mathrm{e}^{-\beta E_7}], \\ \rho_{33} &= \frac{1}{Z}[c_3^2\mathrm{e}^{-\beta E_6} + c_3'^2\mathrm{e}^{-\beta E_7} + c_3''^2\mathrm{e}^{-\beta E_8}], \\ \rho_{44} &= \frac{1}{Z}\left[\frac{\mathrm{e}^{-\beta B}}{\xi^2}\{\mathrm{e}^{-\beta\gamma}4J^2 + \mathrm{e}^{\beta\gamma}(b+\gamma)^2\}\right], \\ \rho_{48} &= \rho_{84} = \frac{1}{Z}[c_5c_7\mathrm{e}^{-\beta E_6} + c_5'c_7'\mathrm{e}^{-\beta E_7}], \\ \rho_{55} &= \frac{1}{Z}[c_5^2\mathrm{e}^{-\beta E_6} + c_5'^2\mathrm{e}^{-\beta E_7}], \\ \rho_{59} &= \rho_{95} = \frac{1}{Z}\left[-4\sinh(\beta\gamma)\frac{(b+\gamma)J}{\xi^2}\mathrm{e}^{\beta B}\right], \\ \rho_{66} &= \frac{1}{Z}\left[\frac{\mathrm{e}^{\beta B}}{\xi^2}\{\mathrm{e}^{-\beta\gamma}(b+\gamma)^2 + \mathrm{e}^{\beta\gamma}4J^2\}\right], \\ \rho_{77} &= \frac{1}{Z}[c_7^2\mathrm{e}^{-\beta E_6} + c_7'^2\mathrm{e}^{-\beta E_7} + c_7''^2\mathrm{e}^{-\beta E_8}], \\ \rho_{88} &= \frac{1}{Z}\left[\frac{\mathrm{e}^{\beta B}}{\xi^2}\{\mathrm{e}^{-\beta\gamma}4J^2 + \mathrm{e}^{\beta\gamma}(b+\gamma)^2\}\right], \\ \rho_{99} &= \frac{1}{Z}\mathrm{e}^{-\beta(J_z-2B)}. \end{aligned} \tag{15}$$

After the partially transposed matrix $\boldsymbol{\rho}^{\mathrm{T_A}}$ is obtained, which is also real and symmetric as the original matrix $\boldsymbol{\rho}^{\mathrm{T}}$, the negativity can be calculated from Eq. (13) as

$$\begin{aligned} N = {} & \Big[\sqrt{\rho_{11}^2+\rho_{51}^2+\rho_{91}^2} + \sqrt{\rho_{22}^2+\rho_{62}^2} \\ & + \sqrt{\rho_{33}^2} + \sqrt{\rho_{44}^2+\rho_{84}^2} + \sqrt{\rho_{15}^2+\rho_{55}^2+\rho_{95}^2} \\ & + \sqrt{\rho_{26}^2+\rho_{66}^2} + \sqrt{\rho_{77}^2} + \sqrt{\rho_{48}^2+\rho_{88}^2} \\ & + \sqrt{\rho_{19}^2+\rho_{59}^2+\rho_{99}^2} - 1\Big]/2. \end{aligned} \tag{16}$$

Our findings are illustrated and discussed in detail in the next section.

3. Results and conclusions

In this last section we illustrate our findings for the thermal entanglement of a two-qutrit spin-1 anisotropic Heisenberg XXZ chain in an inhomogeneous magnetic field and compare them whenever possible.

The first figure, Fig.1, is obtained in three-dimensional planes consisting of (J_z, T, N) for different values of magnetic field B with zero inhomogeneity, i.e. $b = 0$, and $J = 1.0$. It is found that the negativity declines as magnetic field increases and the negativity usually prefers to have three values, about 0, 0.5 and 1. As seen from Fig.1(a) for $B = 0$, N values are all about 1.0 for higher values of J_z, but as J_z decreases N decreases gradually. But when $J_z = -2.0$, N sharply decreases to a value of 0.097, which may be called threshold value N_{th}, below which N is always zero. From Fig. 1(b) where $B = 1.0$, we first see a portion of N about 1.0 for higher J_z, but as J_z decreases we see another portion of N around 0.5, a further decrease in J_z takes N to the final portion with $N = 0$. Figure 1(c) where $B = 3.0$ shows that the portion of $N = 1.0$ decreases while the portions with $N = 0$ and 0.5 increase. When $B = 5.0$, Fig. 1(d) shows that the portion with $N = 1.0$ disappears and only the portions with $N = 0$ and 0.5 remain there but the latter is only seen at higher values of J_z within a smaller interval then the previous. It is now clear that as B increases further first the portion of N with 0.5 disappears and the further increase in B diminishes the N. This result is in agreement with those in Ref. [27], [31], and [36].

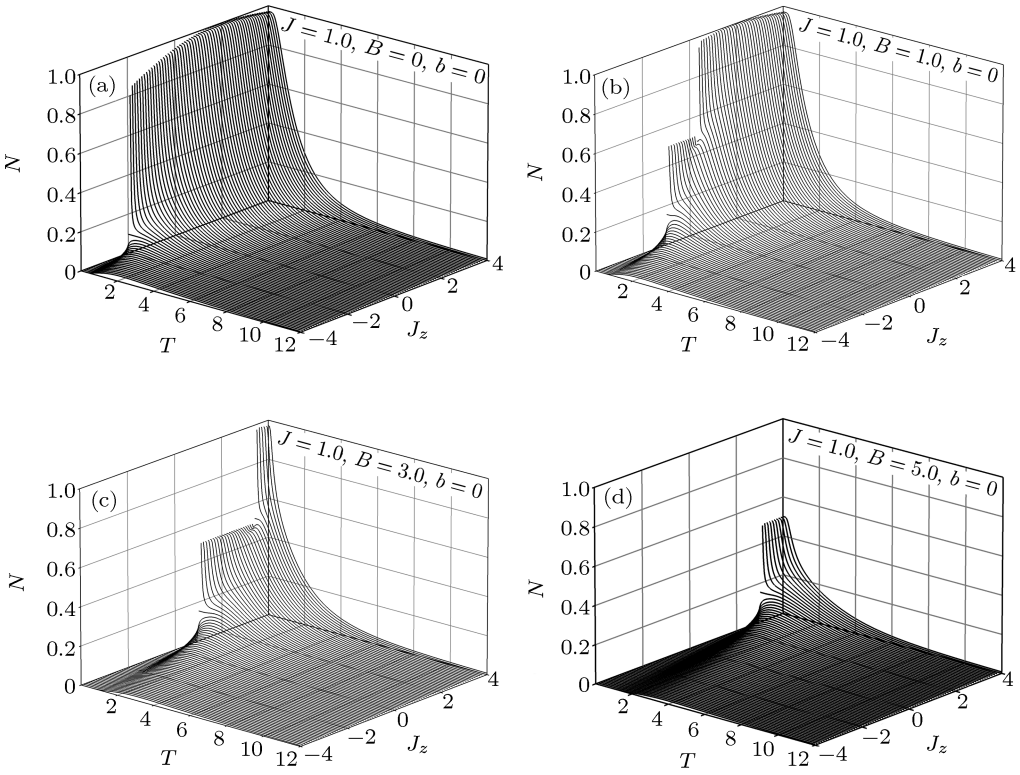


Fig. 1. Variations of negativity N with J_z and T for different values of B when $J = 1.0$ and $b = 0$.

The next illustration is obtained again in three-dimensions with axes (b, T, N) for $J = J_z = 1.0$ and for different values of B as shown in Fig. 2. At lower values of B and at zero b, $N = 1.0$, but as b is increased N declines and eventually vanishes, see Fig. 2(a). When B is equal to 2.0, N presents two peaks; one is at $b = 0$ and $N = 0.5$ and the other is at $b = 1.2$ and N=0.5981. There is a minimum between these peaks at $b = 1.0$ and $N = 0.4219$. The further increase in b after the second peak causes N to decline as shown in Fig. 2(b). Figure 2(c) for $B = 3.0$ shows that N exhibits two peaks as in Fig. 2(b), but now N starts from lower values at lower values of b in comparison with the values of the peaks. The last figure, Fig. 2(d), is also similar to

Figs. 2(b) and 2(c), but now N starts from zero before the appearance of these peaks.

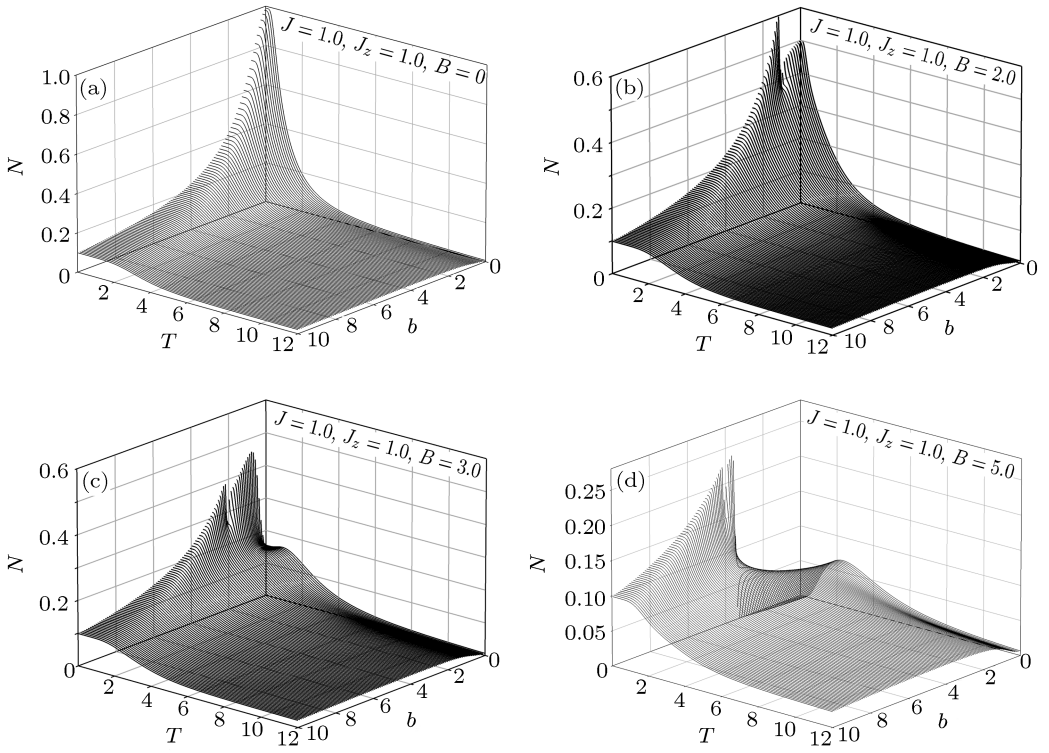
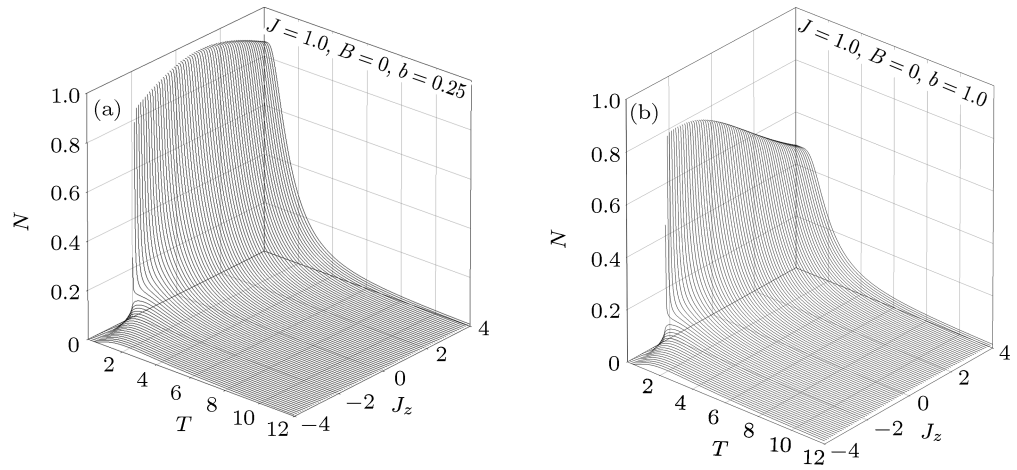


Fig. 2. Variations of negativity N with b and T for different values of B when $J = 1.0$ and $J_z = 1.0$.

Again the dependences of N on temperature and J_z are plotted in Figs. 3(a)–3(c) and 3(d)–3(f) with $B = 0$ and 2.0, respectively, for different values of b. In Fig. 3(a)–3(c), we see that N is high for $b = 0.25$ at higher values of J_z, but when $J_z < -2.0$, N starts from zero at zero temperature. When $b = 1.0$, N is low at higher value of J_z, but as J_z decreases further first N increases then it starts from zero at zero temperature as shown in Fig. 3(a). Eventually, for $b = 4.0$, N is low at higher values of J_z, but as J_z decreases further N increases in the range of J_z between 4.0 and -4.0. Figures 3(d)–3(f) are similar to Figs. 3(a)–3(c) , but now we see that the portion of N about 0.5 is also seen somewhere between the values $J_z = 4.0$ and -4.0.



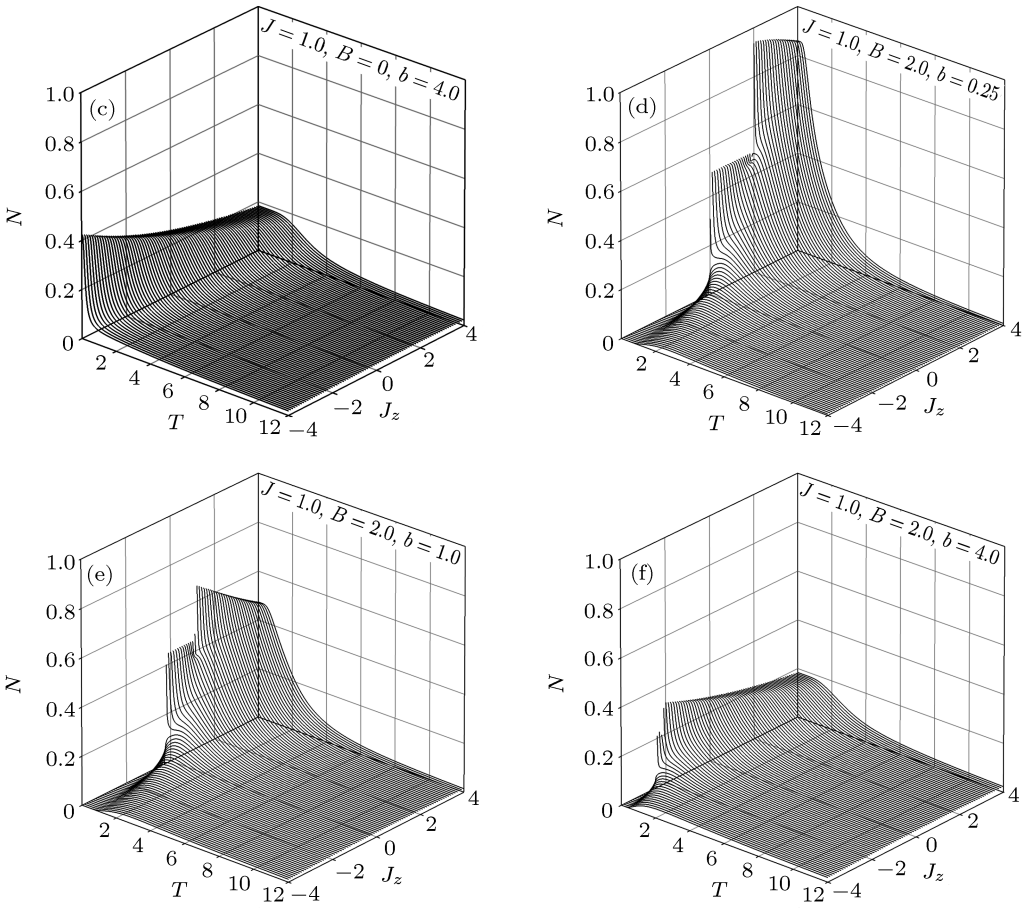


Fig. 3. Variations of negativity N with J_z and T with $J = 1.0$, $B = 0$, 2.0 and $b = 0.25, 1.0, 4.0$.

The property that is very clear from the figures but we have not mentioned so far is that N always decreases when temperature increasing and they vanish at some definite temperatures. It is also clear that the N persists to higher temperatures for higher values of J_z and lower values of B and b. As we have mentioned, this is the first work with the inclusion of the inhomogeneous magnetic field so it is not possible to make any comparison.

As a summary, the thermal entanglement of a two-qutrit spin-1 anisotropic Heisenberg XXZ chain in an inhomogeneous magnetic field is studied in detail. The effects of external magnetic field (B), parameter b which controls the inhomogeneity of B, and bilinear interaction parameters $J_x = J_y \neq J_z$ on the thermal variation of the negativity are studied in detail. The mathematical formulation of the problem is dealt with, and it is worthy of the efforts.

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