Fast generation of cluster states in a linear ion trap*

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We propose a practical scheme to generate cluster states by simultaneously accomplishing two-qubit conditional gating on an array of equidistant ions by using transverse modes. Our operation is robust to heating and insensitive to Lamb–Dicke parameter. Meanwhile, as it is carried out in a geometric quantum computing fashion, our scheme enables the fast and high-fidelity generation of cluster states. The experimental feasibility is discussed with sophisticated ion trap techniques.

Keywords: preparation of cluster states, quantum computation, linear ion trap

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1. Introduction

The cluster state[1,2] is a prerequisite for one-way quantum computing (QC).[3] Due to the peculiar feature of processing quantum information by local operations, one-way QC has attracted much attention, and it has been considered as a promising idea for future scalable QC even for achieving some special gates.[4] Up to now, there have been a lot of proposals for generating cluster states using optical setup,[5] quantum dots,[6] superconducting qubits,[7] or cavity quantum electrodynamics (cavity QED).[8−11] Relevant experiments, however, have been achieved only in a couple of systems, such as the systems with the neutral atoms trapped in an optical lattice[12] and with free photons.[13,14]

The good candidate system for producing cluster states should have the qubits that are well controlled and subject to negligible decoherence. This is because the one-way QC depends strongly on the quality of the generated cluster state and it takes time to carry out the single-qubit operations in series for implementing the one-way QC. Moreover, the candidate system should be able to generate the two-dimensional (2D) cluster states which are essential to genuine quantum computing tasks.[15] It has been shown that any multi-dimensional cluster state could in principle be decomposed into some 2D cluster states,[16] implying that the 2D cluster state is fundamental and crucial to the one-way QC.

The present work focuses on a fast operation of the cluster state with an array of ultracold ions in a linear trap. Trapped ions have been considered as one of the most promising candidates for future QC devices for some years.[17,18] This potential has been verified by experimental demonstrations for simple quantum algorithms with trapped ions[19] and for entangling eight ultracold ions in a linear trap.[20] Scalable QC could be reached by entangling spatially remote ions by emitted photons[21] and by shuttling the ions between different traps.[22] Due to the full controllability and the negligible decoherence, trapped ions is obviously suited to the one-way QC. As shown in recent proposals,[16,23−25] large-sized and multi-dimensional cluster states could be achieved in linear traps by the smart design of laser pulses and/or trapping potential as well as by shuttling the ions.

2. CPF gates on two nearest-neighbour ions

The key step in our scheme is to produce an effective nearest-neighbour coupling on the ion pairs, based on which the controlled phase flip (CPF) gating could be carried out. The CPF gate was introduced first in Ref. [26]. Although the implementation of our idea requires adiabatic operations, as the CPF gating acts si-

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multaneously on all the ions in the array, and also in a geometric QC fashion, we may have the fast and high-fidelity generation of the one-dimensional (1D) cluster state with an array of the equidistant trapped ions. With some additional operations, 2D cluster state is available. Another outstanding feature of our scheme is to employ the transverse modes of the ion trap, which also favours a fast implementation due to the larger frequencies of the transverse modes than those of the longitudinal modes usually used in conventional schemes.

We consider N ultracold ions equidistantly confined as a line in a linear ion trap plus an additional anharmonic potential along the z-axis.[27] We assume that the ions are tightly bound in the x–y plane (i.e., in the transverse or the radial direction), but weakly bound in the z-axis (i.e., in the longitudinal direction). The Lagrangian of the trapped ions in one of the transverse directions (i.e., along the x- or the y-axis) may be written as[28,29]

$$L = \frac{M}{2}\left[\sum_{m=1}^{N}(\dot{q}_m)^2 - \omega_r^2 \sum_{m,n=1}^{N} \boldsymbol{A}_{mn} q_m q_n\right], \quad (1)$$

where q_k denotes the displacement of the kth ion from the equilibrium position, M is the mass of the ion, ω_r is the trapping frequency of the transverse mode, N is the number of the ions, and $A_{mn} = B_{mn} + \omega_r^2/\omega_z^2$, with

$$B_{mn} = \begin{cases} -\sum\limits_{k=1,k\neq m}^{N} \dfrac{1}{d^3\,|m-k|^3}, & \text{if } m = n, \\ \dfrac{1}{d^3\,|m-k|^3}, & \text{if } m \neq n, \end{cases} \quad (2)$$

with d being the dimensionless distance and defined as $|z_{0,m} - z_{0,m+1}| \,/ \sqrt[3]{e^2/4\pi\varepsilon_0 M\omega_z^2}$, $z_{0,m}$ the equilibrium position of the mth ion along the z-axis, and ω_z the longitudinal mode frequency. In our case with the equidistant separation of the ions, the eigenvalues and the eigenvectors of the matrix $\boldsymbol{A}_{mn}$ are different from in the case of conventional linear trap.[28,29] As an example, we show the differences among four trapped ions in Table 1. Throughout the paper, we employ the ED case for our scheme.

Table 1. Numerically calculated eigenvalues and eigenvectors of matrix $\boldsymbol{A}_{mn}$ for four equidistantly trapped ions (labeled by ED) in comparison with those for the trapped ions in usual trap (labeled by UED) without the an harmonic axis potential.

different case		first mode ($p=1$)	second mode ($p=2$)	third mode ($p=3$)	fourth mode ($p=4$)
ED	eigenvalues	$\dfrac{\omega_r^2}{\omega_z}$	$\sqrt{\left(\dfrac{\omega_r}{\omega_z}\right)^2 - \dfrac{0.86}{d^3}}\omega_r$	$\sqrt{\left(\dfrac{\omega_r}{\omega_z}\right)^2 - \dfrac{2.25}{d^3}}\omega_r$	$\sqrt{\left(\dfrac{\omega_r}{\omega_z}\right)^2 - \dfrac{3.46}{d^3}}\omega_r$
	eigenvectors	(0.5,0.5,0.5,0.5)	(–0.66, –0.25, 0.66, 0.25)	(0.5, –0.5, –0.5, 0.5)	(–0.25, 0.66, –0.66, 0.25)
UED	eigenvalues	$\dfrac{\omega_r^2}{\omega_z}$	$\sqrt{\left(\dfrac{\omega_r}{\omega_z}\right)^2 - 2.5}\omega_r$	$\sqrt{\left(\dfrac{\omega_r}{\omega_z}\right)^2 - 5.31}\omega_r$	$\sqrt{\left(\dfrac{\omega_r}{\omega_z}\right)^2 - 8.81}\omega_r$
	eigenvectors	(0.5, 0.5, 0.5, 0.5)	(–0.67, –0.21, 0.67, 0.21)	(0.5, –0.5, –0.5, 0.5)	(–0.21, 0.67, –0.67, 0.21)

We consider the ground and the excited states of the trapped ions, where the qubits are encoded in the hyperfine (or fine) levels $|0\rangle$ and $|1\rangle$ of the ground state, separated by ω_0. The qubits are coupled through vibrational modes under the radiation of two lasers with frequency difference $\mu = \omega_A - \omega_B$. The Hamiltonian of the system could be effectively written in units of $\hbar = 1$ as[30]

$$H_{\text{eff}} = \frac{1}{2}\sum_{m,p}\Omega_m \sin(\mu t + \delta\varphi_m)\eta_p b_m^p (a_p^+ + a_p)\sigma_m^z + \sum_{p=1}^{N}\omega_p a_p^+ a_p, \quad (3)$$

where Ω_m is the effective Rabi frequency of the mth ion due to the laser radiation, $\delta\phi_m$ is the phase difference between the two lasers, $\eta_p = |\delta k|\sqrt{1/2M\omega_p}$ with δk being the wave vector difference between the two lasers beams, b_m^p (ω_p) the mth component of pth eigenvector (pth eigenvalue) of the matrix $\boldsymbol{A}_{mn}$, and a_p^+ (a_p) the creation (annihilation) operator of the pth mode. In a rotating frame with respect to ω_p, Eq. (3) turns into

$$H'_{\text{eff}} = \frac{1}{2}\sum_{m=1}^{N}\Omega_m \sin(\mu t + \delta\varphi_m)\eta_p b_m^p \times (a_p^+ \,\mathrm{e}^{\mathrm{i}\omega_p t} + a_p \,\mathrm{e}^{-\mathrm{i}\omega_p t})\sigma_m^z, \quad (4)$$

which yields the evolution operator

$$U_0 = \exp\left\{ \mathrm{i} \sum_{m,p} (\phi_p^m a_p^+ + \mathrm{h.c.}) \sigma_m^z + \mathrm{i} \sum_{m,n,p} \phi_p^{m,n} \sigma_m^z \sigma_n^z \right\}, \quad (5)$$

where $m < n$, $\phi_p^m = \int_0^\tau \Omega_m \sin(\mu t + \delta\phi_m) \eta_p b_m^p \, \mathrm{e}^{\mathrm{i}\omega_p t} \mathrm{d}t$ and $\phi_p^{m,n} = \int_0^\tau \int_0^{t_2} \Omega_m \Omega_n \eta_p^2 b_m^p b_n^p \, \mathrm{e}^{\mathrm{i}\omega_p t} \sin(\mu t_1 + \delta\phi_m) \times \sin(\mu t_2 + \delta\phi_n) \sin\omega_p (t_2 - t_1) \mathrm{d}t_1 \mathrm{d}t_2/4$. The latter involves the contribution from the geometric phase which was used for geometric QC in Ref. [31]. For our purpose, however, we require the operation time $\tau \gg 1/\mu$ with $\mu \ll \omega_N$, where ω_N is the frequency of the transverse zigzag mode, i.e., the transverse mode with the lowest energy. This implies that besides the necessary condition to accomplish the above geometric phase, we need a further condition for slower operations. As a result, the integrals in the above $\phi_p^{m,n}$ can be simplified and the evolution operator reduces to

$$U_1 = \exp\left\{ \mathrm{i} \sum_{m,p} (\alpha_p^m a_p^+ - \alpha_p^{m*} a_p) \sigma_m^z + \mathrm{i} \sum_{m<n} J_{m,n} \tau \sigma_m^z \sigma_n^z \right\} \quad (6)$$

with $\alpha_p^m = \phi_p^m \, \mathrm{e}^{\mathrm{i}\pi/2}$, and the spin–spin coupling coefficient $J_{m,n} = \sum_p \Omega_m \Omega_n \eta_p^2 b_m^p b_n^p / 8\omega_p$. As α_p^m can be adjusted to be very small and the term regarding σ_m^z has no contribution to the entanglement between qubits, we may focus only on the spin–spin coupling terms below. In the case of $\omega_r^2/\omega_z^2 \gg B_{mn}$, the coupling coefficient can be approximated as

$$J_{m,n} = -\frac{\Omega_m \Omega_n \delta k^2}{16 M d^3 \omega_r^2} \frac{1}{|m-n|^3}, \quad (7)$$

where δk is the difference between the wave vectors of the two lasers. Equation (7) indicates an effective nearest-neighbour coupling, i.e.,

$$U_2 = \exp\left\{ \mathrm{i} \sum_m J_{m,m+1} \sigma_m^z \sigma_{m+1}^z \tau \right\}, \quad (8)$$

based on which we could accomplish the CPF gating simultaneously on the neighbouring ion pairs[16] and thereby straightforwardly achieving the 1D cluster state on an array of equidistant trapped ions.

3. Preparation of cluster states

Specifically, we consider an array of trapped ions initially prepared to be in a superposition state $\Pi_m |+\rangle_m = \Pi_m (|0\rangle_m + |1\rangle_m)/\sqrt{2}$. Executing the CPF gating simultaneously on the array of the ions yields

$$|\Phi\rangle_N = \frac{1}{2^{N/2}} \otimes_{m=1}^N (|0\rangle_m + |1\rangle_m \, \sigma_z^{m+1}), \quad (9)$$

where we have employed the conventional restriction $\sigma_z^{N+1} = 1$ and considered the action of some additional single-qubit σ_z operations. Equation (9) is the standard form of the 1D cluster state defined in Ref. [1]. To generate a 2D cluster state, we may carry out the CPF gating on some non-nearest-neighbour qubits as shown in Fig. 1 for generating a simple 2D cluster state with four trapped ions. A practical 2D cluster state should be more complicated.

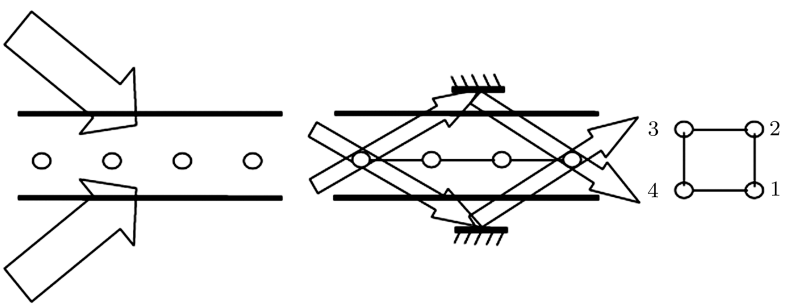


Fig. 1. Generation of 2D cluster state with four trapped ions in a linear trap, where the left-hand plot shows the coupling of the ions with transverse modes by two laser beams globally radiating all the ions, achieving the 1D cluster state, the middle plot is for a CPF gating on the 1st and the 4th ions by individual laser radiation, and the right-hand plot denotes a 2D cluster state of the four trapped ions.

To assess how good the approximation is when the nearest-neighbour coupling replaces Eq. (6) (with $J_{m,n}$ being in the form of Eq. (7)), we consider the fidelity of the 1D cluster state generated by Eq. (8). In our case,

the infidelity comes from two parts, i.e., the entanglement with motional modes and the non-nearest-neighbour coupling. Therefore the fidelity can be defined as

$$F = \mathrm{Tr}_M \left[\langle \Psi | \, U_{\mathrm{idea}}^{+} [U \, |\Psi\rangle \, \langle \Psi| \otimes \prod_p \rho_p U^{+}] U_{\mathrm{idea}} \, |\Psi\rangle \right], \tag{10}$$

where the trace is over all the motional states, $|\Psi\rangle = \Pi_m[(|0\rangle_m + |1\rangle_m)/\sqrt{2}]$, U means the evolution operator regarding the realistic coupling Eq. (6), $U_{\mathrm{idea}} = \exp\{\mathrm{i}\sum_m \pi\sigma_m^z\sigma_{m+1}^z/4\}$, and ρ_p represents the state of the pth normal mode in thermal equilibrium with the average phonon number $\bar{n}_p$.

We have plotted the fidelity in Fig. 2 with four trapped ions as an example using the parameters $d = 1$ ($\sim$5 μm obtained without assuming $\hbar = 1$ for $^{40}\mathrm{Ca}^{+}$ with $\omega_z = 1$ MHz) and $\omega_r/\omega_z = 10$. For simplicity, we assume the identical Rabi frequencies for the four trapped ions to be identical, i.e., $\Omega_m = \Omega$ ($m = 1$, 2, 3, 4). In our calculation, the value of $\tau\omega_r$ is assumed to vary from 0 to 80, and it is found that the fidelity decreases with the increase of the Rabi frequency. This can be understood as the involvement of more non-nearest-neighbour couplings in the faster evolution of the effective Hamiltonian Eq. (6). Therefore, an appropriate choice of the experimental parameters is essential to a high-fidelity implementation of our scheme.

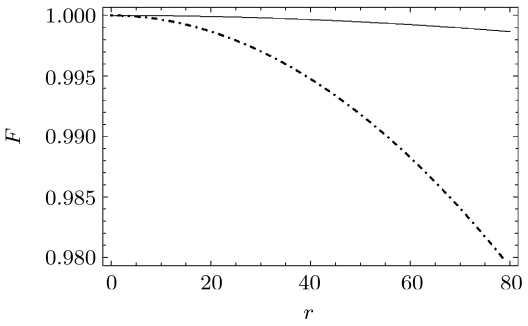


Fig. 2. Curves for fidelity of the multi-qubit gate for four calcium ions with respect to the dimensionless gate time, where dash-dot curve is for the case of $\Omega = 0.1\omega_r$, solid curve is for the case of $\Omega = 0.05\omega_r$, and the values of other parameters are $\mu = 0.01\omega_r$, $\bar{n}_p = 0.5$, $d = 1$, and $\omega_r/\omega_z = 10$.

4. Discussion and conclusion

Now we briefly address something about the feasibility of our proposal with currently reachable technique. We may take $^{40}\mathrm{Ca}^{+}$ as an example, for which we encode qubits in the Zeeman sublevels of $4S_{1/2}$ as shown in Ref. [32]. The two lasers are radiated to couple the qubit levels into an excited level $P_{1/2}$ by detunings. As the axial frequency employed for quantum information processing could be $2\pi \times 1$ MHz,[33] we may assume the frequency of the transverse mode to be $2\pi \times 10$ MHz and the frequency difference between the two lasers is 5 MHz. Therefore, the implementation time for generating a 1D cluster state is on the order of hundreds of microsecs (e.g., 850 μs for $\Omega = 2\pi \times 1$ MHz and $\delta k = 3 \times 10^7$ m^{-1}), which is shorter than the lifetime of the vibrational mode. Since the excited level $4P_{1/2}$ is not really populated during our implementation, we may neglect the atomic decay in our calculation. Considering a time $T_1 = 850$ μs and a little bit longer time $T_2 = 1000$ μs in executing CPF gating on two nearest-neighbour and non-nearest-neighbour ions, respectively, we may achieve a simple 2D cluster state by $T_1 + T_2 < 2$ ms, which is much shorter than the decoherence time of the vibrational modes.

Equations (7) and (8) represent the main result of this work, by which we may accomplish the CPF gating on many trapped ions simultaneously and thereby we could have a faster generation of the desired cluster states than previously proposed schemes.[16,23–25] Moreover, due to the strong confinement in the transverse direction of the trap, our gating is more robust to heating and to Lamb–Dicke limit with respect to the longitudinal trapping. As a result, it is unnecessary to cool the ions down strictly to the vibrational ground state for accomplishing our scheme.

To execute one-way quantum computing based on the generated cluster state, we may carry out the single-qubit operations by laser-radiating the two qubit states with using Raman beams,[32] and achieve the readout by a usual electronic shelving amplification method through coupling the qubit levels into a metastable $D_{5/2}$ level. Our scheme could be straightforwardly applied to other candidate ions, such as $^{9}\mathrm{Be}^{+}$,[33] $^{111}\mathrm{Cd}^{+}$,[34] and $^{171}\mathrm{Yb}^{+}$.[35]

In summary, we have proposed a potentially practical scheme for a fast implementation of the multi-qubits gate to achieve 1D and 2D cluster states with

an array of trapped ions. We have assessed the quality of the generated cluster state and discussed the feasibility with the sophisticated ion trap techniques. In principle, a larger sized cluster state could be achieved by using four trapped ions as building blocks, with the technique of mapping entanglement from the emitted photons[21] or by moving some ions.[22]

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