Concurrence, tangle and fully entangled fraction*

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We show that although we cannot distil a singlet from many pairs of bound entangled states, the concurrence and the tangle of two entangled quantum states are always strictly larger than those of one of them, even both entangled quantum states are bound entangled. We present a relation between the concurrence and the fidelity of optimal teleportation. We also give new upper and lower bounds for concurrence and tangle.

Keywords: concurrence, entanglement, teleportation
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1. Introduction

Quantum entanglement plays a crucial role in quantum information processing.[1] Quantum entangled states have become the key ingredient in the rapidly expanding field of quantum information science, with remarkable prospective applications such as quantum teleportation, quantum cryptography, quantum dense coding and parallel computing.

However, it has been shown that not all of the quantum entangled states are useful in quantum information processing. There exist bound entangled states from which no pure entangled states can be distilled under local operation and classical communication (LOCC).[2] With bound entangled states, the entangled resource teleportation cannot be performed better than with a classical channel, even if conclusive teleportation is allowed.[3] It has been shown that bound entangled states can enhance the fidelity of teleportation for non-bound entangled states.[4,5] However, a bound entangled state can never enhance the teleportation fidelity of another state which is also bound entangled.[5]

An important problem in quantum information theory is to detect quantum entanglement. A series of excellent results has been obtained on the separability criteria and the evaluation of measures of quantum entanglement such as entanglement of formation (EOF)[6] and concurrence.[7−11] The calculation of entanglement of formation or concurrence is complicated except for 2×2 systems or for states with special forms. For general quantum states with higher dimensions or in the multipartite case, it seems to be very difficult to obtain analytical formulas. Instead, the lower and the upper bounds of concurrence[12−16] and EOF[17] have been estimated.

In the rest of this paper, we show in Section 2 that although we cannot distil pure entangled states from any bound entangled states, the concurrence and the tangle of two entangled states will always be strictly larger than those of one of them, even the two entangled states are both bound entangled. We study the relation between the fidelity of optimal teleportation and concurrence in Section 3. We investigate bounds for concurrence and tangle in Section 4. New lower and upper bounds for concurrence and tangle are derived, which can be used not only for estimating entanglement, but also for investigating separability. The subadditivity properties of concurrence and tangle are proved in Section 5. We give conclusions and remarks in the last section.

2. Concurrence and tangle of two entangled quantum states

The concurrence and the tangle are two well defined entanglement measures satisfying the standard properties usually regarded as essential for a good entanglement measure (see, for example, Ref. [18]).

Let $\mathcal{H}_A$ (resp. $\mathcal{H}_B$) be an M (resp. N)-dimensional complex vector space with $|i\rangle$, $i =$

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$1,\ldots,M$ (resp. $|j\rangle$, $j=1,\ldots,N$) used as an orthonormal basis. We assume $M \leq N$ for convenience. A general pure state on $\mathcal{H}_A \otimes \mathcal{H}_B$ is of the form

$$|\Psi\rangle = \sum_{i=1}^{M}\sum_{j=1}^{N} a_{ij}|i\rangle \otimes |j\rangle, \tag{1}$$

where $a_{ij} \in \mathbb{C}$ satisfies normalization

$$\sum_{i=1}^{M}\sum_{j=1}^{N} a_{ij}a_{ij}^{*} = 1.$$

For bipartite pure quantum state $|\psi\rangle$ the concurrence is defined as[7−9]

$$C(|\psi\rangle) = \sqrt{2(1-\mathrm{Tr}\rho_A^2)}, \tag{2}$$

where $\rho_A = \mathrm{Tr}_B|\psi\rangle\lambda\psi|$, while the tangle is defined as[19,20]

$$\tau(|\psi\rangle) = C^2(|\psi\rangle) = 2(1-\mathrm{Tr}\rho_A^2). \tag{3}$$

The definition is extended to general mixed states $\rho = \sum_i p_i|\psi_i\rangle\lambda\psi_i|$ by the convex roof,

$$C(\rho) = \min_{\{p_i,|\psi_i\rangle\}} \sum_i p_i C(\psi_i), \tag{4}$$

$$\tau(\rho) = \min_{\{p_i,|\psi_i\rangle\}} \sum_i p_i \tau(\psi_i). \tag{5}$$

Let $\rho = \sum_{ijkl}\rho_{ij,kl}|ij\rangle\langle kl| \in \mathcal{H}_A \otimes \mathcal{H}_B$ and $\sigma = \sum_{i'j'k'l'}\sigma_{i'j',k'l'}|i'j'\rangle\langle k'l'| \in \mathcal{H}_{A'} \otimes \mathcal{H}_{B'}$, and $\rho\otimes\sigma = \sum_{ijkl,i'j'k'l'}\rho_{ij,kl}\sigma_{i'j',k'l'}|ii'\rangle_{AA'}\langle kk'|\otimes|jj'\rangle_{BB'}\langle ll'|$ denote the bipartite state in the bipartite partition AA' and BB'.

Lemma 1 For pure states $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ and $|\varphi\rangle \in \mathcal{H}_{A'} \otimes \mathcal{H}_{B'}$, the inequalities

$$C(|\psi\rangle\otimes|\varphi\rangle) \geq \max\{C(|\psi\rangle), C(|\varphi\rangle)\}, \tag{6}$$

and

$$\tau(|\psi\rangle\otimes|\varphi\rangle) \geq \max\{\tau(|\psi\rangle), \tau(|\varphi\rangle)\} \tag{7}$$

always hold, the equalities hold if and only if at least one of the states, $|\psi\rangle$ and $|\varphi\rangle$, is separable.

Proof Without loss of generality we assume $C(|\psi\rangle) \geq C(|\varphi\rangle)$. First, note that

$$\rho_{AA'}^{|\psi\rangle\otimes|\varphi\rangle} = \rho_A^{|\psi\rangle}\otimes\rho_{A'}^{|\varphi\rangle}, \tag{8}$$

where $\rho_A^{|\psi\rangle} = \mathrm{Tr}_B|\psi\rangle\langle\psi|$, $\rho_{A'}^{|\varphi\rangle} = \mathrm{Tr}_{B'}|\varphi\rangle\langle\varphi|$, and $\rho_{AA'}^{|\psi\rangle\otimes|\varphi\rangle} = \mathrm{Tr}_{BB'}(|\psi\rangle\langle\psi|\otimes|\varphi\rangle\langle\varphi|)$. Let $\rho_A^{|\psi\rangle} = \sum_i \lambda_i|i\rangle\langle i|$ and $\rho_{A'}^{|\varphi\rangle} = \sum_j \pi_j|j\rangle\langle j|$ be the spectral decompositions of $\rho_A^{|\psi\rangle}$ and $\rho_{A'}^{|\varphi\rangle}$, with $\sum_i\lambda_i = 1$ and $\sum_j \pi_j = 1$ respectively. By using Eq. (1) one obtains that

$$\begin{aligned}\mathrm{Tr}[(\rho_{AA'}^{|\psi\rangle\otimes|\varphi\rangle})^2] &= \sum \lambda_i\pi_j\lambda_{i'}\pi_{j'}|ij\rangle\langle ij|i'j'\rangle\langle i'j'| \\ &= \sum \lambda_i^2\pi_j^2,\end{aligned} \tag{9}$$

and

$$\mathrm{Tr}[(\rho_A^{|\psi\rangle})^2] = \sum_i \lambda_i^2. \tag{10}$$

From the definition of concurrence and the normalisation conditions of λ_i and π_j one immediately obtains

$$\begin{aligned}C(|\psi\rangle\otimes|\varphi\rangle) &= \sqrt{2(1-\mathrm{Tr}[(\rho_{AA'}^{|\psi\rangle\otimes|\varphi\rangle})^2])} \\ &\geq \sqrt{2(1-\mathrm{Tr}[(\rho_A^{|\psi\rangle})^2])} \\ &= C(|\psi\rangle).\end{aligned} \tag{11}$$

If one of the states $|\psi\rangle$ and $|\varphi\rangle$, say $|\varphi\rangle$, is separable, then the rank of $\rho_{A'}^{|\varphi\rangle}$ must be one, which means that there is only one item in the spectral decomposition in $\rho_{A'}^{|\varphi\rangle}$. From the normalisation condition of π_j we obtain $\mathrm{Tr}[(\rho_{AA'}^{|\psi\rangle\otimes|\varphi\rangle})^2] = \mathrm{Tr}[(\rho_A^{|\psi\rangle})^2]$. Hence inequality (11) becomes an equality.

On the other hand, if both $|\psi\rangle$ and $|\varphi\rangle$ are not separable, there must be at least two items in the decomposition of their reduced density matrices $\rho_A^{|\psi\rangle}$ and $\rho_{A'}^{|\varphi\rangle}$, which means that $\mathrm{Tr}[(\rho_{AA'}^{|\psi\rangle\otimes|\varphi\rangle})^2]$ is strictly larger than $\mathrm{Tr}[(\rho_A^{|\psi\rangle})^2]$.

Inequality (7) also holds because for pure quantum state ρ, $\tau(\rho) = C^2(\rho)$. □

By using the lemma, we have Theorem 1.

Theorem 1 For any quantum mixed states $\rho \in \mathcal{H}_A \otimes \mathcal{H}_B$ and $\sigma \in \mathcal{H}_{A'} \otimes \mathcal{H}_{B'}$, inequalities

$$C(\rho\otimes\sigma) \geq \max\{C(\rho), C(\sigma)\} \tag{12}$$

and

$$\tau(\rho\otimes\sigma) \geq \max\{\tau(\rho), \tau(\sigma)\} \tag{13}$$

hold. They become equalities if and only if at least one of the states, ρ and σ, is separable.

Proof We assume $C(\rho) \geq C(\sigma)$ for convenience. Let $\rho = \sum_i p_i\rho_i$ and $\sigma = \sum_j q_j\sigma_j$ be the optimal decompositions such that $C(\rho\otimes\sigma) = \sum_i p_iq_jC(\rho_i\otimes\sigma_j)$. By using the inequality in Lemma 1 we have

$$\begin{aligned}C(\rho\otimes\sigma) &= \sum_{ij} p_iq_jC(\rho_i\otimes\sigma_j) \\ &\geq \sum_{ij} p_iq_jC(\rho_i) \\ &= \sum_i p_iC(\rho_i) \geq C(\rho).\end{aligned} \tag{14}$$

Case 1 If one of the states ρ and σ, say σ, is separable, i.e. σ can be written as $\sigma = \sum_j q_j \sigma_j$, where $\sum_j q_j = 1$ and σ_j are the density matrices of separable pure states. Assume $\rho = \sum_i p_i \rho_i$ to be an optimal decomposition of ρ such that $C(\rho) = \sum_i p_i C(\rho_i)$. Using Lemma 1 we have

$$\begin{aligned} C(\rho \otimes \sigma) &\leq \sum_{ij} p_i q_j C(\rho_i \otimes \sigma_j) \\ &= \sum_{ij} p_i q_j C(\rho_i) \\ &= \sum_i p_i C(\rho_i) \\ &= C(\rho). \end{aligned} \tag{15}$$

Inequalities (14) and (15) show that if σ is separable, then $C(\rho \otimes \sigma) = C(\rho)$.

Case 2 If both ρ and σ are not separable, using Lemma 1 we have

$$\begin{aligned} C(\rho \otimes \sigma) &= \sum_{ij} p_i q_j C(\rho_i \otimes \sigma_j) \\ &> \sum_{ij} p_i q_j C(\rho_i) \\ &= \sum_i p_i C(\rho_i) \geq C(\rho), \end{aligned} \tag{16}$$

i.e. inequality (12) is strictly an inequality.

The inequality (13) for tangle τ can be proved similarly. □

Remark In Ref. [5] the author showed that any entangled state ρ can enhance the teleportation power of a state σ. This holds even if state ρ is bound entangled. But if ρ is bound entangled, the corresponding state σ must be free entangled (distillable). From Theorem 1, we see that even both ρ and σ are bound entangled states, the concurrence and the tangle can be still strictly larger than that of one state.

3. Relation between concurrence and fully entangled fraction

Quantum entangled states are the key resources in quantum teleportation.[21−24] As shown in Ref. [25], the optimal teleportation fidelity is related to the concurrence of a two-qubit quantum state. For a high-dimensional case, the optimal fidelity of teleportation with quantum state $\rho \in \mathcal{H}_A \otimes \mathcal{H}_B$ as an entangled resource, and with dimensions $M = N = d$, is given by[3,26]

$$F(\rho) = \frac{d}{d+1}\mathcal{F}(\rho) + \frac{1}{d+1}, \tag{17}$$

where $\mathcal{F}(\rho)$ is the fully entangled fraction of ρ and defined as

$$\mathcal{F}(\rho) = \max_{\phi \in \epsilon} \langle \phi | \rho | \phi \rangle, \tag{18}$$

where ϵ denotes the set of $d \times d$-dimensional maximally entangled states.

Theorem 2 For any bipartite quantum state $\rho \in \mathcal{H}_A \otimes \mathcal{H}_B$ with dimensions $M = N = d$, we have

$$C(\rho) \geq \sqrt{\frac{2d}{d-1}} \left[\mathcal{F}(\rho) - \frac{1}{d} \right]. \tag{19}$$

Proof It is shown that for any pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, the following inequality holds:[27]

$$C(|\psi\rangle) \geq \sqrt{\frac{2d}{d-1}} \left(\max_{|\phi\rangle \in \varepsilon} |\langle \psi | \phi \rangle|^2 - \frac{1}{d} \right). \tag{20}$$

Let $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ be the optimal decomposition such that $C(\rho) = \sum_i p_i C(|\psi_i\rangle)$. We have

$$\begin{aligned} C(\rho) &= \sum_i p_i C(|\psi_i\rangle) \\ &\geq \sum_i p_i \sqrt{\frac{2d}{d-1}} \left(\max_{|\phi\rangle \in \varepsilon} |\langle \psi_i | \phi \rangle|^2 - \frac{1}{d} \right) \\ &\geq \sqrt{\frac{2d}{d-1}} \left(\max_{|\phi\rangle \in \varepsilon} \sum_i p_i |\langle \psi_i | \phi \rangle|^2 - \frac{1}{d} \right) \\ &= \sqrt{\frac{2d}{d-1}} \left(\max_{|\phi\rangle \in \varepsilon} \langle \phi | \rho | \phi \rangle - \frac{1}{d} \right) \\ &= \sqrt{\frac{2d}{d-1}} \left(\mathcal{F}(\rho) - \frac{1}{d} \right), \end{aligned}$$

which ends the proof. □

Inequality (19) shows a relation between the lower bound of concurrence and the fully entangled fraction (thus the optimal teleportation fidelity), namely the fully entangled fraction of quantum state ρ is limited by its concurrence. Moreover inequality (19) also gives a lower bound for concurrence which is obviously closer than that in Ref. [27].

4. Bounds of concurrence and tangle for bipartite quantum systems

In this section we derive new lower and upper bounds of concurrence and tangle for arbitrary quantum states.

We see that inequalities (12) and (13) can be regarded as lower bounds, respectively, for C and τ of a certain state that can be achieved with the help of another state. In fact, there have been many lower and upper bounds for concurrence and tangle.[12−16,28−32] Here we list just several important ones that will be used in the following. In Ref. [12] a lower bound for a bipartite state $\rho \in \mathcal{H}_A \otimes \mathcal{H}_B$, with dimensions $M \leq N$, has been obtained as

$$C(\rho) \geq \sqrt{\frac{2}{M(M-1)}}[\max(||\mathcal{T}_A(\rho)||_{KF}, ||R(\rho)||_{KF}) - 1], \tag{21}$$

where $\mathcal{T}_A$, R and $||\cdot||_{KF}$ represent the partial transpose, the realignment, and the trace norm (sum of the singular values), respectively.

In Refs. [14], [28] and [29], from the separability criteria related to local uncertainty relation, covariance matrix and correlation matrix, the following lower bounds for bipartite concurrence are obtained:

$$C(\rho) \geq \frac{2||C(\rho)||_{KF} - (1 - \mathrm{Tr}\{\rho_A^2\}) - (1 - \mathrm{Tr}\{\rho_B^2\})}{\sqrt{2M(M-1)}} \tag{22}$$

and

$$C(\rho) \geq \sqrt{\frac{8}{M^3N^2(M-1)}}\left(||\boldsymbol{T}(\rho)||_{KF} - \frac{\sqrt{MN(M-1)(N-1)}}{2}\right), \tag{23}$$

where the entries of the matrices C and $\boldsymbol{T}$ are given by $C_{ij} = \langle \lambda_i^A \otimes \lambda_j^B\rangle - \langle \lambda_i^A \otimes I_N\rangle\langle I_M \otimes \lambda_j^B\rangle$, and $T_{ij} = (MN/2)\langle \lambda_i^A \otimes \lambda_j^B\rangle$, with $\lambda_k^{A/B}$ denoting the normalised generator of SU(M/N) that satisfies $\mathrm{Tr}\{\lambda_k^{A/B}\lambda_l^{A/B}\} = \delta_{kl}$ and $\langle X\rangle = \mathrm{Tr}\{\rho X\}$ representing the expection value of X. It is shown that the lower bounds (22) and (23) are independent of (21). Besides, in Ref. [15], a lower bound for tangle has been derived as

$$\tau(\rho) \geq \frac{8}{MN(M+N)}\left(||\boldsymbol{T}(\rho)||_{\mathrm{HS}}^2 - \frac{MN(M-1)(N-1)}{4}\right), \tag{24}$$

where $||X||_{\mathrm{HS}} = \sqrt{\mathrm{Tr}(XX^\dagger)}$ denotes the Frobenius (Hilbert–Schmidt) norm. Experimentally measurable lower and upper bounds for concurrence have been presented in Refs. [16] and [32] as follows:

$$\sqrt{2(\mathrm{Tr}[\rho^2] - \mathrm{Tr}[\rho_A^2])} \leq C(\rho) \leq \sqrt{2(1 - \mathrm{Tr}[\rho_A^2])}. \tag{25}$$

Since the convexity of $C^2(\rho)$, $\tau(\rho) \geq C^2(\rho)$ always holds. In Refs. [19] and [20] the authors point out that for two qubits quantum systems, tangle τ is always equal to the square of concurrence C^2, as a decomposition $\{p_i, |\psi_i\rangle\}$ achieving the minimum in Eq. (4) will have the property that $C(|\psi_i\rangle) = C(|\psi_j\rangle)$ for all i, j. But for higher-dimensional systems we do not have similar equations. Therefore it is meaningful to derive valid upper bound for tangle and lower bound for concurrence. In the following we derive an effective upper bound for tangle, which can be used to estimate the entanglement of quantum states. We also derive new lower bound for concurrence which is better than those in inequalities (21)–(23) for certain quantum states.

Theorem 3 For any quantum state $\rho \in \mathcal{H}_A \otimes \mathcal{H}_B$, we have

$$\tau(\rho) \leq \min\{2(1 - \mathrm{Tr}(\rho_A^2)), 2(1 - \mathrm{Tr}(\rho_B^2))\}, \tag{26}$$

$$C(\rho) \geq \sqrt{\frac{8}{MN(M+N)}}\left(||\boldsymbol{T}(\rho)||_{\mathrm{HS}} - \frac{\sqrt{MN(M-1)(N-1)}}{2}\right), \tag{27}$$

where $\rho_{A/B}$ is the reduced matrix of ρ, and $\boldsymbol{T}(\rho)$ is the correlation matrix of ρ defined in inequality (23).

Proof We assume $1 - \mathrm{Tr}(\rho_A^2) \leq 1 - \mathrm{Tr}(\rho_B^2)$ for convenience. From the definition of τ, we have that for pure state $|\psi\rangle$, $\tau(|\psi\rangle) = 2(1 - \mathrm{Tr}(\rho_A^{|\psi\rangle})^2)$. Let $\rho = \sum_i p_i\rho_i$ be the optimal decomposition such that $\tau(\rho) = \sum_i p_i\tau(\rho_i)$. We have

$$\tau(\rho) = \sum_i p_i\tau(\rho_i) = \sum_i p_i 2[1 - \mathrm{Tr}(\rho_A^{|\psi_i\rangle})^2] = 2\left[1 - \mathrm{Tr}\sum_i p_i(\rho_A^{|\psi_i\rangle})^2\right] \leq 2[1 - \mathrm{Tr}(\rho_A^2)]. \tag{28}$$

To prove inequality (27), first note that in Ref. [15] the author obtained for pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$,

$$C(|\psi\rangle) = \sqrt{\frac{8}{MN(M+N)}\left(||\boldsymbol{T}(|\psi\rangle)||^2_{\rm HS} - \frac{MN(M-1)(N-1)}{4}\right)}. \tag{29}$$

Using inequality $\sqrt{a-b} \geq \sqrt{a} - \sqrt{b}$ for any $a \geq b$, we have

$$C(|\psi\rangle) \geq \sqrt{\frac{8}{MN(M+N)}}\left(||\boldsymbol{T}(|\psi\rangle)||_{\rm HS} - \frac{\sqrt{MN(M-1)(N-1)}}{2}\right). \tag{30}$$

Now let $\rho = \sum_i p_i\rho_i$ be the optimal decomposition such that $C(\rho) = \sum_i p_i C(\rho_i)$. We obtain

$$\begin{aligned} C(\rho) &= \sum_i p_i C(\rho_i) \geq \sum_i p_i \sqrt{\frac{8}{MN(M+N)}}\left(||\boldsymbol{T}(\rho_i)||_{\rm HS} - \frac{\sqrt{MN(M-1)(N-1)}}{2}\right) \\ &= \sqrt{\frac{8}{MN(M+N)}}\left(\sum_i p_i||T(\rho_i)||_{\rm HS} - \frac{\sqrt{MN(M-1)(N-1)}}{2}\right) \\ &\geq \sqrt{\frac{8}{MN(M+N)}}\left(||T(\rho)||_{\rm HS} - \frac{\sqrt{MN(M-1)(N-1)}}{2}\right), \end{aligned}$$

which ends the proof. □

Measurable upper bound (26), together with the lower bound in inequalities (21)–(25), allows for the better estimation of entanglement for arbitrary quantum states. Moreover, since the upper bound is exactly the value of tangle for pure states, the upper bound can be a good estimation when the state is very weakly mixed. As an example, we consider the 3×3 mixed state $\rho = ((1-p)/9)\boldsymbol{I}_9 + p|\psi\rangle\langle\psi|$, where $|\psi\rangle$ is a randomly generated pure state, $\boldsymbol{I}_9$ is the 9×9 identity matrix. The upper and lower line is the upper and lower bounds for tangle respectively. To compare the validity of the estimation of $\tau(\rho)$, we take $p = 0.981$, 0.993 and 0.998 sequentially. As seen from Fig. 1, for weakly mixed states (with large p), the bounds provide an excellent estimation for tangle. One can also easily find that lower bound (27) is obviously stronger than the value obtained from inequality (23) when $||T||_{\rm KF} \approx ||T||_{\rm HS}$.

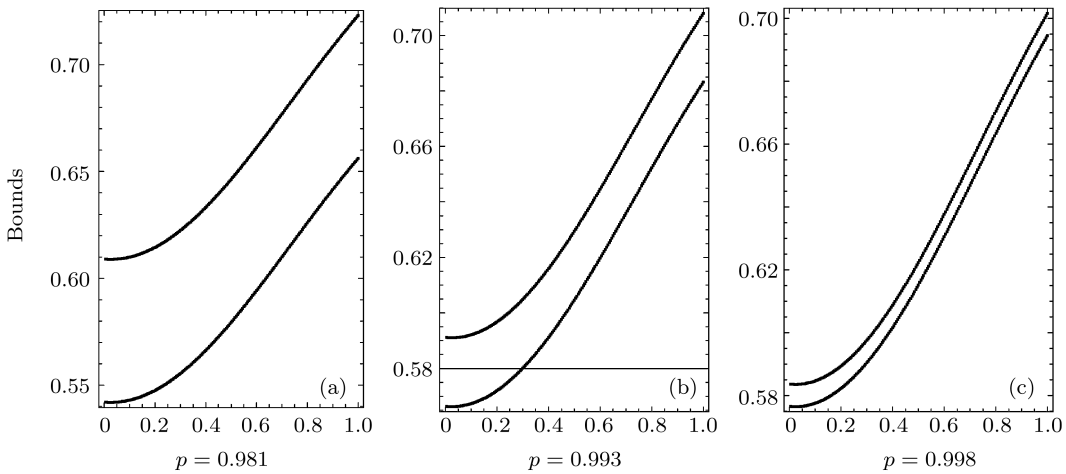


Fig. 1. Upper and lower bounds for $\tau(\rho)$, with $p = 0.981$, 0.993 and 0.998 taken sequentially to show the validity of these bounds.

5. Subadditivity of concurrence and tangle

The additivity is an important property of entanglement measures, though it is usually rather difficult to prove. The strong subadditivity of relative entropy has been proved in Refs. [33] and [34]. In this section, we prove the subadditivity of concurrence and tangle.

Theorem 6 Let ρ and σ be two mixed quantum states in $\mathcal{H}_A \otimes \mathcal{H}_B$. We have

$$C(\rho \otimes \sigma) \leq C(\rho) + C(\sigma)$$

and

$$\tau(\rho \otimes \sigma) \leq \tau(\rho) + \tau(\sigma). \tag{31}$$

Proof We first prove the theorem for pure states. Let $|\psi\rangle$ and $|\phi\rangle$ be two pure states in $\mathcal{H}_A \otimes \mathcal{H}_B$. Assume that $\rho_A^{|\psi\rangle} = \sum_i \lambda_i |i\rangle\langle i|$ and $\rho_A^{|\phi\rangle} = \sum_j \pi_j |j\rangle\langle j|$ are the spectral decompositions of the reduced matrices of $\rho^{|\psi\rangle}$ and $\rho^{|\phi\rangle}$, then we will have

$$\begin{aligned} &\frac{1}{2}[C(|\psi\rangle) + C(|\phi\rangle)]^2 \\ &\geq 1 - \mathrm{Tr}[(\rho_A^{|\psi\rangle})^2] + 1 - \mathrm{Tr}[(\rho_A^{|\phi\rangle})^2] \\ &= 1 - \sum_i \lambda_i^2 + 1 - \sum_j \pi_j^2 \\ &\geq 1 - \sum_{ij} \lambda_i^2 \pi_j^2 \\ &= \frac{1}{2} C^2(|\psi\rangle \otimes |\phi\rangle). \end{aligned} \tag{32}$$

Namely, $C(|\psi\rangle \otimes |\phi\rangle) \leq C(|\psi\rangle) + C(|\phi\rangle)$. For tangle τ, the following inequality can be obtained similarly by changing the first inequality in inequality (32) into an equality: $\tau(|\psi\rangle \otimes |\phi\rangle) \leq \tau(|\psi\rangle) + \tau(|\phi\rangle)$.

Now let $\rho = \sum_i p_i \rho_i$ and $\sigma = \sum_j q_j \sigma_j$ be two mixed states at an optimal decomposition such that $C(\rho) = \sum_i p_i C(\rho_i)$ and $C(\sigma) = \sum_j q_j C(\sigma_j)$, then we will have

$$\begin{aligned} C(\rho) + C(\sigma) &= \sum_{ij} p_i q_j [C(\rho_i) + C(\sigma_j)] \\ &\geq \sum_{ij} p_i q_j C(\rho_i \otimes \sigma_j) \\ &\geq C(\rho \otimes \sigma). \end{aligned} \tag{33}$$

The inequality for τ can be derived similarly. □

6. Conclusions and remarks

We investigate the concurrence and the tangle of quantum states. It is shown that although one can not distil singlets from many bound entangled states, the concurrence (and the tangle) $C(\rho \otimes \sigma)$ (and $\tau(\rho \otimes \sigma)$) is always larger than $\max\{C(\rho), C(\sigma)\}$ ($\max\{\tau(\rho), \tau(\sigma)\}$) respectively. We derive a relation between concurrence and the optimal fidelity of teleportation, which shows that the optimal fidelity of teleportation is limited by the concurrence. We also present new upper and lower bounds for concurrence and tangle, which give rise to the better estimation for entanglement of quantum states. Finally we prove the subadditivity of concurrence and tangle.

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