Implementation of positive-operator-value measurements for single spin qubit via Heisenberg model*

Cheng Liu-Yong(程留永)[a], Shao Xiao-Qiang(邵晓强)[b], Zhang Shou(张 寿)[a]†, and Yeon Kyu-Hwang[c]

[a] *Department of Physics, College of Science, Yanbian University, Yanji* 133002, *China*

[b] *Center for the Condensed-Matter Science and Technology, Department of Physics, Harbin Institute of Technology, Harbin* 150001, *China*

[c] *BK21 Program Physics & Department of Physics, College of Natural Science, Chungbuk National University, Cheonju, Chungbuk* 361-763, *Republic of Korea*

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This paper shows that a proposal for implementing all possible two-operator positive-operator-value measurements of single spin qubit can be obtained via introducing another spin qubit as ancilla. The realization process is accomplished from the free evolution of the Heisenberg XX model by considering nearest-neighbour spin interaction. A controlled-NOT gate, which is a significant operator for this scheme is also constructed and the generalisation to multiple-operator is considered finally.

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1. Introduction

Quantum measurement theory, which connects quantum theoretical computation and quantum experimental measurement, plays an important role in the domain of quantum information processing. Considering an observable in a Hilbert space, after a standard measurement, the possible outcomes are precisely the eigenvalues of the given observable and the initial state should be projected onto the corresponding eigenvector, such a measurement is known as von Neumann measurement.[1] A restriction that operators in the von Neumann measurement process are orthogonal projection operators makes this type inapplicable in some quantum measurement tasks. A generalised type of measurement is represented by a positive-operator-value measurement (POVM),[2] whose operations are nonnegative, Hermitian but unnecessarily orthogonal to each other. There are so many practical utilities for POVM in discrimination of quantum nonorthogonal states and quantum teleportation, that a large amount of attentions in past decades has been focused on the theory and experiment of it in different physical systems for atomic states[3,4] and photon polarisation states.[5−7]

It is well known that the main properties of splendid qubits are long coherence time joining up with rapid operating time. Up to the present, the linear optical system and cavity quantum electrodynamics system, as expectant candidates for quantum computer, are widely researched.[8−11] However, there are some theoretical and technical problems such as decoherence from controlling qubit operation, when one structures large-scale quantum network. In recent years, spin qubit is supposed to be a desirable candidate in quantum information processing because of its scalability and stability. It is then natural that many works are raised for implementing quantum computation and measurement in Heisenberg spin system via controlling spin interaction. In 2003, Bulaevskii and Ortiz explored a process of implementing tunneling measurement of a single spin via employing a tunneling current of electrons as the probe.[12] Later on, they investigated a quantum nondemolition measurements of a single quantum spin using Josephson oscillations.[13] Different from the above measurement types, we here propose an implementation of all possible POVMs of single spin qubit with introducing another spin qubit as ancilla via Heisenberg spin model.

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†Corresponding author. E-mail: szhang@ybu.edu.cn
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2. The POVM

Mathematically, a POVM corresponds to a positive-definite partition of unity operator in the given Hilbert space and can be expressed by a set of Hermitian operators $\{F_i, i \leq n\}$, satisfying

$$\sum_{i=1}^{n} F_i = \sum_{i=1}^{n} M_i^{\dagger} M_i = \boldsymbol{I}, \tag{1}$$

where M_i denotes the Klaus operator and $\boldsymbol{I}$ is the unit matrix. For a quantum system with initial state represented by density matrix $\boldsymbol{\rho}$, the possible outcomes of the measurement correspond to the eigenvalues of the measurement operators. After the POVM, the state of the quantum system immediately becomes

$$\boldsymbol{\rho}_i = \frac{M_i \boldsymbol{\rho} M_i^{\dagger}}{\mathrm{tr}(M_i \boldsymbol{\rho} M_i^{\dagger})} \tag{2}$$

with probability

$$p_i = \mathrm{tr}(M_i \boldsymbol{\rho} M_i^{\dagger}). \tag{3}$$

A universal method to accomplish arbitrary POVM can be derived from the Neumark theorem, which states that one can realize any POVM in given Hilbert space by means of performing appropriate projective measurements onto an expanding space.[14] However, it is unfortunate that along with the increment of the dimension of considered system, operation complexity and failure rate grow more quickly during realisation process. For this reason, schemes with less ancilla dimensions are investigated recently.[6,7,15–17] In Ref. [17], authors determined the minimum dimension of the ancilla system introduced for given POVM by the methods of the tensor product extension and the direct sum extension of the initial Hilbert space, respectively, and according to their consequence, ancilla dimensions in our scheme can be considered as the theoretic minimum value.

3. Implementation of POVM for spin qubit

Here we consider the two-site Heisenberg XX model (see Fig. 1), its Hamiltonain can be described by

$$H = J(s_1^x s_2^x + s_1^y s_2^y), \tag{4}$$

where J is the strength of the exchange interaction, s^i $(i = x, y)$ refers to the component of the spin operator and subscripts of operators indicate the spins being acted on.

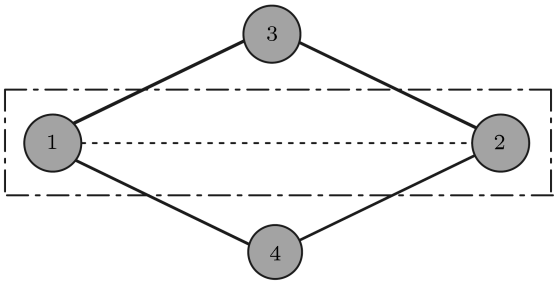


Fig. 1. Diagrammatic illustration of model for implementing two operators POVM. Two spins in dashed line pane indicate the measurement system including spin 1 to be measured and the ancilla spin qubit 2. Dot dashed line between qubits 1 and 2 means the interaction between them will be cancelled when qubit 3 and qubit 4 are introduced.

Conditioned that the initial state of our two spin-$\frac{1}{2}$ system is prepared as $|\Phi_0\rangle_1 \otimes |\downarrow\rangle_2$, where $|\Phi_0\rangle_1$ is arbitrary spin superposition state, we first perform a unitary transformation U on qubit 1. The total state evolves into

$$|\Psi_0\rangle = (a|\uparrow\rangle + b|\downarrow\rangle)_1 |\downarrow\rangle_2, \tag{5}$$

where $|a|^2 + |b|^2 = 1$. And we adopt expressions as $|\uparrow\rangle = |0\rangle, |\downarrow\rangle = |1\rangle$ in the following for convenience. Then for the sake of achieving all possible two operators POVM of qubit 1, a series of operations materialised in the following steps are performed on one or both of the two spins.

(I) Choosing the interaction time $T = 2\theta/J$ under the associated time evolution operator $U(t) = \exp[-\mathrm{i}\, J(s_1^x s_2^x + s_1^y s_2^y)t]$. The state of Eq. (5) will evolve as

$$|\Psi_1\rangle = a\cos\theta|01\rangle_{12} - \mathrm{i}\, a\sin\theta|10\rangle_{12} + b|11\rangle_{12}. \tag{6}$$

(II) In a like manner, controlling the system evolution during time $T = 2\theta/J$ after carrying out a single-qubit operator σ_x on qubit 2, we find that the state of Eq. (6) becomes

$$\begin{aligned} |\Psi_2\rangle = {} & a\cos\theta|00\rangle_{12} - \mathrm{i}\, a\sin\theta|11\rangle_{12} \\ & + b\cos\phi|10\rangle_{12} - \mathrm{i}\, b\sin\phi|01\rangle_{12}. \end{aligned} \tag{7}$$

(III) Performing a CNOT gate on the two spin system considering qubit 1 being target qubit and qubit 2 being control qubit, as a consequence, we find that the state of Eq. (7) changes to

$$
\begin{aligned}
|\Psi_3\rangle &= a\cos\theta|00\rangle_{12} - \mathrm{i}\, a\sin\theta|01\rangle_{12} + b\cos\phi|10\rangle_{12} - \mathrm{i}\, b\sin\phi|11\rangle_{12} \\
&= (a\cos\theta|00\rangle_{12} + b\cos\phi|10\rangle_{12}) - \mathrm{i}(a\sin\theta|01\rangle_{12} + b\sin\phi|11\rangle_{12}). \qquad (8)
\end{aligned}
$$

(IV) Applying an appropriate single-qubit operator could eliminate the phase factor in the second term of Eq. (8) and then give rise to a final state shown as

$$
|\Psi_4\rangle = (a\cos\theta|00\rangle_{12} + b\cos\phi|10\rangle_{12}) + (a\sin\theta|01\rangle_{12} + b\sin\phi|11\rangle_{12}). \qquad (9)
$$

Note that the CNOT gate in step (III) is a key component to our implementation. Since it is difficult to carry out a CNOT gate on two spin qubits directly, decomposing it into one controlled-Z gate plus two Hadamard gates and achieving them step by step becomes a feasible means. The implementation approach proceeds as follows.

First, we perform a Hadamard gate on qubit 1 based on the acquired state $|\Psi_2\rangle$ and obtain a new outcome of Eq. (7) as

$$
\begin{aligned}
|\Psi_2'\rangle = \frac{1}{\sqrt{2}}\big[&a\cos\theta(|00\rangle_{12} + |10\rangle_{12}) - \mathrm{i}\, a\sin\theta(|01\rangle_{12} - |11\rangle_{12}) \\
&+ b\cos\phi(|00\rangle_{12} - |10\rangle_{12}) - \mathrm{i}\, b\sin\phi(|01\rangle_{12} + |11\rangle_{12})\big]. \qquad (10)
\end{aligned}
$$

Secondly, we change the Heisenberg system to achieve the controlled-Z gate by adding an additional qubit 3 with spin-$\frac{1}{2}$ initialized at $|0\rangle$ which interacts with both qubits 1 and 2, while interaction between qubits 1 and 2 vanishes (see Fig. 1). Based on the reconstructed three-site system, we can obtain a serviceable three-qubit unitary operator $\tilde{U}$ proposed by Yung *et al.*,[18] and its effect is equivalent to a swap-phase gate acting on qubits 1 and 2 when the control qubit 3 sets at $|0\rangle$ in our system, expressed as

$$
\begin{aligned}
|\Psi_2'\rangle_{12}|0\rangle_3 \rightarrow \frac{1}{\sqrt{2}}\big[&a\cos\theta(|00\rangle_{12} - |10\rangle_{12}) - \mathrm{i}\, a\sin\theta(-|01\rangle_{12} + |11\rangle_{12}) \\
&+ b\cos\phi(|00\rangle_{12} + |10\rangle_{12}) - \mathrm{i}\, b\sin\phi(-|01\rangle_{12} - |11\rangle_{12})\big]|0\rangle_3. \qquad (11)
\end{aligned}
$$

And we then again introduce a spin qubit 4 with spin-$\frac{1}{2}$ which is in $|0\rangle$, and interactions among qubits 1, 2 and 4 are similar to that among qubits 1, 2 and 3. The total Hamiltonain of this four-spin system is

$$
H' = J(s_1^x s_3^x + s_1^y s_3^y + s_1^x s_4^x + s_1^y s_4^y + s_2^x s_3^x + s_2^y s_3^y + s_2^x s_4^x + s_2^y s_4^y). \qquad (12)
$$

The exact time evolution of the bases for this system is calculated as

$$
\begin{aligned}
|0100\rangle_{1234} \rightarrow\ & \frac{1}{2}[1+\cos(2JT)]|0100\rangle_{1234} - \frac{\mathrm{i}\sin(2JT)}{2}(|0010\rangle_{1234} + |0001\rangle_{1234}) \\
&+ \frac{1}{2}[-1+\cos(2JT)]|1000\rangle_{1234}, \\
|1000\rangle_{1234} \rightarrow\ & \frac{1}{2}[1+\cos(2JT)]|1000\rangle_{1234} - \frac{\mathrm{i}\sin(2JT)}{2}(|0010\rangle_{1234} + |0001\rangle_{1234}) \\
&+ \frac{1}{2}[-1+\cos(2JT)]|0100\rangle_{1234}, \\
|1100\rangle_{1234} \rightarrow\ & \frac{1}{2}[1+\cos(2\sqrt{2}JT)]|1100\rangle_{1234} - \frac{\mathrm{i}\sin(2\sqrt{2}JT)}{2\sqrt{2}}(|0101\rangle_{1234} + |0110\rangle_{1234} \\
&+ |1010\rangle_{1234} + |1001\rangle_{1234}) + \frac{1}{2}[-1+\cos(2\sqrt{2}JT)]|1000\rangle_{1234}. \qquad (13)
\end{aligned}
$$

By taking $JT = 155.505$, the above expressions can be rewrite as

$$
\begin{aligned}
|0100\rangle_{1234} \rightarrow\ & 0.0000147175|0100\rangle_{1234} - 0.00383632\,\mathrm{i}(|0010\rangle_{1234} + |0001\rangle_{1234}) \\
&- 0.999985|1000\rangle_{1234}, \\
|1000\rangle_{1234} \rightarrow\ & 0.0000147175|1000\rangle_{1234} - 0.00383632\,\mathrm{i}(|0010\rangle_{1234} + |0001\rangle_{1234}) \\
&- 0.999985|0100\rangle_{1234}, \\
|1100\rangle_{1234} \rightarrow\ & 0.999966|1100\rangle_{1234} - 0.00409707\,\mathrm{i}(|0101\rangle_{1234} + |0110\rangle_{1234} \\
&+ |1010\rangle_{1234} + |1001\rangle_{1234}) - 0.0000335731|1000\rangle_{1234}. \qquad (14)
\end{aligned}
$$

Namely, we have realized transformations of $|0100\rangle_{1234} \to -|1000\rangle_{1234}, |1000\rangle_{1234} \to -|0100\rangle_{1234}$ and leaving $|1100\rangle_{1234}$ unchanged with a high fidelity. As a result, the state of Eq. (10) approximately changes to

$$\begin{aligned} |\Psi_2^{''}\rangle \;=\; & \frac{1}{\sqrt{2}}[a\cos\theta(|00\rangle_{12}+|10\rangle_{12}) - \mathrm{i}\,a\sin\theta(|01\rangle_{12}+|11\rangle_{12}) \\ & + b\cos\phi(|00\rangle_{12}-|10\rangle_{12}) - \mathrm{i}\,b\sin\phi(|01\rangle_{12}-|11\rangle_{12})]. \end{aligned} \tag{15}$$

After that we perform a Hadamard gate on qubit 1 again and then Eq. (15) evolves into Eq. (8), i.e., the above-mentioned CNOT gate is accomplished successfully.

It is well known that, the singular value decomposition of matrix refers to that the expression $\boldsymbol{A} = \boldsymbol{VDU}$, where $\boldsymbol{U}$ and $\boldsymbol{V}$ are unitary matrices and $\boldsymbol{D}$ is a diagonal matrix, is proper for a square matrix $\boldsymbol{A}$. So Kraus operator M in Eq. (1) can be decomposed as:[6]

$$M_i = V_i D_i U_i. \tag{16}$$

The diagonal elements of D_i are singular values of M_i, i.e., non-negative square roots of eigenvalues of $F_i = M_i^\dagger M_i$.

Based on above statement, we can rewrite Eq. (9) in the matrix transformations as

$$\begin{aligned} |\Phi_0\rangle_1|1\rangle_2 \;\to & \begin{bmatrix} \cos\theta & 0 \\ 0 & \cos\phi \end{bmatrix} (a|0\rangle_1 + b|1\rangle_1)|0\rangle_2 + \begin{bmatrix} \sin\theta & 0 \\ 0 & \sin\phi \end{bmatrix} (a|0\rangle_1 + b|1\rangle_1)|1\rangle_2 \\ & = D_2 U|\Phi_0\rangle_1|0\rangle_2 + D_1 U|\Phi_0\rangle_1|1\rangle_2. \end{aligned} \tag{17}$$

Then we can materially realize any POVM consisting of two measurement operators on spin qubit 1 by choosing appropriate unitary transformation $V_i (i = 1, 2)$ on it and then performing projective measurement on spin qubit 2. The two realized operators are

$$\begin{aligned} F_1 &= M_1^\dagger M_1 = U^\dagger D_1^\dagger V_1^\dagger V_1 D_1 U = U^\dagger D_1^2 U, \\ F_2 &= M_2^\dagger M_2 = U^\dagger D_2^2 U = U^\dagger (I - D_1^2) U = I - F_1. \end{aligned} \tag{18}$$

And all final states after measurement are given by Eq. (2) with probabilities Eq. (3). The complete implementation procedure of all possible two operators POVM of single spin qubit has been carried out at this point.

Our scheme can be generalised to implement n operators POVMs based on procedures of two operators above. We first consider the case of $n = 3$. This requires that the Hamiltonion interaction between qubits 1 and 2 rises again when projective outcome of spin 2 is $|0\rangle$, followed by analogical logic gates in Eqs. (6)–(9). Prospective operators

$$\begin{aligned} F_1 &= U_1^\dagger D_1^2 U_1 = M_1^\dagger M_1, \\ F_2 &= U_1^\dagger \tilde{D}_1^\dagger U_2^\dagger D_2^2 U_2 \tilde{D}_1 U_1 = M_2^\dagger M_2, \\ F_3 &= U_1^\dagger \tilde{D}_1^\dagger U_2^\dagger (I - D_2^2) U_2 \tilde{D}_1 U_1 \\ &= U_1^\dagger \tilde{D}_1^\dagger \tilde{D}_1 U_1 - F_2 = I - F_1 - F_2, \end{aligned} \tag{19}$$

where U_1, D_1 and $\tilde{D}_1$ correspond to U, D_1 and D_2 in Eq. (18) respectively, U_2 and D_2 in the second procedures act as U_1 and D_1 in the first one respectively.

For n operators, straightway, we have

$$F_j = \left[\prod_{i=1}^{j-1} U_i^\dagger \tilde{D}_i^\dagger\right] U_j^\dagger D_j^2 U_j \left[\prod_{i=1}^{j-1} \tilde{D}_{j-\mathrm{i}} U_{j-\mathrm{i}}\right] \tag{20}$$

for $j < n$, and

$$\begin{aligned} F_n \;=\; & \left[\prod_{i=1}^{n-2} U_i^\dagger \tilde{D}_i^\dagger\right] U_{n-1}^\dagger (I - D_{n-1}^2) U_{n-1} \\ & \times \left[\prod_{i=1}^{n-2} \tilde{D}_{n-i} U_{n-i}\right] \\ =\; & \left[\prod_{i=1}^{n-1} U_i^\dagger \tilde{D}_i^\dagger\right] \left[\prod_{i=1}^{n-1} \tilde{D}_{n-i} U_{n-i}\right] \end{aligned} \tag{21}$$

for $j = n$. Overall dimensions of ancilla system in our scheme, expressed as $n-1$, likewise reach the theoretical low bond.

4. Conclusion

To summarise, We have formulated a proposal to implement all possible POVMs of the single spin qubit

via Heisenberg XX model. The realization process assisted by another spin as ancilla qubit mainly includes controlling the evolution of the system and applying relevant quantum logic gates. In the entire process, a key operation of CNOT gate is achieved step by step by altering initial spin system reasonably. It should be pointed that in the context, we have introduced qubits 3 and 4 besides auxiliary qubit 2, nevertheless, both the two qubits 3 and 4 are used to only assist the construction for the CNOT gate between qubits 1 and 2 rather than influencing the total dimensions of the measurement system, thus our proposal submits to the theory of minimum dimension of the ancilla system introduced for realizing POVM. Note that more and more researchers interest themselves in unambiguous state discrimination of nonorthogonal states which have applications in quantum cryptography basically employing the principles and methods of POVM.[19−23] There is good reason to believe that quantum measurement theories, especially POVMs may develop comprehensively and thoroughly in the near future.

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