Exact solution of entanglement of the double Jaynes–Cummings model without rotating wave approximation*

Ren Xue-Zao(任学藻)†, Jiang Dao-Lai(姜道来), Cong Hong-Lu(丛红璐), and Li Lei(黎 雷)

School of Science, Southwest University of Science and Technology, Mianyang 621010, *China*

(Received 12 September 2009; revised manuscript received 16 January 2010)

This paper investigates the influences of atom–field coupling and dipole–dipole coupling for atoms on the entanglement between two atoms by means of concurrence. The results show that the sudden death occurs when the atom–field coupling is strong enough, and the collapse and the revival appear when the dipole–dipole interaction is strong enough.

Keywords: concurrence, sudden death, atom–field coupling, dipole–dipole interaction

PACC: 0365, 4250

1. Introduction

Quantum entanglement is an important concept playing a central role in quantum information science and quantum computation.[1,2] It is an essential resource for teleportation,[3] quantum key distribution[4,5] and controlled quantum logic.[6] Recently, the double Jaynes–Cummings model has been extensively studied and some important results are gained,[7–9] such as entanglement sudden death.[9–12]

The rotating wave approximation (RWA), which neglects counter-rotating terms corresponding to the emission and absorption of virtual photon without energy conservation, is justified for small detuning and small ratio of the atom–field coupling divided by the atomic transition frequency. In the atom–field cavity system, this ratio is typically of the order 10^{-7}–10^{-6}. Recently, the strong coupling is realized in some systems such as superconducting qubit systems[13,14] and semiconductor microcavities.[15] In these systems, we must take the counter rotating wave terms into account.

In this paper, we study the double Janyes-Cummings model without RWA by the method of coherent-state orthogonalization expansion. The influences of atom–field coupling strength and dipole–dipole strength on entanglement are discussed.

2. Model and solution

We consider the double Jaynes–Cummings model without RWA with dipole–dipole interaction, which is described as follows:

$$H' = H'_1 + H'_2 + H'_3, \tag{1}$$

$$H'_1 = \frac{\Omega}{2}[|E\rangle_1\langle E| - |G\rangle_1\langle G|] + \omega_0 a^+ a + g(a^+ + a)(|E\rangle_1\langle G| + |G\rangle_1\langle E|), \tag{2}$$

$$H'_2 = \frac{\Omega}{2}[|E\rangle_2\langle E| - |G\rangle_2\langle G|] + \omega_0 b^+ b + g(b^+ + b)(|E\rangle_2\langle G| + |G\rangle_2\langle E|), \tag{3}$$

$$H'_3 = \eta[(|E\rangle_1\langle G|)(|G\rangle_2\langle E|) + (|G\rangle_1\langle E|)(|E\rangle_2\langle G|)], \tag{4}$$

where $a^+(a)$ denotes the creation (annihilation) operator of the first mode, and the $b^+(b)$ denotes the creation (annihilation) operator of the second mode, ω_0 is the frequency of light field, Ω is the atomic transition frequency between the ground state $|G\rangle$ and excited state $|E\rangle$, g is the strength of atom–light coupling, η is the strength of dipole–dipole interaction.

*Project supported by the Joint Fund of the National Natural Science Foundation of China and the China Academy of Engineering Physics(Grant No. 1097602/A06).

†Corresponding author. E-mail: renxuezao@swust.edu.cn

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Substituting the following equations into Eq. (1):

$$|\uparrow\rangle_1 = \frac{1}{\sqrt{2}}(|E\rangle_1 + |G\rangle_1), \quad |\downarrow\rangle_1 = \frac{1}{\sqrt{2}}(|E\rangle_1 - |G\rangle_1), \tag{5}$$

$$|\uparrow\rangle_2 = \frac{1}{\sqrt{2}}(|E\rangle_2 + |G\rangle_2), \quad |\downarrow\rangle_2 = \frac{1}{\sqrt{2}}(|E\rangle_2 - |G\rangle_2), \tag{6}$$

we obtain

$$H = H_1 + H_2 + H_3, \tag{7}$$

where

$$H_1 = \omega_0 a^+ a + g(a^+ + a)(|\uparrow\rangle_1 \langle\uparrow| - |\downarrow\rangle_1 \langle\downarrow|) + \frac{\Omega}{2}(|\uparrow\rangle_1 \langle\downarrow| + |\downarrow\rangle_1 \langle\uparrow|), \tag{8}$$

$$H_2 = \omega_0 b^+ b + g(b^+ + b)(|\uparrow\rangle_2 \langle\uparrow| - |\downarrow\rangle_2 \langle\downarrow|) + \frac{\Omega}{2}(|\uparrow\rangle_2 \langle\downarrow| + |\downarrow\rangle_2 \langle\uparrow|), \tag{9}$$

$$\begin{aligned} H_3 = {} & \frac{\eta}{2}(|\uparrow\rangle_1 \langle\uparrow| - |\uparrow\rangle_1 \langle\downarrow| + |\downarrow\rangle_1 \langle\uparrow| - |\downarrow\rangle_1 \langle\downarrow|)(|\uparrow\rangle_2 \langle\uparrow| + |\uparrow\rangle_2 \langle\downarrow| - |\downarrow\rangle_2 \langle\uparrow| - |\downarrow\rangle_2 \langle\downarrow|) \\ & + \frac{\eta}{2}(|\uparrow\rangle_1 \langle\uparrow| + |\uparrow\rangle_1 \langle\downarrow| - |\downarrow\rangle_1 \langle\uparrow| - |\downarrow\rangle_1 \langle\downarrow|)(|\uparrow\rangle_2 \langle\uparrow| - |\uparrow\rangle_2 \langle\downarrow| + |\downarrow\rangle_2 \langle\uparrow| - |\downarrow\rangle_2 \langle\downarrow|). \end{aligned} \tag{10}$$

The stationary state of Eq. (7) can be expressed as follows:

$$|\ \rangle = |\varphi_1\rangle\,|\uparrow\uparrow\rangle + |\varphi_2\rangle\,|\uparrow\downarrow\rangle + |\varphi_3\rangle\,|\downarrow\uparrow\rangle + |\varphi_4\rangle\,|\downarrow\downarrow\rangle\,. \tag{11}$$

According to the equation

$$H\,|\ \rangle = E\,|\ \rangle\,, \tag{12}$$

we define four new boson operators as follows:[16−22]

$$A_+ = a + g/\omega_0, \quad A_- = a - g/\omega_0, \quad B_+ = b + g/\omega_0, \quad B_- = b - g/\omega_0. \tag{13}$$

We assume

$$\begin{aligned} |\varphi_1\rangle &= \sum_{m,n} c_{m,n}\,|m,n\rangle_{A_+B_+}, \quad |\varphi_2\rangle = \sum_{m,n} d_{m,n}\,|m,n\rangle_{A_+B_-}, \\ |\varphi_3\rangle &= \sum_{m,n} e_{m,n}\,|m,n\rangle_{A_-B_+}, \quad |\varphi_4\rangle = \sum_{m,n} f_{m,n}\,|m,n\rangle_{A_-B_-}, \end{aligned} \tag{14}$$

where

$$\begin{aligned} &|n\rangle_{A_+} = \frac{1}{\sqrt{n!}}\left(A_+^+\right)^n |0\rangle_{A_+}, \quad |0\rangle_{A_+} = \mathrm{e}^{-\frac{g}{\omega_0}a^+ - \frac{g^2}{2\omega_0^2}}\,|0\rangle, \quad |n\rangle_{A_-} = \frac{1}{\sqrt{n!}}\left(A_-^+\right)^n |0\rangle_{A_-}, \\ &|0\rangle_{A_-} = \mathrm{e}^{\frac{g}{\omega_0}a^+ - \frac{g^2}{2\omega_0^2}}\,|0\rangle, \quad |n\rangle_{B_+} = \frac{1}{\sqrt{n!}}\left(B_+^+\right)^n |0\rangle_{B_+}, \quad |0\rangle_{B_+} = \mathrm{e}^{-\frac{g}{\omega_0}b^+ - \frac{g^2}{2\omega_0^2}}\,|0\rangle, \\ &|n\rangle_{B_-} = \frac{1}{\sqrt{n!}}\left(B_-^+\right)^n |0\rangle_{B_-}, \quad |0\rangle_{B_-} = \mathrm{e}^{\frac{g}{\omega_0}b^+ - \frac{g^2}{2\omega_0^2}}\,|0\rangle. \end{aligned} \tag{15}$$

From Eqs. (11)–(15) we obtain

$$\begin{aligned} &\left[(\omega_0 k - g^2/\omega_0) + (\omega_0 l - g^2/\omega_0) + \frac{\eta}{2}\right] c_{kl} + \frac{\Omega}{2}\sum_{m,n} d_{mnA_+}\langle l|n\rangle_{A_-}\,\delta_{k,m} + \frac{\Omega}{2}\sum_{m,n} e_{mnB_+}\langle k|m\rangle_{B_-}\,\delta_{l,n} \\ &- \frac{\eta}{2}\sum_{m,n} f_{m,nA_+}\langle l|n\rangle_{A_-B_+}\langle k|m\rangle\,B_- = Ec_{kl}, \end{aligned} \tag{16}$$

$$\begin{aligned} &\left[(\omega_0 k - g^2/\omega_0) + (\omega_0 l - g^2/\omega_0) - \frac{\eta}{2}\right] d_{k,l} + \frac{\Omega}{2}\sum_{m,n} c_{mnA_-}\langle l|n\rangle_{A_+}\,\delta_{k,m} + \frac{\Omega}{2}\sum_{m,n} f_{mnB_+}\langle k|m\rangle_{B_-}\,\delta_{l,n} \\ &+ \frac{\eta}{2}\sum_{m,n} e_{m,nA_-}\langle l|n\rangle_{A_+B_+}\langle k|m\rangle_{B_-} = Ed_{kl}, \end{aligned} \tag{17}$$

$$\left[(\omega_0 k - g^2/\omega_0) + (\omega_0 l - g^2/\omega_0) - \frac{\eta}{2}\right] e_{k,l} + \frac{\Omega}{2}\sum_{m,n} c_{mnB_-} \langle k|m\rangle_{B_+} \delta_{l,n} + \frac{\Omega}{2}\sum_{m,n} f_{mnA_+} \langle l|n\rangle_{A_-} \delta_{k,m}$$
$$+ \frac{\eta}{2}\sum_{m,n} d_{mnA_+} \langle l|n\rangle_{A_-B_-} \langle k|m\rangle_{B_+} = E e_{kl}, \tag{18}$$

$$\left[(\omega_0 k - g^2/\omega_0) + (\omega_0 l - g^2/\omega_0) + \frac{\eta}{2}\right] f_{k,l} + \frac{\Omega}{2}\sum_{m,n} d_{mnB_-} \langle k|m\rangle_{B_+} \delta_{l,n} + \frac{\Omega}{2}\sum_{m,n} e_{mnA_-} \langle l|n\rangle_{A_+} \delta_{k,m}$$
$$- \frac{\eta}{2}\sum_{m,n} c_{mnA_-} \langle l|n\rangle_{A_+B_-} \langle k|m\rangle_{B_+} = E f_{k,l}, \tag{19}$$

where

$$\begin{aligned}
&{}_{A_-}\langle m|n\rangle_{A_+} = (-1)^m D_{mn}\left(\frac{2g}{\omega_0}\right), \quad {}_{A_+}\langle m|n\rangle_{A_-} = (-1)^n D_{mn}\left(\frac{2g}{\omega_0}\right),\\
&{}_{B_-}\langle m|n\rangle_{B_+} = (-1)^m D_{mn}\left(\frac{2g}{\omega_0}\right), \quad {}_{B_+}\langle m|n\rangle_{B_-} = (-1)^n D_{mn}\left(\frac{2g}{\omega_0}\right),
\end{aligned} \tag{20}$$

$$D_{mn}(x) = \mathrm{e}^{-\frac{x^2}{2}} \sum_{r=0}^{\min[m,n]} \frac{(-1)^r \sqrt{m!n!}\, x^{m+n-2r}}{(m-r)!(n-r)!r!}. \tag{21}$$

From Eqs. (16)–(21) we can obtain $\{c_n^i\}$, $\{d_n^i\}$, $\{e_n^i\}$, $\{f_n^i\}$ and the energy spectrum $\{E^i\}$ $(i = 1, \ldots, 4(N+1)^2)$. The initial state of our system is chosen as follows:

$$|\psi(0)\rangle = (a\,|EE\rangle + b\,|EG\rangle + c\,|GE\rangle + d\,|GG\rangle) \otimes |\alpha\rangle\,|\beta\rangle\,, \tag{22}$$

where $a^2 + b^2 + c^2 + d^2 = 1$, and $|\alpha\rangle$, $|\beta\rangle$ are coherent states. Substituting Eqs. (5) and (6) into Eq. (22), we finally obtain

$$\begin{aligned}
|\psi(0)\rangle = \frac{1}{2}[&(a+b+c+d)\,|\uparrow\uparrow\rangle + (a-b+c-d)\,|\uparrow\downarrow\rangle + (a+b-c-d)\,|\downarrow\uparrow\rangle\\
&+ (a-b-c+d)\,|\downarrow\downarrow\rangle] \otimes |\alpha\rangle\,|\beta\rangle\,.
\end{aligned} \tag{23}$$

We can also expand our initial state as follows:

$$\begin{aligned}
|\psi(0)\rangle = \sum_{i=1}^{4(N+1)^2} k_i \Bigg[&\sum_{m,n=0}^{N} (c_{mn}^i\,|mn\rangle_{A_+B_+}\,|\uparrow\uparrow\rangle + d_{mn}^i\,|mn\rangle_{A_+B_-}\,|\uparrow\downarrow\rangle)\\
&+ e_{mn}^i\,|mn\rangle_{A_-B_+}\,|\downarrow\uparrow\rangle + f_{mn}^i\,|mn\rangle_{A_-B_-}\,|\downarrow\downarrow\rangle \Bigg] \otimes |\alpha\rangle\,|\beta\rangle\,.
\end{aligned} \tag{24}$$

Comparing Eqs. (23) and (24), we can obtain

$$\sum_{i=1}^{4(N+1)^2} k_i c_{mn}^i = \frac{1}{2}(a+b+c+d)_{A_+} \langle m|\alpha\rangle_{B_+} \langle n|\beta\rangle\,, \tag{25}$$

$$\sum_{i=1}^{4(N+1)^2} k_i d_{mn}^i = \frac{1}{2}(a-b+c-d)_{A_+} \langle m|\alpha\rangle_{B_-} \langle n|\beta\rangle\,, \tag{26}$$

$$\sum_{i=1}^{4(N+1)^2} k_i e_{mn}^i = \frac{1}{2}(a+b-c-d)_{A_-} \langle m|\alpha\rangle_{B_+} \langle n|\beta\rangle\,, \tag{27}$$

$$\sum_{i=1}^{4(N+1)^2} k_i f_{mn}^i = \frac{1}{2}(a-b-c+d)_{A_-} \langle m|\alpha\rangle_{B_-} \langle n|\beta\rangle\,. \tag{28}$$

Solving Eqs. (25)–(28), we can obtain $\{k_i\}$ $(i = 1, \ldots, 4(N+1)^2)$, and the time evolution of the wave function

$$\begin{aligned}
|\psi(t)\rangle = \sum_{i=1}^{4(N+1)^2} k_i\, \mathrm{e}^{-\mathrm{i}E_i t} \sum_{m,n=0}^{N} &[c_{mn}^i\,|mn\rangle_{A_+B_+}\,|\uparrow\uparrow\rangle + d_{mn}^i\,|mn\rangle_{A_+B_-}\,|\uparrow\downarrow\rangle\\
&+ e_{mn}^i\,|mn\rangle_{A_-B_+}\,|\downarrow\uparrow\rangle + f_{mn}^i\,|mn\rangle_{A_-B_-}\,|\downarrow\downarrow\rangle].
\end{aligned} \tag{29}$$

The density matrix of the system is $\rho = |\psi(t)\rangle\langle\psi(t)|$. Using the above results, we can conveniently investigate the entanglement for the two bipartite partitions. The entanglement can be suitably measured by concurrence,[23] which may be calculated explicitly from the density matrix,

$$C_{12}(t) = \max\left(0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\right), \quad (30)$$

where λ_i are the eigenvalues of the matrix $\xi = \rho(\sigma_y^1 \otimes \sigma_y^2)\rho^*(\sigma_y^1 \otimes \sigma_y^2)$, arranged in decreasing order. Here σ_y is the well-known Pauli matrix and ρ^* denotes the complex conjugation of ρ. Concurrence C_{12} varies from 0 for a disentangled state to 1 for a maximally entangled state.

3. The influence of atom–field coupling on entanglement

Figures 1(a) and (b) show the time evolution of entanglement with weak atom–field coupling. In this case, the results of without-RWA are in agreement with that of RWA. Obviously, there is no sudden death phenomenon as shown in Fig. 1(a) with the initial atomic state $|\psi_{\rm atom}\rangle = \cos\theta\,|EG\rangle + \sin\theta\,|GE\rangle$. Unlike the previous case, Fig. 1(b) shows that entanglement can fall abruptly to zero(the lower two curves) with the initial atomic state $|\psi_{\rm atom}\rangle = \cos\theta\,|EE\rangle + \sin\theta\,|GG\rangle$, which agrees with the result of Ref. [9]. Figures 1(c) and 1(d) show the time evolution of entanglement with the increase of atom–field coupling for $\Omega/\omega_0 = 1$, $\theta = \pi/4$, $\eta/\omega_0 = 0$, $\alpha = \beta = 0$. It is obvious that there is no sudden death phenomenon when $g/\omega_0 = 0.01$ (the solid curve). The sudden death of entanglement occurs with the increase of atom–field coupling strength, and will remain zero for a period of time before entanglement recovers(the dashed curve and the dotted curve). The length of the time interval for the zero entanglement depends on the atom–field coupling strength. Comparing the three curves, which are shown in Fig. 1(c) and 1(d), we find that the stronger the atom–field coupling strength is, the longer the state stays in the disentangled separable state.

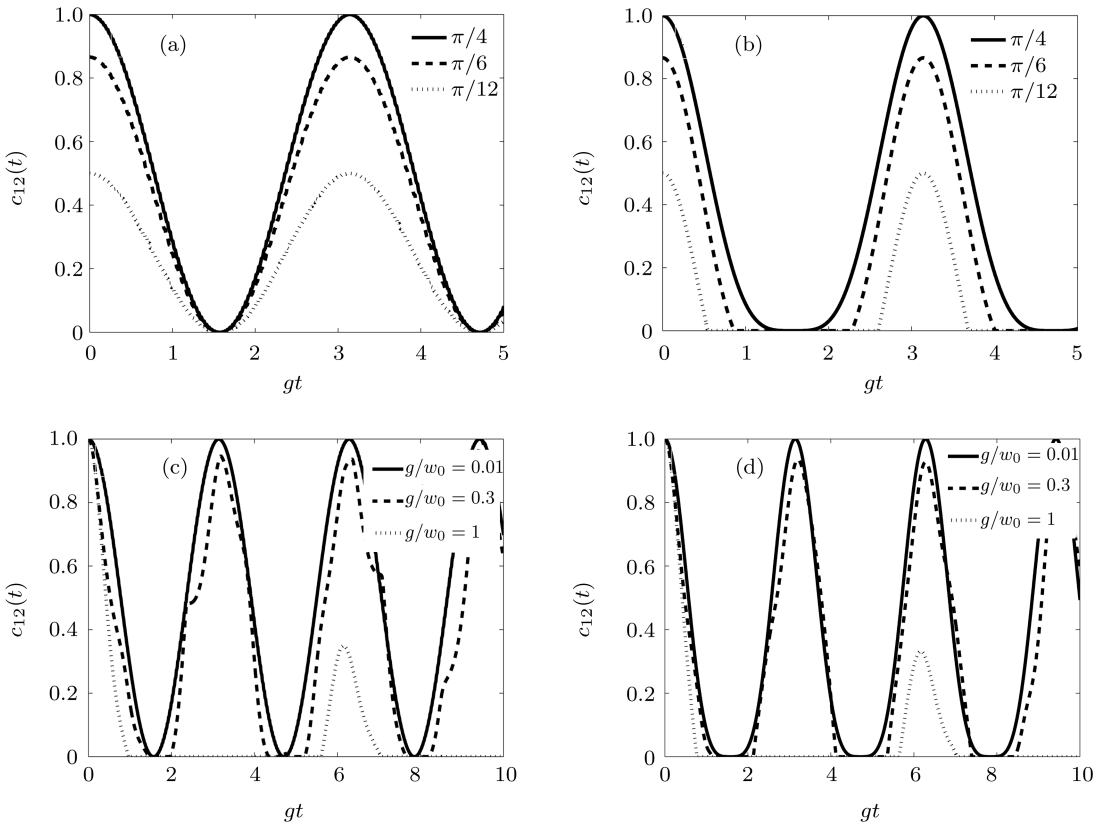


Fig. 1. The influence of atom–field coupling on the time evolution of entanglement, $\Omega/\omega_0 = 1$, $g/\omega_0 = 0.01$, $\eta/\omega_0 = 0$, $\alpha = \beta = 0$: (a) $a = d = 0$, $b = \cos\theta$, $c = \sin\theta$; (b) $a = \cos\theta$, $d = \sin\theta$, $b = c = 0$; (c) $a = d = 0$, $b = \cos\theta$, $c = \sin\theta$; (d) $a = \cos\theta$, $d = \sin\theta$, $b = c = 0$.

4. The influence of dipole–dipole interaction on entanglement

Figure 2 shows the time evolution of entanglement, the initial state of the cavity field is coherent state, with $\Omega/\omega_0 = 1$, $g/\omega_0 = 0.01$, $a = d = 0$, $b = \cos\theta$, $c = \sin\theta$, $\theta = \pi/4$, $\alpha^2 = \beta^2 = 8$, and (a) $\eta/\omega_0 = 0$, (b) $\eta/\omega_0 = 0.15$, (c) $\eta/\omega_0 = 0.3$, (d) $\eta/\omega_0 = 0.45$. From Fig. 2(a) we find that the entanglement sudden death occurs and recovers with small fluctuation. Figure 2(b) shows the time evolution of the concurrence as $\eta/\omega_0 = 0.15$, the collapse and the revival of entanglement appear, and the average of concurrence becomes larger than that of Fig. 2(a), which means that the entanglement of the two atoms is enhanced. Physically, the dipole–dipole interaction is considered to be the effective detuning. The stronger the dipole–dipole interaction is, the larger the detuning is. The interaction between the field and the atoms will be weakened with the increase of η/ω_0, and the entanglement will be enhanced. Further increasing the dipole–dipole coupling strength (Figs. 2(c) and 2(d)), we find that the average of the concurrence becomes larger and larger, and the period of the collapse and the revival becomes longer. The entanglement cannot collapse completely in the collapse regime, but behaves as small zigzag-shaped oscillation. This means that although the atom–field coupling is so weak, the counter rotating wave terms cannot be neglected when the dipole–dipole interaction is strong enough.

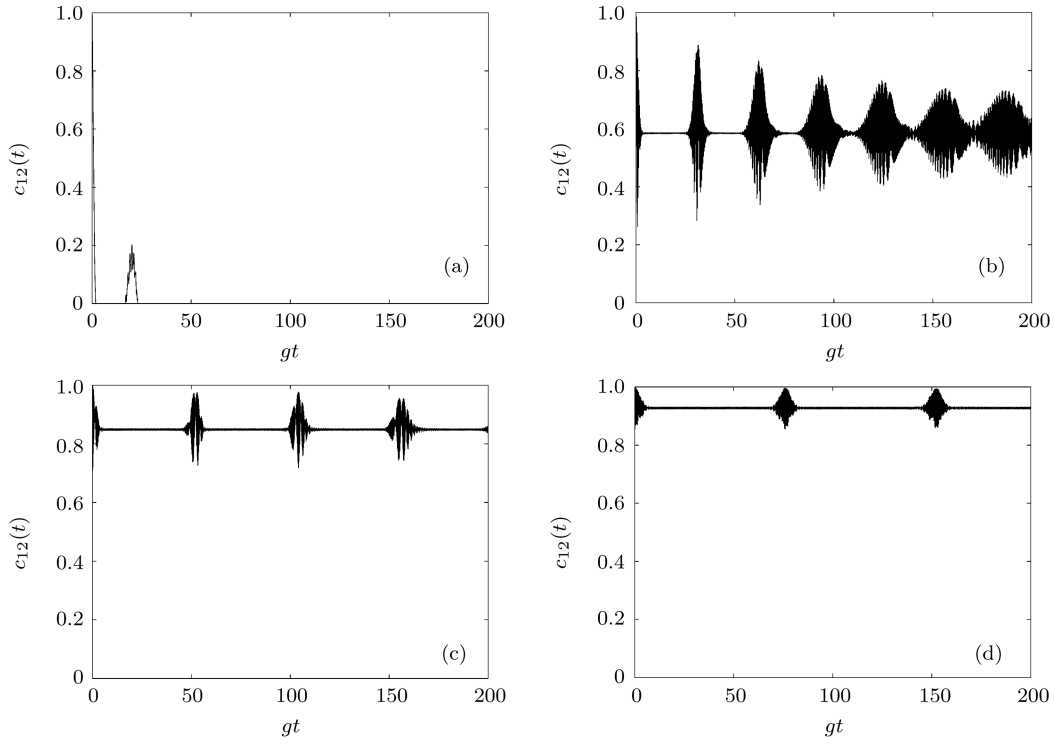


Fig. 2. The influence of dipole–dipole interaction on the evolution of entanglement. $\alpha^2 = \beta^2 = 8, \Omega/\omega_0 = 1, g/\omega_0 = 0.01, a = d = 0, b = \cos\theta, c = \sin\theta, \theta = \pi/4$: (a) $\eta/\omega_0 = 0$; (b) $\eta/\omega_0 = 0.15$; (c) $\eta/\omega_0 = 0.3$; (d) $\eta/\omega_0 = 0.45$.

5. Conclusion

In summary, we investigate the entanglement of two coupled atoms. Based on the numerical results, we have discussed the influences of the atom–field coupling and the dipole–dipole interaction as follows:

(i) For the cavity initially in the vacuum, we investigate the influence of the atom–field coupling. We have investigated the disentanglement of two noninteracting atoms coupling to a two-mode cavity resonantly without RWA. We show that the entanglement evolution depends on the ratio of the atom–field coupling. There is no periodical behaviour in without-RWA.

(ii) For the cavity initially in a coherent state, we investigate the influence of the dipole–dipole interaction. When the dipole–dipole coupling increases, the collapse and the revival of entanglement occurs, and the influence of the counter-rotating terms strengthens. The larger the dipole–dipole interaction is, the stronger the entanglement of the two atoms becomes.

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