Fusion and fission solitons for the (2+1)-dimensional generalized Breor–Kaup system*

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With a projective equation and a linear variable separation method, this paper derives new families of variable separation solutions (including solitory wave solutions, periodic wave solutions, and rational function solutions) with arbitrary functions for (2+1)-dimensional generalized Breor–Kaup (GBK) system. Based on the derived solitary wave excitation, it obtains fusion and fission solitons.

Keywords: projective equation, GBK system, variable separation solutions, fusion and fission solitons

PACC: 0340, 0200, 0365

1. Introduction

Modern soliton theory is one of the most important aspects of nonlinear science, which are widely applied in many natural sciences such as chemistry, biology, mathematics, communication, and particularly in almost all branches of physics like fluid dynamics, plasma physics, field theory, optics, and condensed matter physics, etc.[1−7] Searching for an analytical exact solution to a nonlinear system has long been an important and interesting topic in nonlinear science both for physicists and mathematicians, and various methods for obtaining exact solutions of nonlinear system have been proposed, for example, the bilinear method, the standard Painlevé truncated expansion, the method of 'coalescence of eigenvalue' or 'wavenumbers', the homogenous balance method, and the mapping method, etc.[8−13] The mapping approach is a kind of classic, efficient and well-developed method to solve nonlinear evolution equations, whose remarkable characteristic is that we can have many different ansatzs and therefore, a large number of solutions.[14−24] In this paper, with a projective equation ($\phi' = \sigma\phi + \phi^2$) and a linear variable separation approach, a new family of exact solutions with arbitrary functions of the (2+1)-dimensional generalized Broek–Kaup system (GBK) is derived, based on the derived solutions, we find a few novel localized excitations, such as fusion and fission solitons.

The (2+1)-dimensional GBK system is

$$H_t - H_{xx} + 2HH_x + U_x + aU + bG = 0, \quad (1)$$

$$G_t + 2(GH)_x + G_{xx} + 4a(G_x - H_{xy}) + 4b(G_y - H_{yy}) + c(G - 2H_y) = 0, \quad (2)$$

$$U_y - G_x = 0, \quad (3)$$

where a, b, and c are arbitrary constants.

The GBK system is recently derived from a typical (1+1)-dimensional Broer–Kaup (BK) system by means of the Painlevé analysis.[25] Obviously, when $a = b = c = 0$, the GBK system will degenerate to the usual (2+1)-dimensional BK system,[26−28] which can be derived from inner parameter-dependent symmetry constraint of Kadomtsev–Petviashvili model.[3] Using some suitable dependent and independent variable transformations, Chen and Li[29] have proved that the (2+1)-dimensional BK system can be transformed to the (2+1)-dimensional dispersive long-water wave equation[30] and (2+1)-dimensional Ablowitz–Kaup–Newell–Segur system.[31] Actually, the (2+1)-dimensional BK system has been widely investigated by many researchers.[32−34] However, to the best of our knowledge, the GBK system has not been discussed quite well though Huang and Zhang[35] derived some special soliton-like solitons to a simplified GBK system ($b = 0$) by a variable-coefficient projective Riccati equation method, and Zhang and Chen[36] obtained certain semifolded structures via a Painlevé–Bäcklund transformation. In the following parts of

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this paper, we will derive some new soliton solutions to the GBK system by a projective equation and variable separation method. The basic idea of the algorithm is as follows. For a given nonlinear partial differential equation (NPDE) with the independent variables $x = (x_0 = t, x_1, x_2, \ldots, x_m)$, and the dependent variable u:

$$P(u, u_t, u_{x_i}, u_{x_i x_j}, \ldots) = 0, \tag{4}$$

where P is in general a polynomial function of its arguments, and the subscripts denote the partial derivatives, the solution can be assumed as

$$u = \sum_{i=0}^{n} \{A_i(x)\phi^i[q(x)]\} \tag{5}$$

with

$$\phi' = \sigma\phi + \phi^2, \tag{6}$$

where σ is a constant and the prime denotes the differentiation with respect to q. To determine u explicitly, one may substitute Eqs. (5) and (6) into the given NPDE and collect coefficients of polynomials of ϕ, then eliminate each coefficient to derive a set of partial differential equations of $A_i(x)$ and q, and solve the system of partial differential equations to obtain A_i and q. Finally, Eq. (6) possesses the general solution

$$\phi = \begin{cases} -\frac{1}{2}\sigma\left[1 + \tanh\left(\frac{1}{2}\sigma q\right)\right], & \sigma \neq 0, \\ -\frac{1}{2}\sigma\left[1 + \coth\left(\frac{1}{2}\sigma q\right)\right], & \sigma \neq 0, \\ -\frac{1}{2}\sigma\left[1 - \tan\left(\frac{1}{2}\sigma q\right)\right], & \sigma \neq 0, \\ -\frac{1}{2}\sigma\left[1 + \cot\left(\frac{1}{2}\sigma q\right)\right], & \sigma \neq 0, \\ \frac{-1}{q}, & \sigma = 0. \end{cases} \tag{7}$$

Substituting $A_i(x)$ q and Eq. (7) into Eq. (5), one can obtain the exact solutions of the given NPDE.

2. Exact solutions of the (2+1)-dimensional GBK system

Now, we apply the projective equation and the variable separation method for the (2+1)-dimensional GBK system. First, let us make a transformation for Eqs. (1), (2), and (3). Differentiating Eq. (1) with respect to variable y once and substituting Eq. (3) into (1) we obtain

$$(H_t - H_{xx} + 2HH_x)_y + G_{xx} + aG_x + bG_y = 0, \tag{8}$$

$$G_t + 2(GH)_x + G_{xx} + 4a(G_x - H_{xy}) + 4b(G_y - H_{yy}) + c(G - 2H_y) = 0. \tag{9}$$

Hereafter, we take the following Bäcklund transformations for Eqs. (8) and (9),[36]

$$H = \frac{\varphi_x}{\varphi} + H_0, \quad G = 2\frac{\varphi_{xy}}{\varphi} - 2\frac{\varphi_x\varphi_y}{\varphi^2}, \tag{10}$$

where $\varphi = \varphi(x, y, t)$ and $H_0 = H_0(x,\ t)$ are arbitrary functions of indicated arguments. Based on Eqs. (10) and (3), we have

$$G = 2H_y, \tag{11}$$

and

$$U = 2H_x + A(x, t), \tag{12}$$

where $A(x,\ t)$ is an arbitrary integral function of $(x,\ t)$. Substituting Eq. (11) for Eqs. (8) and (9) yields an identical NPDE

$$(H_t + H_{xx} + 2aH_x + 2bH_y + 2HH_x)_y = 0, \tag{13}$$

which is which is equivalent to

$$H_{yt} + H_{xxy} + 2aH_{xy} + 2bH_{yy} + 2(HH_x)_y = 0. \tag{14}$$

By the balancing procedure for Eq. (14), Eq. (5) becomes

$$H = f + g\phi(q), \tag{15}$$

where f, g, and q are functions of (x, y, t) to be determined. Substituting Eqs. (15) and (6) into Eq. (14) and collecting coefficients of polynomials of ϕ, then setting each coefficient to zero, we have

$$\begin{aligned} f &= -\frac{1}{2}\frac{q_x^2\sigma + q_{xx} + q_t + 2bq_y + 2aq_x}{q_x}, \\ g &= -\frac{1}{2}q_x \end{aligned} \tag{16}$$

with the function q in a linear variable separation form similar to Ref. [37],

$$q = \chi(x, t) + \varphi(y - 2bt), \tag{17}$$

where $\chi \equiv \chi(x, t),\ \varphi \equiv \varphi(y - 2bt)$ are two arbitrary variable separation functions of (x, t) and of $(y - 2bt)$, respectively. Based on the solutions of Eq. (6), one can obtain the exact solutions for the GBK system.

Case 1 For $\sigma \neq 0$, we can derive the following solitary wave solutions and periodic wave solutions of the GBK system:

$$H_1 = -\frac{1}{2}\frac{\chi_{xx}+\chi_t+\varphi_t+2a\chi_x+2b\varphi_y-\chi_x^2\sigma\tanh\left[\frac{1}{2}\sigma(\chi+\varphi)\right]}{\chi_x}, \tag{18}$$

$$G_1 = \frac{1}{2}\frac{-2\varphi_{yt}-4b\varphi_{yy}+\chi_x^2\varphi_y\sigma^2-\chi_x^2\varphi_y\sigma^2\tanh\left[\frac{1}{2}\sigma(\chi+\varphi)\right]^2}{\chi_x}, \tag{19}$$

$$U_1 = \frac{1}{2\chi_x^2}\left\{-2\chi_x\chi_{xxx}-2\chi_x\chi_{xt}+2\chi_x^2\sigma\tanh\left[\frac{1}{2}\sigma(\chi+\varphi)\right]\chi_{xx}+2\chi_{xx}^2 + \chi_x^4\sigma^2\mathrm{sech}\left[\frac{1}{2}\sigma(\chi+\varphi)\right]^2+2\chi_{xx}(\chi_t+\varphi_t+2b\varphi_y)\right\}+A(x,t), \tag{20}$$

$$H_2 = -\frac{1}{2}\frac{\chi_{xx}+\chi_t+\varphi_t+2a\chi_x+2b\varphi_y-\chi_x^2\sigma\coth\left[\frac{1}{2}\sigma(\chi+\varphi)\right]}{\chi_x}, \tag{21}$$

$$G_2 = \frac{1}{2}\frac{-2\varphi_{yt}-4b\varphi_{yy}+\chi_x^2\varphi_y\sigma^2-\chi_x^2\varphi_y\sigma^2\coth\left[\frac{1}{2}\sigma(\chi+\varphi)\right]^2}{\chi_x}, \tag{22}$$

$$U_2 = \frac{1}{2\chi_x^2}\left\{-2\chi_x\chi_{xxx}-2\chi_x\chi_{xt}+2\chi_x^2\sigma\coth\left[\frac{1}{2}\sigma(\chi+\varphi)\right]\chi_{xx}+2\chi_{xx}^2 - \chi_x^4\sigma^2\mathrm{csch}\left[\frac{1}{2}\sigma(\chi+\varphi)\right]^2+2\chi_{xx}(\chi_t+\varphi_t+2b\varphi_y)\right\}+A(x,t), \tag{23}$$

$$H_3 = -\frac{1}{2}\frac{\chi_{xx}+\chi_t+\varphi_t+2a\chi_x+2b\varphi_y+\chi_x^2\sigma\tan\left[\frac{1}{2}\sigma(\chi+\varphi)\right]}{\chi_x}, \tag{24}$$

$$G_3 = -\frac{1}{2}\frac{2\varphi_{yt}+4b\varphi_{yy}+\chi_x^2\varphi_y\sigma^2+\chi_x^2\varphi_y\sigma^2\tan\left[\frac{1}{2}\sigma(\chi+\varphi)\right]^2}{\chi_x}, \tag{25}$$

$$U_3 = -\frac{1}{2\chi_x^2}\left\{2\chi_x\chi_{xxx}+2\chi_x\chi_{xt}+2\chi_x^2\sigma\tan\left[\frac{1}{2}\sigma(\chi+\varphi)\right]\chi_{xx}-2\chi_{xx}^2 + \chi_x^4\sigma^2\mathrm{sec}[\frac{1}{2}\sigma(\chi+\varphi)]^2-2\chi_{xx}(\chi_t+\varphi_t+2b\varphi_y)\right\}+A(x,t). \tag{26}$$

$$H_4 = -\frac{1}{2}\frac{\chi_{xx}+\chi_t+\varphi_t+2a\chi_x+2b\varphi_y-\chi_x^2\sigma\cot\left[\frac{1}{2}\sigma(\chi+\varphi)\right]}{\chi_x}, \tag{27}$$

$$G_4 = -\frac{1}{2}\frac{2\varphi_{yt}+4b\varphi_{yy}+\chi_x^2\varphi_y\sigma^2+\chi_x^2\varphi_y\sigma^2\cot\left[\frac{1}{2}\sigma(\chi+\varphi)\right]^2}{\chi_x}, \tag{28}$$

$$U_4 = -\frac{1}{2\chi_x^2}\left\{2\chi_x\chi_{xxx}+2\chi_x\chi_{xt}-2\chi_x^2\sigma\cot\left[\frac{1}{2}\sigma(\chi+\varphi)\right]\chi_{xx}-2\chi_{xx}^2 + \chi_x^4\sigma^2\mathrm{csc}\left[\frac{1}{2}\sigma(\chi+\varphi)\right]^2-2\chi_{xx}(\chi_t+\varphi_t+2b\varphi_y)\right\}+A(x,t) \tag{29}$$

with three arbitrary functions $\chi(x,t)$, $\varphi(y-2bt)$ and $A(x,t)$.

Case 2 For $\sigma=0$, we can derive the following variable separation solution of the GBK system:

$$H_5 = -\frac{1}{2}\frac{\chi_{xx}+\chi_t+\varphi_t+2a\chi_x+2b\varphi_y}{\chi_x}+\frac{\chi_x}{\chi+\varphi}, \tag{30}$$

$$G_5 = -\frac{\varphi_{yt}+2b\varphi_{yy}}{\chi_x}-\frac{2\chi_x\varphi_y}{(\chi+\varphi)^2}, \tag{31}$$

$$U_5 = \frac{1}{\chi_x^2}(\chi_{xx}(1+\chi_t+\varphi_t+2b\varphi_y)-\chi_x(\chi_{xt}+\chi_{xxx}))+\frac{2\chi_{xx}}{\chi+\varphi}-\frac{2\chi_x^2}{(\chi+\varphi)^2}+A(x,t) \tag{32}$$

with three arbitrary functions $\chi(x,t)$, $\varphi(y-2bt)$ and $A(x,t)$.

3. Fusion and fission solitons in the GBK system

In this section, we mainly discuss the interactions between two solitons in the (2+1)-dimensional GBK system. For simplification, we only discuss the field H_2 of Eq. (21), namely

$$H = H_2 = -\frac{1}{2}\frac{\chi_{xx}+\chi_t+\varphi_t+2a\chi_x+2b\varphi_y-\chi_x^2\sigma\coth\left[\frac{1}{2}\sigma(\chi+\varphi)\right]}{\chi_x}. \tag{33}$$

The interactions between solitons may be regarded as completely elastic. For instaince, when $\chi(x,t)$ and $\varphi(y-2bt)$ are considered to be

$$\chi = 2\mathrm{sech}(x+t)+\mathrm{sech}(x-t), \quad \varphi = \mathrm{sech}(y-2bt), \tag{34}$$

and $a=1$, $b=1/2$ in Eq. (33), we can obtain the interactions between two dromion solitons. Figure 1 shows an evolutional profile corresponding to the physical quantity H expressed by Eq. (33). From Fig. 1 and through detailed analysis, we find that the shapes, amplitudes and velocities of the two dromion solitons are completely conserved after their interactions.

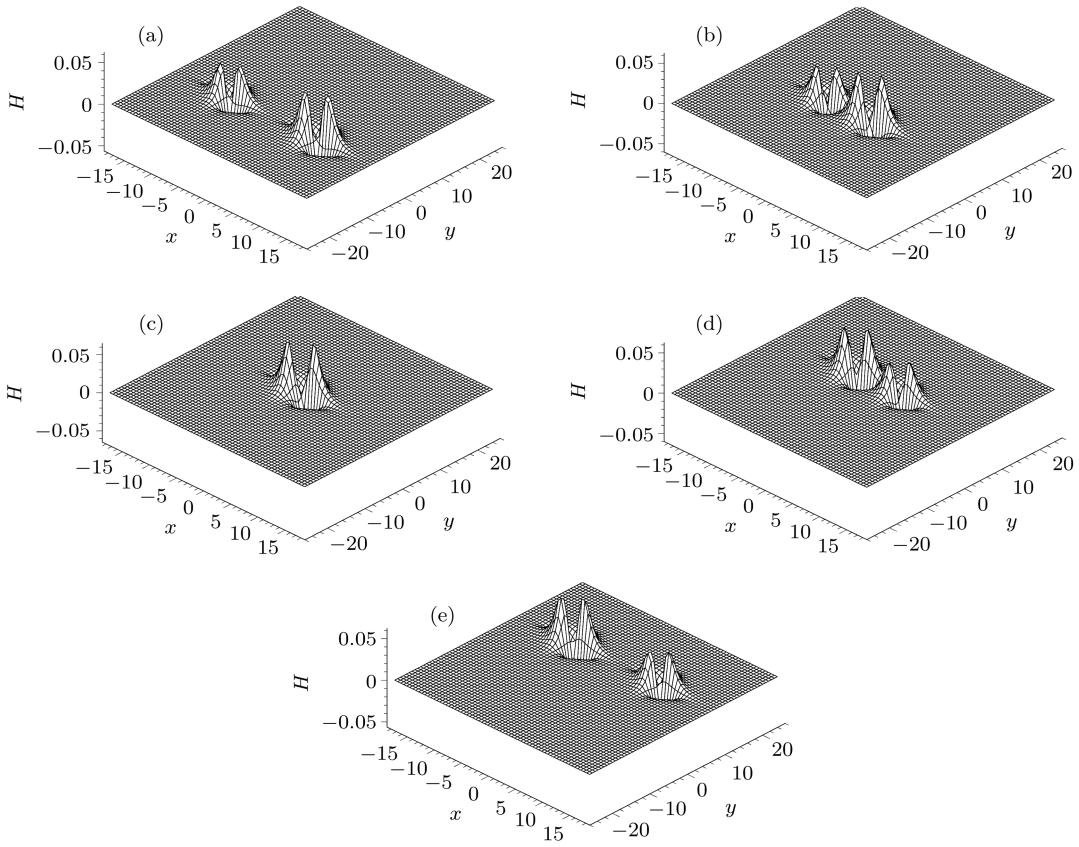


Fig. 1. The evolutional profile of two solitonic structures for the solution of H Eq. (33) with the condition Eq. (34) at different times: (a) $t=-8$; (b) $t=-4$; (c) $t=0$; (d) $t=4$; and (e) $t=8$.

Now we focus our attention on the following intriguing fusion and fission phenomena for the solitory wave solutions H. For instance, when we select the $\chi(x,t)$ and $\varphi(y-2bt)$ to be

$$\chi = 1+0.8\exp(x+t)+0.6\mathrm{sech}(x-t), \quad \varphi = 1+0.5\exp(y-2bt), \tag{35}$$

and substitute Eq. (35) with $a = 1$, $b = 1/2$ into Eq. (33), we can obtain the fusion phenomena of two dromion solitons. From Fig. 2, we can clearly see that the two single dromion solitons fuse to one soliton finally. Along with this line, considering $\chi(x,t)$ and $\varphi(y-2bt)$ to be

$$\chi = 0.5 + 0.1\exp(x-t) + 0.2\mathrm{sech}(x+t), \quad \varphi = 0.5 + 0.6\mathrm{sech}(y-2bt), \tag{36}$$

and $a = 1$, $b = 1/2$ for the solution of H Eq. (33), we can obtain another new type of solitary excitation with apparently different properties presented in Fig. 3. From Fig. 3, we can find that one single dromion soliton fissions into two solitons.

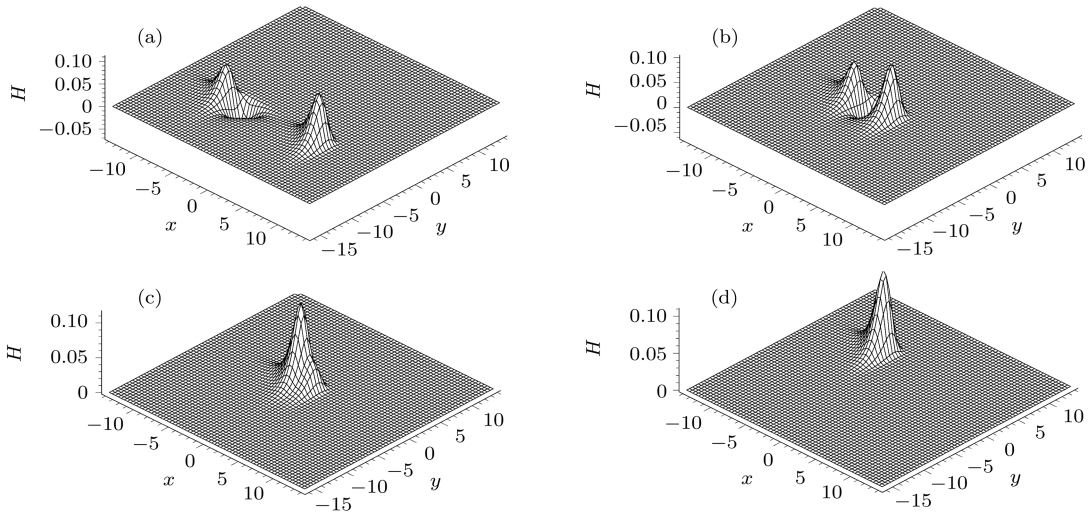


Fig. 2. The evolutional plot of two dromion solitons into one soliton for the solution of H Eq. (33) under the condition Eq. (35) at different times: (a) $t = -6$; (b) $t = -2$; (c) $t = 0$; and (d) $t = 5$.

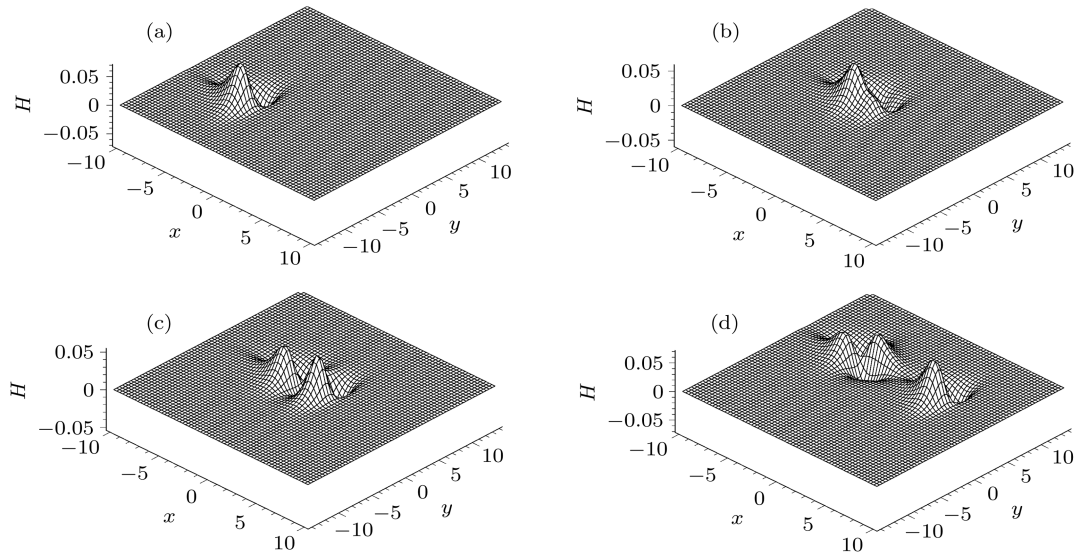


Fig. 3. One single dromion soliton fission into two solitons evolutional plot of the solution of H Eq. (33) under the condition Eq. (36) at different times: (a) $t = -4$; (b) $t = 0$; (c) $t = 1$; and (d) $t = 4$.

4. Summary and discussion

In summary, with the help of a projective equation and a linear variable separation method, the (2+1)-dimensional generalized BK system is derived. Based on the derived solitary wave solution of H Eq. (33), we find some new localized excitations—the fusion and fission soliton solutions of GBK system. Usually, the interaction between soliton solutions of integrable models is completely elastic, that

is to say, the amplitude, velocity and wave shape of a soliton are not changed after their interaction. However, for some specific cases, the soliton interactions may be fusion and fission. From the above analysis, we can see that these intriguing phenomena may occur in a higher-dimensional soliton system if we choose appropriate initial conditions and/or boundary conditions for the given system, which are similar to those in some (1+1)-dimensional cases studied by several authors.[38−40] In this paper, although we have given out some soliton fusion and fission phenomena in the (2+1)-dimensional case, it is obvious that there are still many significant and interesting problems to be further discussed. As Wang *et al.*[39] have pointed out in (1+1)-dimensional cases, these problems are as follows. What is the necessary and sufficient condition for soliton fusion and fission? What is for soliton elastic and nonelastic interactions? What is the general equation for the distribution of the energy and momentum after soliton fusion and soliton fission? How can we use the soliton fusion and soliton fission of integrable models to investigate the observed soliton fusion and soliton fission in the experiments? These are all the pending issues to be further studied.

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