Conformal invariance and conserved quantities of Appell systems under second-class Mei symmetry*

Luo Yi-Ping(罗一平)† and Fu Jing-Li(傅景礼)

Department of Physics, Zhejiang Sci-Tech University, Hangzhou 310018, *China*

(Received 2 March 2010; revised manuscript received 23 March 2010)

In this paper we introduce the new concept of the conformal invariance and the conserved quantities for Appell systems under second-class Mei symmetry. The one-parameter infinitesimal transformation group and infinitesimal transformation vector of generator are described in detail. The conformal factor in the determining equations under second-class Mei symmetry is found. The relationship between Appell system's conformal invariance and Mei symmetry are discussed. And Appell system's conformal invariance under second-class Mei symmetry may lead to corresponding Hojman conserved quantities when the conformal invariance satisfies some conditions. Lastly, an example is provided to illustrate the application of the result.

Keywords: second-class Mei symmetry, conformal invariance, conserved quantity, Appell system
PACC: 0320

1. Introduction

The symmetry is a core concept in modern physics, it makes a general reference to the standard symmetry and overall symmetry. In theoretical physics, conformal invariance is a symmetry under dilatation (scale invariance) and the special conformal transformations. It was introduced into physics by Bateman and Cunningham shortly after Einstein's first paper on special relativity.[1,2] In 1997, Galiullin studied the conformal invariance of Birkhoff equations and their relation with the Lie symmetry.[3] In 2001, Robert investigated the conformal Hamiltonian systems by geometric methods, in which the author discussed the geometric structure of conformal systems with invariance and pointed out the difference of conformal dynamics from arbitrary symmetric systems.[4] Recently, the conformal invariance developed rapidly in dynamical mechanical systems, Cai and Liu *et al.* studied the conformal invariance and conserved quantities of mechanical systems.[5−15] Since Mei presented a new symmetry—the form invariance, more and more attention has been paid to research on Mei symmetry and conserved quantity, which was quickly extended to all types of mechanical systems.[16−30]

However, until now the research on the conformal invariance and conserved quantities of Appell systems under second-class Mei symmetry has not been reported yet. In the present paper, the new concept of conformal invariance and conserved quantities of Appell systems under second-class Mei symmetry are presented firstly. In fact, the aim for studying the conformal invariance and conserved quantities of Appell systems under second-class Mei symmetry is to seek the one-parameter group of infinitesimal transformation of equations. By infinitesimal transformation, we can get the conformal factor of conformal invariance and the conserved quantities of Appell systems under second-class Mei symmetry. Therefore, the study has important significance not only geometrically but also mechanically.

2. Second-class Mei symmetry of Appell systems

Suppose that a mechanical system consists of N particles with masses m_i and its position vectors $\boldsymbol{r}_i$, and let the position of the system be determined by n generalised coordinates $q_s(s = 1, \dots, n)$, the energy of acceleration for the system is expressed as

$$S = S(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}) = \frac{1}{2} m_i \ddot{\boldsymbol{r}}_i^2, \tag{1}$$

and the non-potential generalised forces is

$$Q_s = Q_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}). \tag{2}$$

*Project supported by the National Natural Science Foundation of China (Grant Nos. 10672143 and 60575055).
†Corresponding author. E-mail: zjstulyp@126.com
© 2010 Chinese Physical Society and IOP Publishing Ltd

The Appell equations are written in the form

$$\frac{\partial S}{\partial \ddot{q}_s} = Q_s, \quad (s = 1, \dots, n). \tag{3}$$

Introducing a function $\tilde{L}$ as

$$\tilde{L} = S - \sum_{s=1}^{n} \ddot{q}_s Q_s, \tag{4}$$

Eq. (3) becomes

$$\frac{\partial \tilde{L}}{\partial \ddot{q}_s} = 0, \quad (s = 1, \dots, n). \tag{5}$$

Equation (5) is also called the dynamic equation of the Appell system. The function $\tilde{L}$ and Lagrange function L have the following relation:

$$\frac{\partial \tilde{L}}{\partial \ddot{q}_s} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s}. \tag{6}$$

From Eq. (6) we can get

$$F_s \equiv A_{sk}(t, \boldsymbol{q})\ddot{q}_k + B_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) - Q_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \tag{7}$$

where

$$A_{sk} = \frac{\partial^2 L}{\partial \dot{q}_s \partial \dot{q}_k}, \quad B_s = \frac{\partial^2 L}{\partial \dot{q}_s \partial q_k}\dot{q}_k + \frac{\partial^2 L}{\partial \dot{q}_s \partial t} - \frac{\partial L}{\partial q_s}. \tag{8}$$

Supposing the system to be non-singular, we can solve all generalised accelerations as

$$\ddot{q}_s = \alpha_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}). \tag{9}$$

Let us introduce the infinitesimal transformations of time and generalised coordinates as

$$t^* = t + \Delta t, \quad q_s^*(t^*) = q_s(t) + \Delta q_s. \tag{10}$$

Their expandsion forms are

$$\begin{aligned} t^* &= t + \varepsilon\xi_0(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \\ q_s^*(t^*) &= q_s(t) + \varepsilon\xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \end{aligned} \tag{11}$$

where ε is the small parameter and ξ_0, ξ_s are the generators of the one-parameter infinitesimal transformation group. From Eq. (11), we have

$$\begin{aligned} \frac{\mathrm{d}q_s^*}{\mathrm{d}t^*} &= \frac{\mathrm{d}q_s + \varepsilon \mathrm{d}\xi_s}{\mathrm{d}t + \varepsilon \mathrm{d}\xi_0} \\ &= \dot{q}_s + \varepsilon\left(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0\right) + O\left(\varepsilon^2\right), \end{aligned} \tag{12}$$

and

$$\frac{\mathrm{d}^2 q_s^*}{\mathrm{d}t^{*2}} = \ddot{q}_s + \varepsilon\left[\left(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0\right)^{\bullet} - \ddot{q}_s\dot{\xi}_0\right] + O\left(\varepsilon^2\right). \tag{13}$$

Let function $\tilde{L}$ be $\tilde{L}^*$ under the infinitesimal transformations Eq. (11). Taking the Taylor expansion of $\tilde{L}^*$ with respect to $(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})$, one can easily obtain

$$\begin{aligned} \tilde{L}^* &= \tilde{L}\left(t^*, \boldsymbol{q}^*, \frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}, \frac{\mathrm{d}^2\boldsymbol{q}^*}{\mathrm{d}t^{*2}}\right) \\ &= \tilde{L}\left(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}\right) + \varepsilon\left\{\frac{\partial \tilde{L}}{\partial t}\xi_0 + \frac{\partial \tilde{L}}{\partial q_s}\xi_s + \frac{\partial \tilde{L}}{\partial \dot{q}_s}\left(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0\right) + \frac{\partial \tilde{L}}{\partial \ddot{q}_s}\left[\left(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0\right)^{\bullet} - \ddot{q}_s\dot{\xi}_0\right]\right\} + O\left(\varepsilon^2\right), \end{aligned} \tag{14}$$

and

$$\frac{\partial \tilde{L}^*}{\partial \ddot{q}_k} = \varepsilon\frac{\partial}{\partial \ddot{q}_k}\left[\frac{\partial \tilde{L}}{\partial t}\xi_0 + \frac{\partial \tilde{L}}{\partial q_s}\xi_s + \frac{\partial \tilde{L}}{\partial \dot{q}_s}\left(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0\right)\right] + O(\varepsilon^2) = \frac{\partial \tilde{L}}{\partial \ddot{q}_k} + O(\varepsilon^2) = O\left(\varepsilon^2\right). \tag{15}$$

Definition 1 If the form of Appell equations (5) keeps an invariance under the infinitesimal transformations (11), then the invariance is called the second-class Mei symmetry of Appell equations.

Taking the infinitesimal generators as

$$\tilde{X}^{(0)} = \xi_0\frac{\partial}{\partial t} + \xi_s\frac{\partial}{\partial q_s}, \tag{16}$$

its first expended vector

$$\tilde{X}^{(1)} = \tilde{X}^{(0)} + \left(\frac{\bar{\mathrm{d}}\xi_s}{\mathrm{d}t} - \dot{q}_s\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right)\frac{\partial}{\partial \dot{q}_s}, \tag{17}$$

and its second expended vector as following

$$\tilde{X}^{(2)} = \tilde{X}^{(1)} + \left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}\xi_s}{\mathrm{d}t} - \dot{q}_s\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right) - \ddot{q}_s\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right]\frac{\partial}{\partial \ddot{q}_s}, \tag{18}$$

where

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{q}_s\frac{\partial}{\partial q_s} + \alpha_s\frac{\partial}{\partial \dot{q}_s} + \dot{\alpha}_s\frac{\partial}{\partial \ddot{q}_s}. \tag{19}$$

Thus Eq. (14) is expressed as

$$\tilde{L}^* = \tilde{L}\left(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}\right) + \varepsilon\tilde{X}^{(2)}(\tilde{L}) + O\left(\varepsilon^2\right). \tag{20}$$

Substituting Eq. (14) into (15) and neglecting terms that are $O\left(\varepsilon^2\right)$or higher, we obtain

$$\frac{\partial}{\partial \ddot{q}_s}\tilde{X}^{(2)}\left(\tilde{L}\right)=\tilde{X}^{(1)}\left(Q_s\right). \tag{21}$$

Equation (21) is called the criterion equations of second-class Mei symmetry for Appell systems. Then, we have the following criterion.

Criterion 1 For Appell equations (5), if the infinitesimal generators ξ_0, ξ_s satisfy the Eq. (21), then invariance of Eq. (5) under the infinitesimal transformations Eqs. (11) is called the second-class Mei symmetry.

3. Conformal invariance of second-class Mei symmetry for Appell systems

Definition 2 In infinitesimal transformations of infinitesimal generators $\xi_0(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ and $\xi_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})$, if F_s satisfies the following expressions

$$\tilde{X}^{(2)}(F_s)=U_s^l(F_l), \tag{22}$$

then the infinitesimal transformations have conformal invariance of the corresponding Appell system (5), where the expressions of Eq. (22) is called the determinate equation of Eq. (7), and U_s^l is called the conformal factor.

In order to get the general expression of the conformal factor of conformal invariance, we calculate the difference

$$\tilde{X}^{(2)}(F_s)-\tilde{X}^{(2)}\left(F_s\right)|_{F_s=0}. \tag{23}$$

As

$$\dot{\xi}_s=\frac{\partial \xi_s}{\partial t}+\frac{\partial \xi_s}{\partial q_k}\dot{q}_k, \tag{24}$$

$$\ddot{\xi}_s=\frac{\partial^2 \xi_s}{\partial t^2}+2\frac{\partial^2 \xi_s}{\partial q_k \partial t}\dot{q}_k+\frac{\partial^2 \xi_s}{\partial q_k \partial q_j}\dot{q}_k\dot{q}_j+\frac{\partial \xi_s}{\partial q_k}\ddot{q}_k,\quad (s=0,1,\ldots,n;\quad j,k=1,\ldots n), \tag{25}$$

therefore

$$\begin{aligned}
\tilde{X}^{(2)}(F_s) &= A_{sk}(\ddot{\xi}_k-2\,\ddot{q}_k\dot{\xi}_0-\dot{q}_k\ddot{\xi}_0)+\tilde{X}^{(1)}[(B_s-Q_s)+A_{sk}\ddot{q}_k]\\
&= A_{sk}(\ddot{\xi}_k-2\,\ddot{q}_k\dot{\xi}_0-\dot{q}_k\ddot{\xi}_0)+\tilde{X}^{(0)}[A_{sk}\ddot{q}_k+(B_s-Q_s)]+(\dot{\xi}_k-\dot{q}_k\dot{\xi}_0)\frac{\partial\left(B_s-Q_s\right)}{\partial \dot{q}_k}\\
&= A_{sk}\left(\frac{\partial^2 \xi_k}{\partial t^2}+2\frac{\partial^2 \xi_k}{\partial q_r \partial t}\,\dot{q}_r+\frac{\partial^2 \xi_k}{\partial q_r \partial q_j}\,\dot{q}_r\dot{q}_j+\frac{\partial \xi_k}{\partial q_r}\ddot{q}_r\right)-2A_{sk}\ddot{q}_k\left(\frac{\partial \xi_0}{\partial t}+\frac{\partial \xi_0}{\partial q_r}\dot{q}_r\right)\\
&\quad +A_{sk}\dot{q}_k\left(\frac{\partial^2 \xi_0}{\partial t^2}+2\frac{\partial^2 \xi_0}{\partial q_r \partial t}\,\dot{q}_r+\frac{\partial^2 \xi_0}{\partial q_r \partial q_j}\,\dot{q}_r\dot{q}_j+\frac{\partial \xi_0}{\partial q_r}\ddot{q}_r\right)+\tilde{X}^{(0)}[A_{sk}\ddot{q}_k+(B_s-Q_s)]\\
&\quad +\left[\frac{\partial \xi_k}{\partial t}+\frac{\partial \xi_k}{\partial q_r}\dot{q}_r-\dot{q}_r\left(\frac{\partial \xi_0}{\partial t}+\frac{\partial \xi_0}{\partial q_r}\dot{q}_r\right)\right]\frac{\partial\left(B_s-Q_s\right)}{\partial \dot{q}_k}\\
&= A_{sk}\left(\frac{\partial \xi_k}{\partial q_r}-\dot{q}_k\frac{\partial \xi_0}{\partial q_r}\right)\ddot{q}_r-2A_{sk}\ddot{q}_k\left(\frac{\partial \xi_0}{\partial t}+\frac{\partial \xi_0}{\partial q_r}\dot{q}_r\right)+\tilde{X}^{(0)}A_{sk}\ddot{q}_k+C(t,\boldsymbol{q},\dot{\boldsymbol{q}}),
\end{aligned} \tag{26}$$

where $C(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ is the sum of items in which $\ddot{q}_s$ are not contained. Similarly there is

$$\tilde{X}^{(2)}(F_s\,)|_{F_s=0}=A_{sk}\left(\frac{\partial \xi_k}{\partial q_r}-\dot{q}_k\frac{\partial \xi_0}{\partial q_r}\right)\alpha_r-2A_{sk}\alpha_k\left(\frac{\partial \xi_0}{\partial t}+\frac{\partial \xi_0}{\partial q_r}\dot{q}_r\right)+\tilde{X}^{(0)}A_{sk}\alpha_k+C(t,\boldsymbol{q},\dot{\boldsymbol{q}}), \tag{27}$$

where $\alpha_k=-A^{sk}B_k$. From Eqs. (26) and (27), we can get

$$\begin{aligned}
&\tilde{X}^{(2)}(F_s)-\tilde{X}^{(2)}\left(F_s\right)|_{F_s=0}\\
&= A_{sk}\left(\frac{\partial \xi_k}{\partial q_r}-\dot{q}_k\frac{\partial \xi_0}{\partial q_r}\right)(\ddot{q}_r-\alpha_r)-2A_{sk}(\ddot{q}_k-\alpha_k)\left(\frac{\partial \xi_0}{\partial t}+\frac{\partial \xi_0}{\partial q_r}\dot{q}_r\right)+\tilde{X}^{(0)}A_{sk}(\ddot{q}_k-\alpha_k).
\end{aligned} \tag{28}$$

On account of

$$\ddot{q}_k-\alpha_k=\ddot{q}_k+A^{kl}B_l=A^{kl}(A_{lm}\ddot{q}_m+B_l)=A^{kl}F_l, \tag{29}$$

$$\begin{aligned}\tilde{X}^{(2)}(F_s)-\tilde{X}^{(2)}\ (F_s)|_{F_s=0} &= A_{sk}\left(\frac{\partial\xi_k}{\partial q_r}-\dot{q}_k\frac{\partial\xi_0}{\partial q_r}\right)A^{rl}F_l-2A_{sk}A^{kl}\left(\frac{\partial\xi_0}{\partial t}+\frac{\partial\xi_0}{\partial q_r}\dot{q}_r\right)F_l+\tilde{X}^{(0)}A_{sk}A^{kl}F_l\\ &= \left[A_{sk}\left(\frac{\partial\xi_k}{\partial q_r}-\dot{q}_k\frac{\partial\xi_0}{\partial q_r}\right)A^{rl}-2\delta_s^l\left(\frac{\partial\xi_0}{\partial t}+\frac{\partial\xi_0}{\partial q_r}\dot{q}_r\right)+\tilde{X}^{(0)}A_{sk}A^{kl}\right]F_l\\ &= U_s^lF_l. \end{aligned} \tag{30}$$

Furthermore, we can find the conformal factor that the conformal invariance would be second-class Mei symmetry for Appell systems as

$$U_s^l=A_{sk}\left(\frac{\partial\xi_k}{\partial q_r}-\dot{q}_k\frac{\partial\xi_0}{\partial q_r}\right)A^{rl}-2\delta_s^l\left(\frac{\partial\xi_0}{\partial t}+\frac{\partial\xi_0}{\partial q_r}q_r\right)+\tilde{X}^{(0)}A_{sk}A^{kl}. \tag{31}$$

Deduction Suppose the differential equation (7) can be transformed into the standard forms

$$F_s=\ddot{q}_s-\alpha_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})=0 \tag{32}$$

by the action of infinitesimal transformations of one-parameter group, the conformal factor under second-class Mei symmetry of Eq. (32) is

$$U_s^l=\frac{\partial\xi_s}{\partial q_l}-2\dot{\xi}_0-\dot{q}_s\frac{\partial\xi_0}{\partial q_l}. \tag{33}$$

4. Conserved quantities of conformal invariance for second-class Mei symmetry

The conformal invariance of Appell systems under second-class Mei symmetry would also lead to conserved quantities. Let us study Hojmal conserved quantities. In virtue of the special Lie symmetry for Appell systems, a method of finding out the Hojmal conserved quantity is given by the following proposition.

Proposition For the Appell systems given by Eq. (5), if the infinitesimal generators ξ_s corresponding to a Lie symmetry of the system satisfy

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right)=\frac{\partial\alpha_s}{\partial q_k}\xi_k+\frac{\partial\alpha_s}{\partial\dot{q}_k}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k\right), \tag{34}$$

and there exists a function $\mu=\mu(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ satisfying the following condition

$$\frac{\partial\alpha_s}{\partial\dot{q}_s}+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu=0, \tag{35}$$

then the Lie symmetry of Appell systems under second-class Mei symmetry can directly lead to a Hojman conserved quantity as follows:

$$I=\frac{1}{\mu}\frac{\partial}{\partial q_s}(\mu\xi_s)+\frac{1}{\mu}\frac{\partial}{\partial\dot{q}_s}\left(\mu\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right)=\text{const.} \tag{36}$$

Proof

Since

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}I=\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial q_s}\xi_s\right)+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\partial\xi_s}{\partial q_s}\right)+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left[\frac{1}{\mu}\frac{\partial\mu}{\partial\dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s+\frac{\partial}{\partial\dot{q}_s}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right)\right], \tag{37}$$

we can easily verify

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\partial\xi_s}{\partial q_s}\right)=\frac{\partial}{\partial q_s}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right)-\frac{\partial\alpha_k}{\partial q_s}\frac{\partial\xi_s}{\partial\dot{q}_k}, \tag{38}$$

$$\frac{\partial}{\partial\dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right)=\frac{\partial}{\partial\dot{q}_s}\left(\frac{\partial\alpha_s}{\partial q_k}\xi_k\right)+\frac{\partial}{\partial\dot{q}_s}\left(\frac{\partial\alpha_s}{\partial\dot{q}_k}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k\right), \tag{39}$$

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial}{\partial\dot{q}_s}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right)=\frac{\partial}{\partial\dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right)-\frac{\partial}{\partial q_s}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right)-\frac{\partial\alpha_k}{\partial\dot{q}_s}\frac{\partial}{\partial\dot{q}_k}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right). \tag{40}$$

Substituting Eqs. (38) and (39) into Eq. (37), we have

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}I=\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial q_s}\xi_s\right)+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial\dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right)+\frac{\partial^2\alpha_s}{\partial q_k\partial\dot{q}_s}\xi_k+\frac{\partial^2\alpha_s}{\partial\dot{q}_k\partial\dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k. \tag{41}$$

Utilising the condition (35), we have

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial q_s}\xi_k\right)=-\frac{\partial^2\alpha_s}{\partial q_k\partial\dot{q}_s}\xi_k+\frac{1}{\mu}\frac{\partial\mu}{\partial q_s}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right)-\frac{1}{\mu}\frac{\partial\mu}{\partial\dot{q}_k}\frac{\partial\alpha_k}{\partial q_s}\xi_s, \tag{42}$$

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left[\frac{1}{\mu}\frac{\partial\mu}{\partial\dot{q}_s}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right)\right]=-\frac{\partial^2\alpha_s}{\partial\dot{q}_k\partial\dot{q}_s}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k\right)-\frac{1}{\mu}\frac{\partial\alpha_k\partial\mu}{\partial\dot{q}_s\partial\dot{q}_k}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right)+\frac{1}{\mu}\frac{\partial\mu}{\partial\dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right)-\frac{1}{\mu}\frac{\partial\mu}{\partial q_s}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right). \tag{43}$$

Substituting Eqs. (42) and (43) into Eq. (37), and using Eq. (34), we obtain

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}I=\frac{1}{\mu}\frac{\partial\mu}{\partial\dot{q}_s}\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right)-\frac{\partial\alpha_s}{\partial\dot{q}_k}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k\right)-\frac{\partial\alpha_s}{\partial q_k}\xi_k\right]=0. \tag{44}$$

The proof of the proposition is accomplished.

5. Illustration example

As an application of the conformal invariance of Appell systems under second-Mei symmtery, an illustration example is introduced.

A system has the energy of acceleration

$$S=\frac{1}{2}\left(\ddot{q}_1^2+\ddot{q}_2^2\right)+\left(2+\frac{4}{t}\dot{q}_1\right)\ddot{q}_1-q_1, \tag{45}$$

and the non-potential generalised forces

$$Q_1=1+\frac{2}{t}\dot{q}_1,\quad Q_2=-1-\frac{2}{t}\dot{q}_1. \tag{46}$$

Let us study its conformal invariance and conserved quantities under second-Mei symmetry.

The differential equations of motion of the system can be expressed as

$$\ddot{q}_1=\alpha_1=-1-\frac{2}{t}\dot{q}_1,\quad \ddot{q}_2=\alpha_2=-1-\frac{2}{t}\dot{q}_1. \tag{47}$$

Then, we have

$$F=\begin{bmatrix}F_1=\ddot{q}_1+\dfrac{2}{t}\dot{q}_1\\ F_2=\ddot{q}_2+\dfrac{2}{t}\dot{q}_1\end{bmatrix}. \tag{48}$$

From determining Eqs. (34) of Lie symmetry under the infinitesimal transformations, we have

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_1=\frac{\partial\alpha_1}{\partial\dot{q}_1}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_1+\frac{\partial\alpha_1}{\partial\dot{q}_2}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_2, \tag{49}$$

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_2=\frac{\partial\alpha_2}{\partial\dot{q}_1}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_1+\frac{\partial\alpha_2}{\partial\dot{q}_2}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_2. \tag{50}$$

Taking infinitesimal transformation generators

$$\xi_1=\xi_2=\frac{1}{2}\left(\dot{q}_1t+q_1+\frac{1}{2}t^2\right)^2, \tag{51}$$

then we know that the Appell equation of system is Lie symmetric.

On account of

$$\begin{aligned}\tilde{X}^{(2)}(F)&=\left[\frac{\bar{\mathrm{d}}\xi_1}{\mathrm{d}t}\frac{\partial}{\partial\dot{q}_1}+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}\xi_1}{\mathrm{d}t}\right)\frac{\partial}{\partial\ddot{q}_1}+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}\xi_2}{\mathrm{d}t}\right)\frac{\partial}{\partial\ddot{q}_2}\right]\begin{bmatrix}F_1=\ddot{q}_1+\dfrac{2}{t}\dot{q}_1\\ F_2=\ddot{q}_1+\dfrac{2}{t}\dot{q}_1\end{bmatrix}\\&=\begin{pmatrix}\dot{q}_1+q_1+\dfrac{1}{2}t^2 & 0\\ \dot{q}_1+q_1+\dfrac{1}{2}t^2 & 0\end{pmatrix}\begin{bmatrix}F_1=\ddot{q}_1+\dfrac{2}{t}\dot{q}_1\\ F_2=\ddot{q}_1+\dfrac{2}{t}\dot{q}_1\end{bmatrix},\end{aligned} \tag{52}$$

we get conformal factor

$$U_s^l=\begin{pmatrix}\dot{q}_1+q_1+\dfrac{1}{2}t^2 & 0\\ \dot{q}_1+q_1+\dfrac{1}{2}t^2 & 0\end{pmatrix}. \tag{53}$$

Using Eq. (31) we can get

$$\begin{aligned}U_s^l&=A_{sk}\left(\frac{\partial\xi_k}{\partial q_r}-\dot{q}_k\frac{\partial\xi_0}{\partial q_r}\right)A^{rl}-2\delta_s^l\left(\frac{\partial\xi_0}{\partial t}+\frac{\partial\xi_0}{\partial q_r}q_r\right)+\tilde{X}^{(0)}A_{sk}A^{kl}\\&=\begin{pmatrix}\dot{q}_1+q_1+\dfrac{1}{2}t^2 & 0\\ \dot{q}_1+q_1+\dfrac{1}{2}t^2 & 0\end{pmatrix}.\end{aligned} \tag{54}$$

From Eq. (35), we have

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu=\frac{2}{t}, \tag{55}$$

and its solution is

$$\mu=t^2. \tag{56}$$

Using the proposition, the conserved quantity expression (36) yields

$$I=\dot{q}_1t+q_1+\frac{1}{2}t^2=\text{const.} \tag{57}$$

6. Conclusion

The Appell systems exhibit conformal invariance and conserved quantities under second-class Mei symmetry. For an Appell equation, if it possesses the characteristics of conformal invariance of second-class Mei symmetry, then it possesses second-class Mei symmetry. By introducing the concept of the conformal invariance for Appell systems, we can find the conformal factor in the determining equations and the conserved quantities as shown in Eqs. (31) or (33) and (36). The conformal invariance may lead to corresponding Hojman conserved quantities when the conformal invariance satisfies some conditions. The results can be applied to other fields, and lay the foundation for the corresponding study of holonomic system.

References

[1] Bateman H 1909 *Proceedings of the London Mathematical Society* **8** 223
[2] Cunningham E 1909 *Proceedings of the London Mathematical Society* **8** 77
[3] Galiullin A S, Gafarov G G, Malaishka R P and Khwan A M 1997 *Analytical Dynamics of Helmholtz,Birkhoff and Nambu Systems* (Moscow: UFN) (in Russian)
[4] Robert M L and Matthew P 2001 *J. Geom. Phys.* **39** 276
[5] Cai J L and Mei F X 2008 *Acta Phys. Sin.* **57** 5369 (in Chinese)
[6] Cai J L 2009 *Acta Phys. Pol.* A **115** 854
[7] Cai J L, Luo S K and Mei F X 2008 *Chin. Phys.* B **17** 3170
[8] Liu C, Liu S X, Mei F X and Guo Y X 2008 *Acta Phys. Sin.* **57** 6709 (in Chinese)
[9] Liu C, Mei F X and Guo Y X 2008 *Acta Phys. Sin.* **57** 6704 (in Chinese)
[10] Liu C, Zhu N, Mei F X and Guo Y X 2008 *Commun. Theor. Phys.* **50** 1065
[11] Chen X W, Liu C and Mei F X 2008 *Chin. Phys.* B **17** 3180
[12] He G and Mei F X 2008 *Chin. Phys.* B **17** 2764
[13] Cai J L 2008 *Chin. Phys. Lett.* **25** 1523
[14] Cai J L 2009 *Acta Phys. Sin.* **58** 22 (in Chinese)
[15] Liu C, Mei F X and Guo Y X 2009 *Chin. Phys.* B **18** 395
[16] Pang T, Fang J H, Zhang M J, Lin P and Lu K 2009 *Chin. Phys.* B **18** 3150
[17] Fang J H 2009 *Acta Phys. Sin.* **58** 3617 (in Chinese)
[18] Ge W K 2008 *Acta Phys. Sin.* **57** 6714 (in Chinese)
[19] Ge W K 2007 *Acta Phys. Sin.* **56** 1 (in Chinese)
[20] Huang X H, Zhang X B and Shi S Y 2008 *Acta Phys. Sin.* **57** 6056 (in Chinese)
[21] Jia L Q, Zhang Y Y, Cui J C and Luo S K 2009 *Commun. Theor. Phys.* **52** 572
[22] Ding N and Fang J H 2009 *Commun. Theor. Phys.* **52** 12
[23] Xie G X, Cui J C, Zhang Y Y and Jia L Q 2009 *Chin. Phys. Lett.* **26** 631
[24] Zhang M J, Fang J H, Lin P, Lu K and Pang T 2009 *Commun. Theor. Phys.* **52** 561
[25] Chen X W, Zhao Y H and Li Y M 2009 *Chin. Phys.* B **18** 3139
[26] Chen X W, Zhao Y H and Liu C 2009 *Acta Phys. Sin.* **58** 5150 (in Chinese)
[27] Luo Y P 2009 *Int. J. Theor. Phys.* **48** 2665
[28] Zhang Y 2009 *Chin. Phys.* B **18** 4636
[29] Zhang H B, Chen L Q, Liu R W and Gu S L 2005 *Acta Phys. Sin.* **54** 2489 (in Chinese)
[30] Mei F X 2004 *Symmetries and Conserved Quantities of Constrained Mechanical Systems* (Beijing: Beijing Institute of Technology Press) (in Chinese)