Conformal invariance and Hojman conserved quantities for holonomic systems with quasi-coordinates*

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We propose a new concept of the conformal invariance and the conserved quantities for holonomic systems with quasi-coordinates. A one-parameter infinitesimal transformation group and its infinitesimal transformation vector of generators for holonomic systems with quasi-coordinates are described in detail. The conformal factor in the determining equations of the Lie symmetry is found. The necessary and sufficient conditions of conformal invariance, which are simultaneously of Lie symmetry, are given. The conformal invariance may lead to corresponding Hojman conserved quantities when the conformal invariance satisfies some conditions. Finally, an illustration example is introduced to demonstrate the application of the result.

Keywords: quasi-coordinates, conformal invariance, conformal factor, conserved quantity

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1. Introduction

The symmetry principle is one of the top-level principles in physics. The symmetries and the conserved quantities of a dynamical system play an important role in the fields of modern science and technology. The conformal invariance is a type of symmetry. In theoretical physics, conformal invariance is a symmetry under dilatation (scale invariance) and also under the special conformal transformations. It was introduced into physics by Cunningham[1] and Bateman[2] shortly after Einstein's first paper on special relativity. In fact, the conformal invariance means that the equation by the action of the infinitesimal operator equals a conformal multiplier matrix multiplied itself. In 1997, Galiullin *et al.*[3] studied the conformal invariance of Birkhoff equations and their relation with the Lie symmetry. In 2001, Robert and Matthew[4] investigated the conformal Hamiltonian systems by a geometric method, discussed the geometric structure of conformal systems with invariance, and pointed out the difference between conformal dynamics and arbitrary symmetric system. In recent years, Mei studied the conformal invariance and conserved quantities in dynamical mechanical systems, and Cai and Liu *et al.* studied the conformal invariance and conserved quantities, which was quickly extended to all types of mechanical systems.[5−12] Some work was also done to discuss this problem. The theory of conformal invariance developed rapidly and provided us with some direct and useful methods of solving practical problems.[13−27]

Since dynamical equations with quasi-coordinates are much more complex than general dynamical equations, the research of symmetries and conserved quantities must be more difficult correspondingly. However, the study of the conformal invariance for holonomic systems with quasi-coordinates is of significance not only geometrically but also mechanically. Up to now, the researches on the conformal invariance and the conserved quantities for holonomic systems with quasi-coordinates have not been reported yet. In the present paper, we study the conformal invariance and the conserved quantities for holonomic systems with quasi-coordinates. In fact, the aim of studying the conformal invariance of holonomic systems with quasi-coordinates is to seek the one-parameter group of infinitesimal transformation of equations. Here, the conformal invariance of holonomic systems with quasi-coordinates is discussed in detail. The conformal invariance is first defined and the conformal factors of holonomic systems with quasi-coordinates are obtained by infinitesimal transformations. We find the Hojman conserved quantities of holonomic systems with quasi-coordinates by the determining equations of Lie symmetry. Finally an illustration example is introduced to demonstrate the application of the result.

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2. Differential equations of holonomic systems with quasi-coordinates

Suppose that the configuration of the mechanical system is determined by n generalised coordinates $q_s (s = 1, \ldots, n)$. We introduce the quasi-velocities with n degrees and are independent of each other. In general, the quasi-velocities can be written in the form

$$\omega_s = \omega_s(q_k, \dot{q}_k, t) \quad (s, k = 1, \ldots, n), \tag{1}$$

therefore

$$\det\left(\frac{\partial \omega_s}{\partial \dot{q}_k}\right) \neq 0, \tag{2}$$

and we can solve $\dot{q}_s$ as a function of ω_k

$$\dot{q}_s = \dot{q}_s(q_k, \omega_k, t). \tag{3}$$

In a particular case, the quasi-velocities can be taken to be in the linear homogenous form of the generalised velocities

$$\omega_s = a_{sk}(\boldsymbol{q})\dot{q}_k, \tag{4}$$

and

$$\det(a_{sk}) \neq 0. \tag{5}$$

Therefore, we have

$$\dot{q}_s = b_{sk}(\boldsymbol{q})\omega_k, \tag{6}$$

where $b_{sr}a_{rk} = \delta_{sk}$. The motion equations of system are written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L^*}{\partial \omega_s} + \frac{\partial L^*}{\partial \omega_k}\gamma^k_{rs}\omega_r - \frac{\partial L^*}{\partial \pi_s} = P^*_s, \tag{7}$$

where L^* is the Lagrangian function expressed by quasi-velocity and its relations with Lagrangian as follows:

$$L^*(t, q_s, \omega_s) = L(t, q_s, b_{sk}\omega_k), \tag{8}$$

γ^k_{rs} is called Boltzmann three signs, i.e.

$$\gamma^k_{rs} = -\gamma^k_{sr}, \tag{9}$$

and

$$\gamma^k_{rs} = \left(\frac{\partial a_{km}}{\partial q_l} - \frac{\partial a_{kl}}{\partial q_m}\right) b_{lr}b_{ms}. \tag{10}$$

The P^*_s is generalised non-potential force expressed by quasi-velocity and its relation with generalised non-potential force Q_s, and expressed as

$$P^*_s = Q_k b_{ks}. \tag{11}$$

Note that the quasi-coordinates do not appear in the function. The partial derivatives with respect to the quasi-coordinates are defined as

$$\frac{\partial}{\partial \pi_s} = \frac{\partial \dot{q}_k}{\partial \omega_s}\frac{\partial}{\partial q_k}, \tag{12}$$

where $\dot{q}_k$ has the form of expression (3).

Suppose that Eq. (7) is non-singular, that is,

$$\det\left(\frac{\partial^2 L^*}{\partial \omega_s \partial \omega_k}\right) \neq 0. \tag{13}$$

From Eq. (7) we can obtain

$$\begin{aligned} F_s \equiv\ & A_{sk}(t, \boldsymbol{q})\dot{\omega}_k + B_s(t, \boldsymbol{q}, \boldsymbol{\omega}) \\ & - P^*_s(t, \boldsymbol{q}, \boldsymbol{\omega}) = 0, \end{aligned} \tag{14}$$

where

$$\begin{aligned} A_{sk} &= \frac{\partial^2 L^*}{\partial \omega_s \partial \omega_k}, \\ B_s &= \frac{\partial^2 L^*}{\partial \omega_s \partial \pi_k}\omega_k + \frac{\partial^2 L^*}{\partial \omega_s \partial t} - \frac{\partial L^*}{\partial \pi_s}. \end{aligned} \tag{15}$$

Using Eq. (15), we can solve all generalised accelerations $\dot{\omega}_s$ as follows:

$$\dot{\omega}_s = \alpha_s(t, \boldsymbol{q}, \boldsymbol{\omega}). \tag{16}$$

3. Definition and criterion of conformal invariance

The aim of studying the conformal invariance of holonomic systems with quasi-coordinates is to seek the sets of transformations with independent or dependent variables corresponding to the conformal invariance. We study one-parameter group of infinitesimal transformation of Eq. (7) or (14) and take the infinitesimal transformations of time and generalised coordinate as

$$t^* = t + \Delta t, \quad \text{and} \quad \pi^*_s(t^*) = \pi_s(t) + \Delta \pi_s, \tag{17}$$

their expended forms are as follows:

$$t^* = t + \varepsilon \xi_0(t, \boldsymbol{q}, \boldsymbol{\omega}),$$

and

$$\pi^*_s(t^*) = \pi_s(t) + \varepsilon \xi_s(t, \boldsymbol{q}, \boldsymbol{\omega}), \tag{18}$$

where ε is the small parameter; ξ_0 and ξ_s are the generators of the one-parameter infinitesimal transformation group.

Taking the infinitesimal generators as

$$X^{(0)} = \xi_0 \frac{\partial}{\partial t} + \xi_s \frac{\partial}{\partial \pi_s}, \tag{19}$$

its first expended vector is

$$X^{(1)} = X^{(0)} + (\dot{\xi}_s - \omega_s\dot{\xi}_0 - \gamma^s_{rt}\omega_r\xi_t)\frac{\partial}{\partial\omega_s}, \quad (20)$$

and its second expended vector is

$$\begin{aligned} &X^{(2)} \\ &= X^{(1)} + [(\dot{\xi}_s - \omega_s\dot{\xi}_0 - \gamma^k_{rt}\omega_r\xi_t)^{\cdot} - \dot{\omega}_s\dot{\xi}_0]\frac{\partial}{\partial\dot{\omega}_s}. \end{aligned} \quad (21)$$

Definition In the infinitesimal transformations of infinitesimal generators $\xi_0(t, \boldsymbol{q}, \boldsymbol{\omega})$ and $\xi_s(t, \boldsymbol{q}, \boldsymbol{\omega})$, if F_s satisfies the following condition:

$$X^{(2)}(F_s) = l^l_s(F_l), \quad (22)$$

then this kind of invariance is called the conformal invariance of Eq. (14) under one-parameter infinitesimal transformation (18). Equation (22) is called the determining equation of conformal invariance for holonomic systems with quasi-coordinates, and l^l_s is called the conformal factor.

In order to obtain the general expression of conformal factor of conformal invariance, we calculate the difference

$$X^{(2)}(F_s) - X^{(2)}\ (F_s)|_{F_s=0}\,. \quad (23)$$

Since

$$\dot{\xi}_s = \frac{\partial\xi_s}{\partial t} + \frac{\partial\xi_s}{\partial\pi_k}\omega_k, \quad (s = 0, 1 \ldots n), \quad (24)$$

$$\begin{aligned} \ddot{\xi}_s = &\frac{\partial^2\xi_s}{\partial t^2} + 2\frac{\partial^2\xi_s}{\partial\pi_k\partial t}\omega_k + \frac{\partial^2\xi_s}{\partial\pi_k\partial\pi_j}\omega_k\omega_j \\ &+ \frac{\partial\xi_s}{\partial\pi_k}\dot{\omega}_k, (k, j = 1, \ldots n), \end{aligned} \quad (25)$$

therefore

$$\begin{aligned} X^{(2)}(F_s) &= A_{sk}(\ddot{\xi}_k - 2\,\dot{\omega}_k\dot{\xi}_0 - \omega_k\ddot{\xi}_0) + X^{(1)}(A_{sk})\dot{\omega}_k + X^{(1)}(B_s - P^*_s) \\ &= A_{sk}(\ddot{\xi}_k - 2\,\dot{\omega}_k\dot{\xi}_0 - \omega_k\ddot{\xi}_0) + X^{(0)}(A_{sk})\dot{\omega}_k + X^{(0)}(B_s - P^*_s) + (\dot{\xi}_k - \omega_k\dot{\xi}_0)\frac{\partial(B_s - P^*_s)}{\partial\omega_k} \\ &= A_{sk}\left(\frac{\partial^2\xi_k}{\partial t^2} + 2\frac{\partial^2\xi_k}{\partial\pi_k\partial t}\omega_r + \frac{\partial^2\xi_k}{\partial\pi_r\partial\pi_j}\omega_r\omega_j + \frac{\partial\xi_k}{\partial\pi_r}\dot{\omega}_r\right) - 2A_{sk}\dot{\omega}_k\left(\frac{\partial\xi_0}{\partial t} + \frac{\partial\xi_0}{\partial\pi_r}\omega_r\right) \\ &\quad - A_{sk}\omega_k\left(\frac{\partial^2\xi_0}{\partial t^2} + 2\frac{\partial^2\xi_0}{\partial\pi_r\partial t}\omega_r + \frac{\partial^2\xi_0}{\partial\pi_r\partial\pi_j}\omega_r\omega_j + \frac{\partial\xi_0}{\partial\pi_r}\dot{\omega}_r\right) + X^{(0)}(A_{sk})\dot{\omega}_k + X^{(0)}(B_s - P^*_s) \\ &\quad + \left[\frac{\partial\xi_k}{\partial t} + \frac{\partial\xi_k}{\partial\pi_r}\omega_r - \omega_r\left(\frac{\partial\xi_0}{\partial t} + \frac{\partial\xi_0}{\partial\pi_r}\omega_r\right)\right]\frac{\partial(B_s - P^*_s)}{\partial\omega_k} \\ &= A_{sk}\left(\frac{\partial\xi_k}{\partial\pi_r} - \omega_r\frac{\partial\xi_0}{\partial\pi_r}\right)\dot{\omega}_r - 2A_{sk}\dot{\omega}_k\left(\frac{\partial\xi_0}{\partial t} + \frac{\partial\xi_0}{\partial\pi_r}\omega_r\right) + X^{(0)}(A_{sk})\dot{\omega}_k + C(t, \boldsymbol{q}, \boldsymbol{\omega}), \end{aligned} \quad (26)$$

where $C(t, \boldsymbol{q}, \boldsymbol{\omega})$ is the sum of items in which $\dot{\omega}_s$ is not contained. Treating alike leads to

$$X^{(2)}(F_s\ )|_{F_s=0} = A_{sk}\left(\frac{\partial\xi_k}{\partial\pi_r} - \omega_r\frac{\partial\xi_0}{\partial\pi_r}\right)\alpha_r - 2A_{sk}\alpha_k\left(\frac{\partial\xi_0}{\partial t} + \frac{\partial\xi_0}{\partial\pi_r}\omega_r\right) + X^{(0)}(A_{sk})\alpha_k + C(t, \boldsymbol{q}, \boldsymbol{\omega}), \quad (27)$$

where $\alpha_k = -A^{sk}B_k$. From Eqs. (26) and (27), we have the difference

$$\begin{aligned} &X^{(2)}(F_s) - X^{(2)}\ (F_s)|_{F_s=0} \\ &= A_{sk}\left(\frac{\partial\xi_k}{\partial\pi_r} - \omega_r\frac{\partial\xi_0}{\partial\pi_r}\right)(\dot{\omega}_r - \alpha_r) - 2A_{sk}(\dot{\omega}_k - \alpha_k)\left(\frac{\partial\xi_0}{\partial t} + \frac{\partial\xi_0}{\partial\pi_r}\omega_r\right) \\ &\quad + X^{(0)}(A_{sk})(\dot{\omega}_r - \alpha_k), \quad (s, k, r = 1, \ldots, n). \end{aligned} \quad (28)$$

On account of

$$\dot{\omega}_k - \alpha_k = \dot{\omega}_k + A^{kl}B_l = A^{kl}(A_{lm}\dot{\omega}_m + B_l) = A^{kl}F_l, \quad (29)$$

we have

$$\begin{aligned} X^{(2)}(F_s) - X^{(2)}\ (F_s)|_{F_s=0} &= A_{sk}\left(\frac{\partial\xi_k}{\partial\pi_r} - \omega_r\frac{\partial\xi_0}{\partial\pi_r}\right)A^{rl}F_l - 2A_{sk}A^{kl}F_l\left(\frac{\partial\xi_0}{\partial t} + \frac{\partial\xi_0}{\partial\pi_r}\omega_r\right) + X^{(0)}(A_{sk})A^{kl}F_l \\ &= \left[A_{sk}\left(\frac{\partial\xi_k}{\partial\pi_r} - \omega_r\frac{\partial\xi_0}{\partial\pi_r}\right)A^{rl} - 2\delta^l_s\left(\frac{\partial\xi_0}{\partial t} + \frac{\partial\xi_0}{\partial\pi_r}\omega_r\right) + X^{(0)}(A_{sk})A^{kl}\right]F_l \\ &= \beta^l_sF_l, \quad (s, k, r, l = 1, \cdots n), \end{aligned} \quad (30)$$

where

$$\beta_s^l = A_{sk}\left(\frac{\partial \xi_s}{\partial \pi_r} - \omega_r \frac{\partial \xi_0}{\partial \pi_r}\right) A^{rl} - 2\delta_s^l \left(\frac{\overset{+}{\partial \xi_0}}{\partial t} + \frac{\partial \xi_0}{\partial \pi_r}\omega_r\right) X^{(0)}(A_{sk})A^{kl}. \tag{31}$$

If expression (14) is of Lie symmetry by the infinitesimal point transformation of one-parameter, thus

$$X^{(2)}\ (F_s)|_{F_s=0} = 0. \tag{32}$$

From Eqs. (30) and (32) it follows that

$$(l_s^l - \beta_s^l)(F_l) = X^{(2)}\ (F_s)|_{F_s=0}\,, \tag{33}$$

and then

$$l_s^l = \beta_s^l. \tag{34}$$

Equation (34) is called the necessary and sufficient conditions of the conformal invariance, and it is simultaneously of Lie symmetry under the infinitesimal point transformation (18) for holonomic systems with quasi-coordinates. Thus, there is the following proposition.

Proposition 1 For holonomic systems with quasi-coordinates, the necessary and sufficient conditions that the conformal invariance would be of the Lie symmetry, are that Eq. (14) is the conformal factor satisfying Eq. (34).

We have the following deduction.

Deduction Suppose that differential Eq. (14) can transform into the standard forms

$$F_s \equiv \dot{\omega}_s - \alpha_s = 0, \tag{35}$$

by the action of the infinitesimal point transformations of one-parameter group, the necessary and sufficient conditions of conformal invariance which are simultaneously of Lie symmetry, are as follows:

$$l_s^l = \frac{\partial \xi_s}{\partial \pi_l} - 2\dot{\xi}_0 - \omega_s \frac{\partial \xi_0}{\partial \pi_l}. \tag{36}$$

4. Conformal invariance and Hojman conserved quantities

The conformal invariance of holonomic systems with quasi-coordinates can also lead to conserved quantities under certain conditions. By the action of special infinitesimal transformations when there exists the conformal factor which makes the differential equations have conformal invariance and Lie symmetry simultaneously, then the Hojman conserved quantities can be educed. Assume that the infinitesimal transformations of time and generalised coordinate have forms (18), the conformal factor is simplified into the following form:

$$\begin{aligned} l_s^l =\ & A_{sk}\left(\frac{\partial \xi_s}{\partial \pi_r} - \omega_r \frac{\partial \xi_0}{\partial \pi_r}\right) A^{rl} \\ & - 2\delta_s^l \left(\frac{\partial \xi_0}{\partial t} + \frac{\partial \xi_0}{\partial \pi_r}\omega_r\right) + X^{(0)}(A_{sk})A^{kl}. \end{aligned} \tag{37}$$

If the infinitesimal generator $\xi_s(t, \boldsymbol{q}, \boldsymbol{\omega})$ satisfies the following condition:

$$\begin{aligned} \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\overline{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) =\ & \frac{\partial \alpha_s}{\partial \pi_k}\xi_k + \frac{\partial \alpha_s}{\partial \omega_k}\left(\frac{\overline{\mathrm{d}}}{\mathrm{d}t}\xi_k - \gamma_{rt}^k \omega_r \xi_t\right) \\ & + \frac{\overline{\mathrm{d}}}{\mathrm{d}t}(\gamma_{rt}^s \omega_r \xi_t), \end{aligned} \tag{38}$$

where

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \omega_k \frac{\partial}{\partial \pi_k} + \alpha_k \frac{\partial}{\partial \omega_k}, \tag{39}$$

differential Eq. (7) is of Li symmetry.

Proposition 2 For holonomic systems with quasi-coordinates, if the infinitesimal generator ξ_i satisfies Eq. (39) and

$$\frac{\partial}{\partial \pi_s}(\gamma_{rt}^s)\omega_r \xi_t = 0, \tag{40}$$

and there exists a function $\mu = \mu(t, \boldsymbol{q}, \boldsymbol{\omega})$ which satisfies following condition:

$$\frac{\partial \alpha_s}{\partial \omega_s} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln \mu = 0, \tag{41}$$

then Hojman conserved quantities caused directly by Lie symmetry is in the following form

$$\begin{aligned} & I_{\mathrm{H}} \\ =\ & \frac{\partial \tau}{\partial t} + \frac{1}{\mu}\frac{\partial (\mu \xi_s)}{\partial \pi_s} \\ & + \frac{1}{\mu}\frac{\partial}{\partial \omega_s}\left[\mu\left(\frac{\overline{\mathrm{d}}}{\mathrm{d}t}\xi_s - \gamma_{rt}^s \omega_r \xi_t\right)\right] - \frac{\overline{\mathrm{d}}}{\mathrm{d}t}\tau \\ =\ & \text{const.} \end{aligned} \tag{42}$$

Its proof can be found in Refs. [28] and [29].

5. Illustration example

As an application of the conformal invariance of holonomic systems with quasi-coordinates, an illustration example is introduced.

We study the conformal invariance and the conserved quantities for holonomic systems with quasi-coordinates. The systems with two freedom degrees can be written in the form

$$L = \frac{1}{2}(q_1^2\dot{q}_2^2 + \dot{q}_2^2). \tag{43}$$

The non-potential generalised forces are

$$Q_1 = q_1\dot{q}_1^2 - q_1^2(\dot{q}_1 + \dot{q}_2), \quad \text{and} \quad Q_2 = Q_2(t, q_1). \tag{44}$$

Let

$$\omega_1 = q_1\dot{q}_1, \quad \text{and} \quad \omega_2 = \dot{q}_2, \tag{45}$$

then we have

$$\begin{aligned} &a_{11} = q_1, \quad a_{12} = a_{21} = 0, \quad a_{22} = 1, \\ &b_{11} = \frac{1}{q_1}, \quad b_{12} = b_{21} = 0, \quad b_{22} = 0, \\ &\gamma_{rt}^s = 0. \end{aligned} \tag{46}$$

On account of

$$\begin{aligned} L^* &= \frac{1}{2}(\omega_1^2 + \omega_2^2), \\ P_1^* &= Q_1 b_{11} + Q_2 b_{21} = \frac{\omega_1^2}{q_1^2} - q_1\omega_2 - \omega_1, \\ P_2^* &= Q_1 b_{12} + Q_2 b_{22} = Q_2(t, q_1), \end{aligned} \tag{47}$$

from Eq. (7), we can obtain

$$\begin{aligned} \dot{\omega}_1 &= \frac{\omega_1^2}{q_1^2} - \omega_1 - q_1\omega_2, \\ \dot{\omega}_2 &= Q_2(t, q_1). \end{aligned} \tag{48}$$

The differential equations of holonomic systems with quasi-coordinates are

$$F = \begin{pmatrix} \dot{\omega}_1 + \omega_1 + q_1\omega_2 - \dfrac{\omega_1^2}{q_1^2} \\ \dot{\omega}_2 - Q_2 \end{pmatrix} = 0. \tag{49}$$

Using determining equations of Lie symmetry, we can obtain

$$\begin{aligned} \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\overline{\mathrm{d}}}{\mathrm{d}t}\xi_1 &= \frac{2\omega_1}{q_1^2}\frac{\overline{\mathrm{d}}}{\mathrm{d}t}\xi_1 - q_1\frac{\overline{\mathrm{d}}}{\mathrm{d}t}\xi_2 - \frac{\overline{\mathrm{d}}}{\mathrm{d}t}\xi_1 \\ &\quad - \frac{2\omega_1^2\xi_1}{q_1^3 q_1} - \omega_2\frac{\xi_1}{q_1}, \\ \frac{\overline{\mathrm{d}}}{\mathrm{d}t}\frac{\overline{\mathrm{d}}}{\mathrm{d}t}\xi_2 &= \frac{\xi_1}{q_1}\frac{\partial Q_2}{\partial q_1} \end{aligned} \tag{50}$$

with a solution

$$\xi_1 = 0, \quad \xi_2 = \frac{1}{2}\left(\frac{\omega_1}{q_1} + q_1 + q_2\right)^2. \tag{51}$$

We have

$$\begin{aligned} X^{(2)} &= X^{(1)} + [(\dot{\xi}_s - \omega_s\dot{\xi}_0 - \gamma_{rt}^k\omega_r\xi_t)^{\cdot} - \dot{\omega}_s\dot{\xi}_0]\frac{\partial}{\partial\dot{\omega}_s} \\ &= \xi_2\frac{\partial}{\partial\pi_2} + \dot{\xi}_2\frac{\partial}{\partial\omega_2} + (\dot{\xi}_2)^{\cdot}\frac{\partial}{\partial\dot{\omega}_2}, \end{aligned} \tag{52}$$

and

$$\begin{aligned} &X^{(2)}(F) \\ &= \left[\xi_2\frac{\partial}{\partial\pi_2} + \dot{\xi}_2\frac{\partial}{\partial\omega_2} + (\dot{\xi}_2)^{\cdot}\frac{\partial}{\partial\dot{\omega}_2}\right] \\ &\quad \times \begin{pmatrix} \dot{\omega}_1 + \omega_1 + q_1\omega_2 - \dfrac{\omega_1^2}{q_1^2} \\ \dot{\omega}_2 - Q_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 1 - \dfrac{\omega_1}{q_1^2} & 1 \end{pmatrix} \begin{pmatrix} \dot{\omega}_1 + \omega_1 + q_1\omega_2 - \frac{\omega_1^2}{q_1^2} \\ \dot{\omega}_2 - Q_2 \end{pmatrix}. \end{aligned} \tag{53}$$

We can obtain the conformal factor

$$l_s^l = \begin{pmatrix} 0 & 0 \\ 1 - \dfrac{\omega_1}{q_1^2} & 1 \end{pmatrix}. \tag{54}$$

Using Eq. (41), we can have

$$\frac{2\omega_1}{q_1^2} - 1 + \frac{\overline{\mathrm{d}}}{\mathrm{d}t}\ln\mu = 0, \tag{55}$$

and

$$\mu = q_1^{-2}\left(\frac{\omega_1}{q_1} + q_1 + q_2\right)\exp t. \tag{56}$$

Using Proposition 2, we obtain the Hojma conserved quantities according to Eq. (42) as follows:

$$I_{\mathrm{H}} = \left(\frac{\omega_1}{q_1} + q_1 + q_2\right)^{-1} = \text{const.} \tag{57}$$

6. Conclusion

We study the conformal invariance and the Hojman conserved quantities for holonomic systems with quasi-coordinates. The holonomic systems with quasi-coordinates exhibit conformal invariance and conserved quantities under infinitesimal transformation. By introducing the concept of the conformal invariance and the conserved quantities for holonomic systems with quasi-coordinates, we can find the conformal factor in the determining equations of the Lie symmetry and the conserved quantities as shown in Eqs. (36), (37) and (42). The conformal invariance may lead to corresponding Hojman conserved quantities when the conformal invariance satisfies some conditions.

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