Weak Noether symmetry for a nonholonomic controllable mechanical system*

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This paper discusses the weak Noether symmetry for a nonholonomic controllable mechanical system of Chetaev type, and presents expressions of three kinds of conserved quantities obtained by using weak Noether symmetry. Finally, the application of these new results is illustrated by an example.

Keywords: weak Noether symmetry, conserved quantity, nonholonomic controllable mechanical system

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1. Introduction

For a mechanical system, there exists a close relation between its integrability and the variations in its symmetries and conserved quantities, thus the relation provides another tool for analysing how the mehanical universe works, and tells us why a phenomenon occurs without giving us the details of the actual process; so the symmetries and conserved quantities of dynamical and mechanical systems play an important role in mechanics and physics.[1−20] The modern methods to find the conserved quantity of a system are those using the following three main kinds of symmetries: the Noether symmetry,[1] the Lie symmetry,[2] and the Mei symmetry (or the form invariance).[3] Now people have done much work about these three symmetries and the conserved quantities obtained directly or indirectly from them.[4−20] The theory of the symmetries and conserved quantities gives a simpler method to solve the problems of dynamical and mechanical systems, so it becomes more important and has the important theoretical and practical significance.

Weak Noether symmetry is a kind of new symmetries, which was first published in the Beijing Institute of Technology Transaction (in Chinese) by Mei and Shui[21] in 2006. The weak Noether symmetry is derived from generalised accelerations in the Noether identity, expressed as functions of time, generalised coordinates and generalised velocities by using differential equations. From Mei's paper, we can generalise the weak Noether symmetry derived from the Noether identity and find more undiscovered conserved quantities, so it has an important theoretical significance. In Refs. [22]–[25], Xie *et al.* studied the weak Noether symmetry of some mechanical systems and obtained Noether conserved quantity or non-Noether conserved quantity. The control theory is very useful in many fields of the modern scientific world. The study of the symmetries and conserved quantities of controllable mechanical systems becomes more and more necessary and makes development in controllable mechanical system.[26−31] In the present paper, we discuss the weak Noether symmetry for a nonholonomic controllable mechanical system of Chetaev type, and presents expressions for the three kinds of conserved quantities produced by weak Noether symmetry. Finally, the application of the new results is shown by an example.

2. Weak Noether symmetry of a system

Here we consider a mechanical system whose configuration is determined by n generalised coordinates $q_s(s = 1, \ldots, n)$. Let its motion be subjected to the g ideal nonlinear nonholonomic constraints of Chetaev type

$$\varphi_\beta(t, q_s, \dot{q}_s, \mu_r, \dot{\mu}_r) = 0, \quad (\beta = 1, \ldots, g), \tag{1}$$

$$\frac{\partial \varphi_\beta}{\partial \dot{q}_s}(\xi_s - \dot{q}_s \xi_0) = 0, \tag{2}$$

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where $\mu_r (r = 1, \ldots, p)$ are controlled parameters. The motion of the mechanical system is controlled by using the controlling parameters μ_r, which are important to a mechanical system whose motion can be described by the differential equations as follows:

$$E_s(L) = Q_s + \sum_{\beta=1}^{g} \lambda_\beta \frac{\partial \varphi_\beta}{\partial \dot{q}_s}, \quad (s = 1, \ldots, n; \beta = 1, \ldots, g), \tag{3}$$

where E_s are the Euler operators

$$E_s = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_s} - \frac{\partial}{\partial q_s}. \tag{4}$$

In general, it is supposed that the system (3) is nonsingular, and before integrating Eq. (3), we can find λ_β as a function of t, q_s, $\dot{q}_s$, μ_r and $\dot{\mu}_r$, thus we can obtain the equations of motion of the holonomic system corresponding to the nonholonomic system expressed by Eqs. (1) and (3)

$$E_s(L) = Q_s + \Lambda_s, \quad (s = 1, \ldots, n), \tag{5}$$

where $\Lambda_s = \Lambda_s(t, q_s, \dot{q}_s, \mu_r, \dot{\mu}_r) = \sum_{\beta=1}^{g} \lambda_\beta (\partial \varphi_\beta / \partial \dot{q}_s)$ are the generalised nonholonomic reactive constraint forces, $L = L(t, q_s, \dot{q}_s)$ is the Lagrangian, $Q_s = Q_s(t, q_s, \dot{q}_s)$ are the nonpotential generalised forces, λ_β are the multipliers of constraint, and the repeated subscripts represent the summation.

From Eqs. (5), we can find all the generalised accelerations

$$\ddot{q}_s = \alpha_s(t, q_s, \dot{q}_s, \mu_r, \dot{\mu}_r), \quad (s = 1, \ldots, n). \tag{6}$$

Introduce infinitesimal transformation in terms of time and generalised coordinates

$$t^* = t + \varepsilon \xi_0(t, q_s, \dot{q}_s), \quad q_s^*(t^*) = q_s(t) + \varepsilon \xi_s(t, q_s, \dot{q}_s), \tag{7}$$

where ε is an infinitesimal parameter, ξ_0 and ξ_s are infinitesimal generators.

The weak Noether symmetry is more prevalent than the Noether symmetry because the condition of the former is weaker.

Definition Under the infinitesimal transformations (7), if the infinitesimal generators ξ_0 and ξ_s together with the gauge function $G_\mathrm{N} = G_\mathrm{N}(t, q_s, \dot{q}_s, \mu_r, \dot{\mu}_r)$, satisfy

$$L\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 + \bar{X}^{(1)}(L) + \frac{\partial L}{\partial \mu_r}\dot{\mu}_r\xi_0 + \frac{\partial L}{\partial \dot{\mu}_r}\ddot{\mu}_r\xi_0 + (Q_s + \Lambda_s)(\xi_s - \dot{q}_s\xi_0) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}G_\mathrm{N} = 0, \tag{8}$$

where

$$\bar{X}^{(1)} = \xi_0\frac{\partial}{\partial t} + \xi_s\frac{\partial}{\partial q_s} + \left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s - \dot{q}_s\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0\right)\frac{\partial}{\partial \dot{q}_s}, \tag{9}$$

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{q}_s\frac{\partial}{\partial q_s} + \alpha_s\frac{\partial}{\partial \dot{q}_s}, \tag{10}$$

then the symmetry is called weak Noether symmetry of the corresponding nonholonomic controllable mechanical system (5).

If ξ_0 and ξ_s also, in addition to Eq. (8), satisfy the restriction equation (2), then the symmetry is called weak Noether symmetry of the nonholonomic controllable mechanical system (1) and (3).

3. Three kinds of conserved quantities

Proposition 1 If the infinitesimal generators, ξ_0 and ξ_s satisfy both Eq. (8) and Eq. (2), then the weak Noether symmetry of the system (1) and (3) can yield the following Noether conserved quantity

$$I_\mathrm{N} = L\xi_0 + \frac{\partial L}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\xi_0) + G_\mathrm{N} = \mathrm{const}. \tag{11}$$

However, we call Eq. (11) the Noether conserved quantity of the corresponding holonomic system (5) if ξ_0 and ξ_s only admit Eq. (8) but do not admit Eq. (2).

So, for a nonholonomic controllable mechanical system of Chetaev type, the type of Noether conserved quantities which the weak Noether symmetry has, is the same as that the Noether symmetry have.

In a similar way, we can have Hojman conserved quantity.

For the transformation (7), taking $\xi_0 = 0$, i.e. setting the time t unchangeable, we obtain the determining equation of Lie symmetry

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s = \frac{\partial \alpha_s}{\partial q_k}\xi_k + \frac{\partial \alpha_s}{\partial \dot{q}_k}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k. \tag{12}$$

In this way, the weak Noether identity (8) will become

$$\frac{\partial L}{\partial q_s}\xi_s + \frac{\partial L}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s + (Q_s + \Lambda_s)\xi_s + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}G_\mathrm{N} = 0. \tag{13}$$

As a consequence, we obtain another proposition.

Proposition 2 If the infinitesimal generator ξ_s satisfies Eqs. (12) and (13), and there exists a function $\tilde{\mu} = \tilde{\mu}(t, q_s, \dot{q}_s, \mu_r, \dot{\mu}_r)$ such that

$$\frac{\partial \alpha_s}{\partial \dot{q}_s} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\tilde{\mu} = 0, \tag{14}$$

then Hojman conserved quantity of corresponding holonomic system (5) is given by

$$I_{\rm H}=\frac{1}{\tilde{\mu}}\left\{\frac{\partial}{\partial q_s}(\tilde{\mu}\xi_s)+\frac{\partial}{\partial\dot{q}_s}\left(\tilde{\mu}\frac{\bar{\rm d}}{{\rm d}t}\xi_s\right)\right\}=\text{const.} \quad (15)$$

It is familiar to us that the criterion equation of form invariance for corresponding holonomic system (5) is

$$\frac{\bar{\rm d}}{{\rm d}t}\frac{\partial\bar{X}^{(1)}(L)}{\partial\dot{q}_s}-\frac{\partial\bar{X}^{(1)}(L)}{\partial q_s}-\bar{X}^{(1)}(Q_s)-\bar{X}^{(1)}(\Lambda_s)=0, \quad (16)$$

and thereby we obtain Proposition 3.

Proposition 3 If Eqs. (16) hold for the infinitesimal generators ξ_0 and ξ_s of weak Noether symmetry, and there exists a gauge function $G_{\rm F}=G_{\rm F}(t,q_s,\dot{q}_s,\mu_r,\dot{\mu}_r)$ to make structure equation hold

$$\bar{X}^{(1)}(L)\frac{\bar{\rm d}}{{\rm d}t}\xi_0+\bar{X}^{(1)}\left[\bar{X}^{(1)}(L)\right]+\frac{\partial\bar{X}^{(1)}(L)}{\partial\mu_r}\dot{\mu}_r\xi_0+\frac{\partial\bar{X}^{(1)}(L)}{\partial\dot{\mu}_r}\ddot{\mu}_r\xi_0+\bar{X}^{(1)}(Q_s+\Lambda_s)(\xi_s-\dot{q}_s\xi_0)+\frac{\bar{\rm d}}{{\rm d}t}G_{\rm F}=0, \quad (17)$$

then a new-type conserved quantity deduced by weak Noether symmetry for corresponding holonomic system (5) is obtained

$$I_{\rm F}=\bar{X}^{(1)}(L)\xi_0+\frac{\partial\bar{X}^{(1)}(L)}{\partial\dot{q}_s}(\xi_s-\dot{q}_s\xi_0)+G_{\rm F}=\text{const.} \quad (18)$$

Now we have obtained three kinds of conserved quantities by using weak Noether symmetry.

4. Weak Noether symmetry and the Noether symmetry of a system

Equations (2) do not include the second derivative of the generalised coordinate $\ddot{q}_s$, so the Noether identity of a nonholonomic controllable mechanical system (1) and (3) has the form as follows:

$$L\dot{\xi}_0+\xi_0\frac{\partial L}{\partial t}+\xi_s\frac{\partial L}{\partial q_s}+(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0)\frac{\partial L}{\partial\dot{q}_s}+\frac{\partial L}{\partial\mu_r}\dot{\mu}_r\xi_0+\frac{\partial L}{\partial\dot{\mu}_r}\ddot{\mu}_r\xi_0+(Q_s+\Lambda_s)(\xi_s-\dot{q}_s\xi_0)+\dot{G}_{\rm N}=0. \quad (19)$$

If the infinitesimal generators ξ_0 and ξ_s do not depend on generalised velocities $\dot{q}_s$, that is

$$\frac{\partial\xi_0}{\partial\dot{q}_s}=\frac{\partial\xi_s}{\partial\dot{q}_k}=0,\quad (s,k=1,2,\ldots,n), \quad (20)$$

then Eqs. (8) are the same as Eqs. (19) formally; otherwise, they are different.

Choose

$$\xi_0=\xi_s=0,\quad (s=1,\ldots,n). \quad (21)$$

It is apparent that the generators admit the restriction equations (2); according to Noether symmetry, we have

$$G_{\rm N}=0,\quad I_{\rm N}=0, \quad (22)$$

which is trivial.

Substituting Eq. (19) into Eq. (8) we obtain

$$\frac{\bar{\rm d}}{{\rm d}t}G_{\rm N}=0, \quad (23)$$

so that we can not only take $G_{\rm N}=0$ but also can choose arbitrary integral. For the latter, we can take the Noether conserved quantity as

$$I_{\rm N}=G_{\rm N}. \quad (24)$$

From the above, we can say that the weak Noether symmetry is more prevalent than the Noether symmetry.

5. Example

The Lagrangian, the nonpotential generalised forces, and the constraint equations of a nonholonomic controllable mechanical system are, respectively, written as

$$L=\frac{1}{2}(\dot{q}_1^2+\dot{q}_2^2), \quad (25)$$

$$Q_1=0,\quad Q_2=0, \quad (26)$$

$$\varphi=\dot{q}_2-\mu q_2=0, \quad (27)$$

where μ is the controlling parameter, but not the integral parameter.

The equations of motion are

$$\ddot{q}_1=0,\ \ddot{q}_2=\dot{\mu}q_2+\mu\dot{q}_2. \quad (28)$$

For this system, the weak Noether identity (8) reads

$$L\frac{\bar{\rm d}}{{\rm d}t}\xi_0+\left(\frac{\bar{\rm d}}{{\rm d}t}\xi_1-\dot{q}_1\frac{\bar{\rm d}}{{\rm d}t}\xi_0\right)\frac{\partial}{\partial\dot{q}_1}+\left(\frac{\bar{\rm d}}{{\rm d}t}\xi_2-\dot{q}_2\frac{\bar{\rm d}}{{\rm d}t}\xi_0\right)\frac{\partial}{\partial\dot{q}_2}+(\dot{\mu}q_2+\mu\dot{q}_2)(\xi_2-\dot{q}_2\xi_0)$$

$$+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}G_{\mathrm{N}}=0. \tag{29}$$

We take the generators in the form

$$\xi_0=0,\ \ \xi_1=\dot{q}_1,\ \xi_2=0,\ G_{\mathrm{N}}=\dot{q}_1. \tag{30}$$

It is easy to see that the generators (30) satisify Eqs. (29) and (2), such that this system is weakly Noether symmetric. Next, we will caculate whether it is Noether symmetric or not. The Noether identity (19) reads

$$\dot{q}_1\dot{\xi}_1+\dot{q}_2\dot{\xi}_2+(\dot{\mu}q_2+\mu\dot{q}_2)\xi_2+\dot{G}_{\mathrm{N}}=0, \tag{31}$$

and its Killing equations are

$$\begin{aligned}&\dot{q}_1\left(\frac{\partial\xi_1}{\partial t}+\frac{\partial\xi_1}{\partial q_1}\dot{q}_1+\frac{\partial\xi_1}{\partial q_2}\dot{q}_2\right)\\&+\dot{q}_2\left(\frac{\partial\xi_2}{\partial t}+\frac{\partial\xi_2}{\partial q_1}\dot{q}_1+\frac{\partial\xi_2}{\partial q_2}\dot{q}_2\right)\\&+(\dot{\mu}q_2+\mu\dot{q}_2)\xi_2\\&=-\frac{\partial G_{\mathrm{N}}}{\partial t}-\frac{\partial G_{\mathrm{N}}}{\partial q_1}\dot{q}_1-\frac{\partial G_{\mathrm{N}}}{\partial q_2}\dot{q}_2,\end{aligned} \tag{32}$$

$$\begin{aligned}&\dot{q}_1\left(\frac{\partial\xi_1}{\partial\dot{q}_1}+\frac{\partial\xi_1}{\partial\dot{q}_2}\right)+\dot{q}_2\left(\frac{\partial\xi_2}{\partial\dot{q}_1}+\frac{\partial\xi_2}{\partial\dot{q}_2}\right)\\&=-\frac{\partial G_{\mathrm{N}}}{\partial\dot{q}_1}-\frac{\partial G_{\mathrm{N}}}{\partial\dot{q}_2}.\end{aligned} \tag{33}$$

This is an inconsistent system of equations, and we fail to look for the Noether conserved quantity. The determining equation of Lie symmetry for corresponding holonomic system (5) is

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s=\frac{\partial\alpha_s}{\partial q_k}\xi_k+\frac{\partial\alpha_s}{\partial\dot{q}_k}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k. \tag{34}$$

Equations (1) under the infinitesimal transformations are

$$\bar{X}^{(1)}(\varphi)=0. \tag{35}$$

The generators (30) satisfy both Eqs. (34) and (35), so the generators are both weak Noether symmetric and Lie symmetric.

We also obtain

$$\begin{aligned}&\bar{X}^{(1)}(L)=0,\\&\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\bar{X}^{(1)}(L)}{\partial\dot{q}_s}-\frac{\partial\bar{X}^{(1)}(L)}{\partial q_s}\\&=\bar{X}^{(1)}(Q_s)+\bar{X}^{(1)}(\Lambda_s)\\&=0.\end{aligned} \tag{36}$$

So, the generators (30) are both weak Noether symmetric and form invariance.

From Proposition 1, we can take Noether conserved quantity as

$$I_{\mathrm{N}}=\dot{q}_1^2+\dot{q}_1. \tag{37}$$

In a similar way, we can have Hojman conserved quantity

$$I_{\mathrm{H}}=\frac{\dot{q}_1}{q_1-\dot{q}_1t}, \tag{38}$$

and the new conserved quantity

$$I_{\mathrm{F}}=\dot{q}_2-\mu q_2. \tag{39}$$

For Eq. (38), we can obtain the Noether generators from the Noether converse therom

$$\bar{\xi}_1=\xi_1-\dot{q}_1\xi_0=\frac{q_1}{(q_1-\dot{q}_1t)^2}, \tag{40}$$

$$\bar{\xi}_2=\xi_2-\dot{q}_2\xi_0=0. \tag{41}$$

They cannot be Eqs. (30), since Eqs. (30) are not the Noether symmetry; in the same way, we can also arrive at the same conclusion via Eqs. (37) and (39).

6. Conclusion

In this paper, the weak Noether symmetry of a nonholonomic controllable mechanical system is introduced; the results in this paper have more general significance, they can be applied to not only a nonholonomic controllable mechanical system but also a nonholonomic mechanical system (when $\mu_r=0$). From the illustrated example, we can see that the weak Noether symmetry is more prevalent than the Noether symmetry, and we can use it to obtain some conserved quantities which cannot be found by Noether symmetry.

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