Meshless analysis of three-dimensional steady-state heat conduction problems*

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Steady-state heat conduction problems arisen in connection with various physical and engineering problems where the functions satisfy a given partial differential equation and particular boundary conditions, have attracted much attention and research recently. These problems are independent of time and involve only space coordinates, as in Poisson's equation or the Laplace equation with Dirichlet, Neuman, or mixed conditions. When the problems are too complex, it is difficult to find an analytical solution, the only choice left is an approximate numerical solution. This paper deals with the numerical solution of three-dimensional steady-state heat conduction problems using the meshless reproducing kernel particle method (RKPM). A variational method is used to obtain the discrete equations. The essential boundary conditions are enforced by the penalty method. The effectiveness of RKPM for three-dimensional steady-state heat conduction problems is investigated by two numerical examples.

Keywords: reproducing kernel particle method, meshless method, steady-state heat conduction problem

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1. Introduction

Partial differential equations arise in connection with various physical and geometrical problems, where the involved functions depend on two or more independent variables, usually on time t and on one or several space variables.[1] A steady-state heat conduction equation is one of the most important partial differential equations in engineering mathematics, because it occurs in connection with gravitational fields, electrostatics fields, incompressible fluid flow, and other areas.[1] Mathematically, the problem is a function which satisfies a given partial differential equation and particular boundary conditions (boundary-value problems). Physically speaking, the problem is independent of time and involves only space coordinates, just as initial-value problems are associated with hyperbolic partial differential equations of the elliptic type.[2,3]

When steady-state heat conduction problems are too complex, it is difficult to find an analytical solution for such problems, so the only choice left is an approximate numerical solution. In this paper, the reproducing kernel particle method (RKPM) is used to obtain the numerical solution of a three-dimensional steady-state heat conduction problem.

In recent years meshless techniques have attracted the attention of researchers. In a meshless (mesh-free) method, a set of scattered nodes are used instead of meshing the domain of the problem. For many engineering problems, such as large deformation and crack growth, it is necessary to deal with extremely large deformations or fractures of the mesh with the remeshing technique. The meshless method is a new and interesting numerical technique which can solve many engineering problems unsuited for conventional numerical methods with a minimum of meshing or no meshing at all.[4]

Some meshless methods have been developed, such as smooth particle hydrodynamics (SPH) methods,[5] radial basis function (RBF),[6] element free Galerkin (EFG) method,[7−9] meshless local Petrov–Galerkin (MLPG) method,[10] RKPM,[11,12] finite point method (FPM),[13,14] meshless method based on local boundary integral equation,[15] boundary element-free method (BEFM),[16,17] meshless method with complex variables,[18−21] and so on.

The heat conduction problems arising in many

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kinds of engineering fields have thus attracted much attention and research recently. It is difficult to find an analytical solution for such problems, so it is important to find the numerical solution. Chen uses the RBF meshless method to solve a simple heat conduction problem,[22] Singh *et al.* have done much work on steady, unsteady, nonlinear heat transfer problems based on EFG method,[23−27] Arefmanesh *et al.* put forward a meshless method for solving heat conduction problem based on MLPG,[28,29] Liu *et al.* use a meshless least-squares method for solving the steady-state heat conduction problem,[30,31] Sladek *et al.* use a meshless local boundary integral equation method (LBIE) for heat conduction analysis,[32,33] Cheng and Liew use RKPM for two-dimensional unsteady heat conduction problem and obtain the convergence analysis of this method.[34]

The RKPM is greatly developed and widely applied in various fields. To the best knowledge of the authors, this method has not been used for a three-dimensional (3D) steady-state heat conduction problem. In this paper, RKPM is first used to obtain the numerical solution of a 3D steady-state heat conduction problem. In this method, the function over the solution domain needs only a set of nodes, it does not require element connectivity. The variational method is used to obtain the corresponding discrete equations and the essential boundary conditions are enforced by a penalty method. The numerical results are presented in comparison with the exact results.

2. The RKPM[10]

In RKPM, we let

$$u^h(\boldsymbol{x}) = \int_\Omega u(\boldsymbol{x}')\bar{w}(\boldsymbol{x}-\boldsymbol{x}')\mathrm{d}\boldsymbol{x}', \tag{1}$$

where $\bar{w}(\boldsymbol{x}-\boldsymbol{x}')$ is the correction kernel function

$$\bar{w}(\boldsymbol{x}-\boldsymbol{x}') = c(\boldsymbol{x};\boldsymbol{x}-\boldsymbol{x}')w(\boldsymbol{x}-\boldsymbol{x}'), \tag{2}$$

where $c(\boldsymbol{x};\boldsymbol{x}-\boldsymbol{x}')$ is the correction function which can be the combination of the polynomial basis.

Let

$$\begin{aligned} c(\boldsymbol{x};\boldsymbol{x}-\boldsymbol{x}') &= \sum_{i=1}^{m} p_i(\boldsymbol{x}-\boldsymbol{x}')b_i(\boldsymbol{x}) \\ &= \boldsymbol{p}^{\mathrm{T}}(\boldsymbol{x}-\boldsymbol{x}')\boldsymbol{b}(\boldsymbol{x}), (\boldsymbol{x}\in\Omega), \end{aligned} \tag{3}$$

where m is the number of terms in the basis, $p_i(\boldsymbol{x}-\boldsymbol{x}')$ are the monomial basis functions, and $b_i(\boldsymbol{x})$ are coefficients of the monomial basis functions.

Generally, the basis can be chosen as follows:

Linear basis:

$$\boldsymbol{p}^{\mathrm{T}} = (1, x_1-x_1', x_2-x_2'), \quad \text{(2D)} \tag{4}$$

$$\boldsymbol{p}^{\mathrm{T}} = (1, x_1-x_1', x_2-x_2', x_3-x_3'). \quad \text{(3D)} \tag{5}$$

Quadratic basis:

$$\boldsymbol{p}^{\mathrm{T}} = \left(1, x_1-x_1', (x_1-x_1')^2\right), \quad \text{(1D)} \tag{6}$$

$$\boldsymbol{p}^{\mathrm{T}} = \left(1, x_1-x_1', x_2-x_2', (x_1-x_1')^2, (x_1-x_1')(x_2-x_2'), (x_2-x_2')^2\right). \quad \text{(2D)} \tag{7}$$

Corresponding to the kernel approximation of Eq. (1), we can obtain the discretization approximation of Eq. (1) by the trapezoidal integral rule as follows:

$$\begin{aligned} u^h(\boldsymbol{x}) &= \sum_{I=1}^{n} \bar{w}(\boldsymbol{x}-\boldsymbol{x}_I)u(\boldsymbol{x}_I)\Delta V_I \\ &= \sum_{I=1}^{n} c(\boldsymbol{x};\boldsymbol{x}-\boldsymbol{x}_I)w(\boldsymbol{x}-\boldsymbol{x}_I)u_I\Delta V_I, \end{aligned} \tag{8}$$

where $w(\boldsymbol{x}-\boldsymbol{x}_I)$ are the weighting functions with property of compact support, $\boldsymbol{x}_I$ are nodes in the domain of the node $\boldsymbol{x}$, ΔV_I is the domain measure which is related to the nodes $\boldsymbol{x}_I$,

$$\sum_{I=1}^{n} \Delta V_I = V. \tag{9}$$

For a 3D problem, V means the total volume of the problem domain.

Equation (8) can be rewritten in the vector form

$$u^h(\boldsymbol{x}) = \boldsymbol{C}(\boldsymbol{x})\boldsymbol{W}(\boldsymbol{x})\boldsymbol{V}\cdot\boldsymbol{u}, \tag{10}$$

where

$$\boldsymbol{u}^{\mathrm{T}} = (u_1, u_2, \ldots, u_n), \tag{11}$$

$$\boldsymbol{W}(\boldsymbol{x}) = \begin{bmatrix} w(\boldsymbol{x}-\boldsymbol{x}_1) & 0 & \dots & 0 \\ 0 & w(\boldsymbol{x}-\boldsymbol{x}_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & w(\boldsymbol{x}-\boldsymbol{x}_n) \end{bmatrix}, \tag{12}$$

$$\boldsymbol{V} = \begin{bmatrix} \Delta V_1 & 0 & \dots & 0 \\ 0 & \Delta V_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \Delta V_n \end{bmatrix}. \tag{13}$$

Let

$$c_I(\boldsymbol{x}) = c(\boldsymbol{x}; \boldsymbol{x}-\boldsymbol{x}_I), \tag{14}$$

then

$$\boldsymbol{C}(\boldsymbol{x}) = (c_1(x), c_2(x), \ \dots, c_n(x)) = \boldsymbol{b}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{P}, \tag{15}$$

where

$$\boldsymbol{P} = \begin{bmatrix} p_1(\boldsymbol{x}-\boldsymbol{x}_1) & p_1(\boldsymbol{x}-\boldsymbol{x}_2) & \dots & p_1(\boldsymbol{x}-\boldsymbol{x}_n) \\ p_2(\boldsymbol{x}-\boldsymbol{x}_1) & p_2(\boldsymbol{x}-\boldsymbol{x}_2) & \dots & p_2(\boldsymbol{x}-\boldsymbol{x}_n) \\ \vdots & \vdots & \ddots & \vdots \\ p_m(\boldsymbol{x}-\boldsymbol{x}_1) & p_m(\boldsymbol{x}-\boldsymbol{x}_2) & \dots & p_m(\boldsymbol{x}-\boldsymbol{x}_n) \end{bmatrix}, \tag{16}$$

$$\boldsymbol{b}^{\mathrm{T}}(\boldsymbol{x}) = (b_1(x), b_2(x), \dots, b_m(x)), \tag{17}$$

where the coefficients $b_i(\boldsymbol{x})$ are determined by the reproducing conditions of the approximation function. In order to obtain the accuracy of n order, we obtain

$$\overline{M}(\boldsymbol{x}) = M(\boldsymbol{x})\boldsymbol{b}(\boldsymbol{x}) = H, \tag{18}$$

where

$$M(\boldsymbol{x}) = \sum_{I=1}^{n} \boldsymbol{p}(\boldsymbol{x}-\boldsymbol{x}_I)\boldsymbol{p}^{\mathrm{T}}(\boldsymbol{x}-\boldsymbol{x}_I) \times w(\boldsymbol{x}-\boldsymbol{x}_I)\Delta V_I, \tag{19}$$

$$\boldsymbol{H} = (1, 0, \dots, 0)^{\mathrm{T}}, \tag{20}$$

$$\boldsymbol{b}(\boldsymbol{x}) = M^{-1}(\boldsymbol{x})\boldsymbol{H}. \tag{21}$$

The approximation $u^h(\boldsymbol{x})$ can then be expressed as

$$u^h(\boldsymbol{x}) = \sum_{I=1}^{n} \Phi_I(\boldsymbol{x})u_I = \boldsymbol{\Phi}(\boldsymbol{x})\boldsymbol{u}, \tag{22}$$

where the shape function $\boldsymbol{\Phi}(\boldsymbol{x})$ is given by

$$\begin{aligned} \boldsymbol{\Phi}(\boldsymbol{x}) &= [\Phi_1(x), \Phi_2(x), \dots, \Phi_n(x)] \\ &= \boldsymbol{C}(\boldsymbol{x})\boldsymbol{W}(\boldsymbol{x})\boldsymbol{V}. \end{aligned} \tag{23}$$

3. The discretion and numerical implementation

A three-dimensional steady-state heat conduction equation in isotropic materials with thermal properties independent of temperature is governed by

$$k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \dot{Q} = 0, \quad (\text{on } V) \tag{24a}$$

with boundary conditions:

$$\text{at surface} \quad x = 0(S_1), \quad T = T_{s_1}, \tag{24b}$$

$$\text{at surface} \quad x = L(S_2), \quad T = T_{s_2}, \tag{24c}$$

$$\text{at surface} \quad y = 0(S_3), \quad k\frac{\partial T}{\partial y}n_y = h(T - T_\infty), \tag{24d}$$

$$\text{at surface} \quad y = W(S_4), \quad -k\frac{\partial T}{\partial y}n_y = h(T - T_\infty), \tag{24e}$$

$$\text{at surface} \quad z = 0(S_5), \quad k\frac{\partial T}{\partial z}n_z = h(T - T_\infty), \tag{24f}$$

$$\text{at surface} \quad z = H(S_6), \quad -k\frac{\partial T}{\partial z}n_z = h(T - T_\infty), \tag{24g}$$

where T is the temperature, k denotes the coefficient of thermal conductivity and ρ represents the density of the material, c is specific heat of the material, $\dot{Q}$ denotes internal heat generation per unit volume. h is convective heat transfer coefficient, T_∞ denotes surrounding fluid temperature, V is the 3D domain. L, W and H denote the length of domain V in the direction of x, y and z, respectively.

The weighted integral form of Eq. (24a) is obtained as follows

$$\int_V w\left[k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \dot{Q}\right]\mathrm{d}V = 0. \tag{25}$$

The weak form of Eq. (25) is

$$\int_V \left[k\nabla^{\mathrm{T}} w \nabla T - w\dot{Q}\right]\mathrm{d}V - \int_S wk\left(\frac{\partial T}{\partial x}n_x + \frac{\partial T}{\partial x}n_y + \frac{\partial T}{\partial x}n_z\right)\mathrm{d}S = 0. \tag{26}$$

The functional $\Pi(T)$ which denotes the total potential energy can be written as

$$\begin{aligned}\Pi(T) = &\frac{1}{2}\int_V \nabla T(k\nabla T)\mathrm{d}V - \int_V T\dot{Q}\mathrm{d}V + \int_{S_3} h\Big(\frac{T^2}{2} - TT_\infty\Big)\mathrm{d}S_3 \\ &+ \int_{S_4} h\Big(\frac{T^2}{2} - TT_\infty\Big)\mathrm{d}S + \int_{S_5} h\Big(\frac{T^2}{2} - TT_\infty\Big)\mathrm{d}S + \int_{S_6} h\Big(\frac{T^2}{2} - TT_\infty\Big)\mathrm{d}S. \end{aligned} \tag{27}$$

Enforcing essential boundary conditions using a penalty method, the modified functional $\Pi^*(T)$ can be obtained as

$$\begin{aligned}\Pi^*(T) = &\frac{1}{2}\int_V \nabla T(k\nabla T)\mathrm{d}V - \int_V T\dot{Q}\mathrm{d}V + \int_{S_3} h\Big(\frac{T^2}{2} - TT_\infty\Big)\mathrm{d}S \\ &+ \int_{S_4} h\Big(\frac{T^2}{2} - TT_\infty\Big)\mathrm{d}S + \int_{S_5} h\Big(\frac{T^2}{2} - TT_\infty\Big)\mathrm{d}S + \int_{S_6} h\Big(\frac{T^2}{2} - TT_\infty\Big)\mathrm{d}S \\ &+ \frac{\alpha}{2}\int_{S_1} (T - T_{S_1})^2\mathrm{d}S + \frac{\alpha}{2}\int_{S_2} (T - T_{S_2})^2\mathrm{d}S. \end{aligned} \tag{28}$$

Taking variation, we have

$$\begin{aligned}\delta\Pi^*(T) = &\int_V \nabla T(k\delta\nabla T)\mathrm{d}V - \int_V \delta T\dot{Q}\mathrm{d}V + \int_{S_3\cup S_4\cup S_5\cup S_6} \delta T h(T - T_\infty)\mathrm{d}S \\ &+ \int_{S_1} \delta T\alpha(T - T_{S_1})\mathrm{d}S + \int_{S_2} \delta T\alpha(T - T_{S_2})\mathrm{d}S, \end{aligned} \tag{29}$$

where α is the penalty factor.

By Eq. (22), we can obtain the approximation of the temperature function

$$T^h(\boldsymbol{x}, t) = \sum_{I=1}^{n} \Phi_I(\boldsymbol{x})T_I(t) = \boldsymbol{\Phi}(\boldsymbol{x})\boldsymbol{T}, \tag{30}$$

where

$$\boldsymbol{\Phi}(\boldsymbol{x}) = (\Phi_1(\boldsymbol{x}), \Phi_2(\boldsymbol{x}), \ldots, \Phi_n(\boldsymbol{x}))\,, \tag{31}$$

$$\boldsymbol{T} = (T_1(t), T_2(t), \ldots, T_n(t))^{\mathrm{T}}\,, \tag{32}$$

$$\boldsymbol{x} = (x, y, z)\,. \tag{33}$$

Substituting Eq. (30) into Eq. (29) yields

$$\begin{aligned}\delta\Pi^*(T) = &\int_V \nabla(\boldsymbol{\Phi}(\boldsymbol{x})\boldsymbol{T})(k\delta\nabla(\boldsymbol{\Phi}(\boldsymbol{x})\boldsymbol{T})\mathrm{d}V - \int_V [\delta(\boldsymbol{\Phi}(\boldsymbol{x})\boldsymbol{T})\dot{Q}\mathrm{d}V \\ &+ \int_{S_3\cup S_4\cup S_5\cup S_6} \delta(\boldsymbol{\Phi}(\boldsymbol{x})\boldsymbol{T})h(\boldsymbol{\Phi}(\boldsymbol{x})\boldsymbol{T} - T_\infty)\mathrm{d}S + \int_{S_1} \delta(\boldsymbol{\Phi}(\boldsymbol{x})\boldsymbol{T})\alpha(\boldsymbol{\Phi}(\boldsymbol{x})\boldsymbol{T} - T_{S_1})\mathrm{d}S \\ &+ \int_{S_2} \delta(\boldsymbol{\Phi}(\boldsymbol{x})\boldsymbol{T})\alpha(\boldsymbol{\Phi}(\boldsymbol{x})\boldsymbol{T} - T_{S_2})\mathrm{d}S. \end{aligned} \tag{34}$$

Since $\delta\Pi^*(T) = 0$ and δT is arbitrary in Eq. (34), the following equations can be obtained

$$\boldsymbol{K}\boldsymbol{T} = \boldsymbol{F}, \tag{35}$$

where

$$\boldsymbol{K} = \int_V \nabla\boldsymbol{\Phi}^{\mathrm{T}}(\boldsymbol{x})k\nabla\boldsymbol{\Phi}(\boldsymbol{x})\mathrm{d}V + \int_{S_1\cup S_2} \boldsymbol{\Phi}^{\mathrm{T}}(\boldsymbol{x})\alpha\boldsymbol{\Phi}(\boldsymbol{x})\mathrm{d}S + \int_{S_3\cup S_4\cup S_5\cup S_6} \boldsymbol{\Phi}^{\mathrm{T}}(\boldsymbol{x})h\boldsymbol{\Phi}(\boldsymbol{x})\mathrm{d}S, \tag{36}$$

$$K_{IJ} = \int_V \nabla\Phi_I^{\mathrm{T}}(\boldsymbol{x})\rho c\nabla\Phi_J(\boldsymbol{x})\mathrm{d}V + \int_{S_1\cup S_2} \Phi_I^{\mathrm{T}}(\boldsymbol{x})\alpha\Phi_J(\boldsymbol{x})\mathrm{d}S + \int_{S_3\cup S_4\cup S_5\cup S_6} \Phi_I^{\mathrm{T}}(\boldsymbol{x})h\Phi_J(\boldsymbol{x})\mathrm{d}\Omega, \tag{37}$$

$$F_I = \int_V \Phi_I(\boldsymbol{x})\dot{Q}\mathrm{d}V + \int_{S_1} \alpha\Phi_I(\boldsymbol{x})T_{S_1}\mathrm{d}S + \int_{S_2} \alpha\Phi_I(\boldsymbol{x})T_{S_2}\mathrm{d}S + \int_{S_3\cup S_4\cup S_5\cup S_6} \Phi_I(\boldsymbol{x})hT_\infty\mathrm{d}\Gamma. \tag{38}$$

4. Numerical example

To test the efficiency of the method of the reproducing kernel particle approximation on a 3D steady-state heat conduction problem, a numerical analysis of the model shown in Fig. 1 has been carried out by using a linear basis function. The results obtained by the RKPM are also compared with analytical methods.

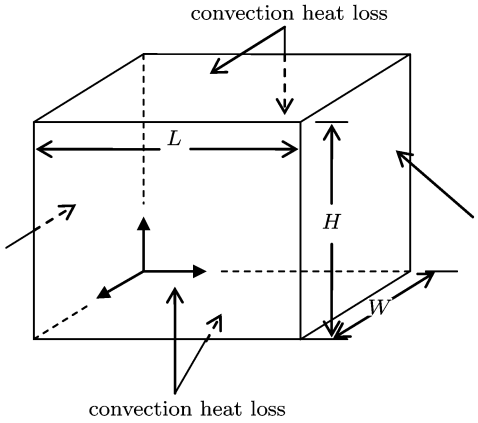


Fig. 1. The 3D model ($L = 1$ m, $W = 2$ m, $H = 3$ m, $k = 1$ W/mK, $h = 0$ W/m^2·K, $T_\infty = 0$ °C, $T_{S_1} = 0$ °C, $T_{S_2} = 0$ °C).

Consider the following problem

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = f(x, y, z),$$
$$0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c, \tag{39}$$

with boundary conditions

$$T(0, y, z) = T(a, y, z) = 0, \tag{40a}$$
$$T(x, 0, z) = T(x, b, z) = 0, \tag{40b}$$
$$T(x, y, 0) = T(x, y, c) = 0, \tag{40c}$$

where

$$f(x, y, z) = -\pi^2 \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) \times \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} \sin\frac{\pi z}{c}. \tag{41}$$

The exact solution of Eq. (39) is

$$T(x, y, z) = \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} \sin\frac{\pi z}{c}. \tag{42}$$

We use the RKPM to solve the above heat conduction problem with penalty factor $\alpha = 4 \times 10^5$ and scaling parameter $d_{\max} = 1.13$ and $a = 1$, $b = 2$, $c = 3$. The back-ground mesh is adopted for integration in this method. Figure 2 shows uniform nodes arrangement for example. The weight function is chosen to be Gauss function and the bases are chosen to be linear. In Fig. 3, the exact solution

$$T(x, y, z) = \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} \sin\frac{\pi z}{c}$$

is plotted in comparison with numerical results when $x = 0.75$ and $y = 1.6$. From Fig. 3, we can conclude that numerical result obtained by RKPM method is in good agreement with the exact solution.

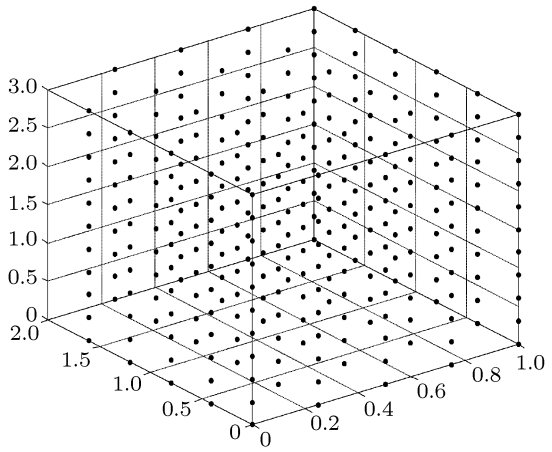


Fig. 2. Nodes arrangement for example.

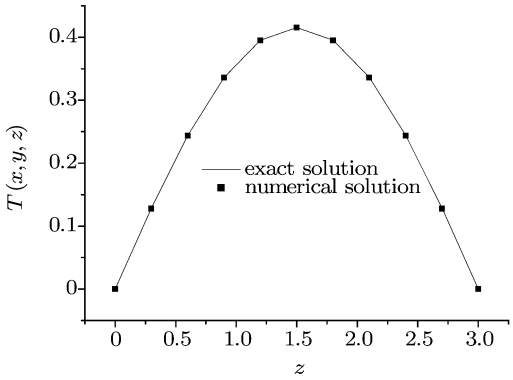


Fig. 3. Numerical solution and exact solution at location $x = 0.75$ and $y = 1.6$.

5. Conclusion

The RKPM was used for solving the 3D steady-state heat conduction problem. The meshless property of RKPM is the most important advantage of this scheme over the traditional mesh dependent techniques such as the finite difference method, finite element method and boundary element method. The meshless nature allows us to solve problems with non-regular geometry. Our numerical results show that the technique is accurate and efficient. The RKPM can be extended for 3D unsteady-state heat conduction problems and nonlinear heat conduction problems.

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