Scattering of scalar light wave from a Gaussian–Schell model medium*

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(Received 5 January 2010; revised manuscript received 26 January 2010)

The scattering of scalar light wave from a random medium with a correlation function of Gaussian–Schell model distribution is studied. It is shown that the properties of the scattered field, i.e., the spectral density and the spectral degree of coherence of the scattered field, are closely related to the properties of the scattering medium, including the scaled effective radius and the scaled correlation length of the correlation function.

Keywords: scattering, Gaussian–Schell model medium, spectral density, coherence

PACC: 4225F, 4225K

1. Introduction

In 1989, Wolf and his collaborators found that the spectrum of light may change when it is scattered from an object.[1] Since then, numerous papers have been published on the subject of light scattering,[2−10] both for continuous media and for a collection of particles, due to the potential application. For example, Shirai and Asakura studied the scattering of light wave from a random medium,[2,3] and Dogariu and Wolf[4] discussed the light scattering from a collection of particles. In addition, the inverse problem, i.e., finding the statistical properties of scattering medium from the properties of the scattered field, is also discussed in Refs. [5]–[7]. Recently, the scattering matrix theory for stochastic scalar field scattered from a random medium has been presented by Korotkova and Wolf,[8] which is also used to discuss the scattering of light wave from a collection of particles, both for deterministic distribution[9] and for random distribution.[10]

When we discuss the scattering of light wave from an object, two kinds of scattering media, i.e., the sphere centred (see, for example, Ref. [8]) and the quasi-homogeneous (see, for example, Ref. [11]), are usually considered. The sphere centred medium has scattering potential $F(\boldsymbol{r},\omega)$ in the form of Gaussian distribution, and the quasi-homogeneous medium features the property that scattering potential $F(\boldsymbol{r},\omega)$ varies much more slowly with position than correlation coefficient $\mu_F(\boldsymbol{r}_1,\boldsymbol{r}_2;\omega)$ varies with difference $\boldsymbol{r}_2-\boldsymbol{r}_1$. In this paper, we consider the scattering of a scalar light wave from a new kind of random medium with a property that both the strength and the correlation coefficient of the correlation function are of Gaussian distribution, which can be referred to as Gaussian–Schell model medium analogous to the definition of the radiation source. We will first derive the formula for the scattering of scalar light wave from a Gaussian–Schell model medium, and then present some numerical results to show the properties of spectral density and the spectral degree of coherence of the scattered field.

2. Theory

Consider a spatially coherent, polychromatic plane wave with spectrum $S^{(\mathrm{i})}(\omega)$, with ω being the frequency, incident on a scattering medium in a direction specified by a real unit vector $\boldsymbol{s}_0$ (see Fig. 1). The cross-spectral density function of the incident field at a pair of points specified by the position vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ may be expressed as (see Subsection 4.1 in Ref. [12].)

$$W^{(\mathrm{i})}(\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{s}_0;\omega) = \langle U^{(\mathrm{i})*}(\boldsymbol{r}_1,\boldsymbol{s}_0;\omega) \times U^{(\mathrm{i})}(\boldsymbol{r}_2,\boldsymbol{s}_0;\omega)\rangle, \quad (1)$$

where the asterisk denotes the complex conjugate and the angular brackets denote ensemble average,

*Project supported by the National Natural Science Foundation of China (Grant No. 10874150), the Natural Science Foundation of Zhejiang Province of China (Grant No. R1090168), and the Program for New Century Excellent Talents in University (Grant No. NCET-07-0760).

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$U^{(\mathrm{i})}(\boldsymbol{r}, \boldsymbol{s}_0; \omega)$ is a member of a statistical ensemble of random function, and given as

$$U^{(\mathrm{i})}(\boldsymbol{r}, \boldsymbol{s}_0; \omega) = a(\omega) \exp(\mathrm{i}\, k \boldsymbol{s}_0 \cdot \boldsymbol{r}), \tag{2}$$

where $a(\omega)$ is, in general, complex and $k = \omega/c$, with c being the light speed in vacuum.

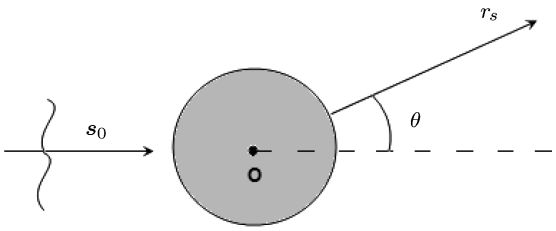


Fig. 1. Illustration of the notations.

Substituting Eq. (2) into Eq. (1), the cross-spectral density function can be expressed as

$$W^{(\mathrm{i})}(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{s}_0; \omega) = S^{(\mathrm{i})}(\omega) \exp(\mathrm{i}\, k \boldsymbol{s}_0 \cdot (\boldsymbol{r}_2 - \boldsymbol{r}_1)), \tag{3}$$

where $S^{(\mathrm{i})}(\omega) = \langle a^*(\omega) a(\omega) \rangle$ is the spectral density of the incident light.

The spectral degree of coherence can be defined as (see Subsection 4.2 in Ref. [12])

$$\mu(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{s}_0; \omega) = \frac{W(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{s}_0; \omega)}{\sqrt{S(\boldsymbol{r}_1, \boldsymbol{s}_0; \omega)} \sqrt{S(\boldsymbol{r}_2, \boldsymbol{s}_0; \omega)}}, \tag{4}$$

where $S(\boldsymbol{r}_j, \boldsymbol{s}_0; \omega) = W(\boldsymbol{r}_j, \boldsymbol{r}_j, \boldsymbol{s}_0, \omega)$ $(j = 1, 2)$ represents the spectral density of the light field.

Let $F(\boldsymbol{r}, \omega)$ be the scattering potential of the medium, and assume that the scattering medium is so weak that the scattering can be analysed in the accuracy of the first-order Born approximation (see Subsubsection 13.1.2 in Ref. [13]). Assume that the correlation function of the scattering potential can be defined as (see Subsubsection 6.3.1 in Ref. [12])

$$C_{\mathrm{F}}(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega) = \langle F^*(\boldsymbol{r}_1, \omega) F(\boldsymbol{r}_2, \omega) \rangle, \tag{5}$$

where the angular brackets denote the average taken over the ensemble of the random medium.

The cross-spectral density function of the scattered field in the far zone, at two points specified by position vectors $r\boldsymbol{s}_1$ and $r\boldsymbol{s}_2$ $(\boldsymbol{s}_1^2 = \boldsymbol{s}_2^2 = 1)$, as shown in Fig. 2, can be expressed as (see Subsubsection 6.3.2 in Ref. [12])

$$W^{(S)}(r\boldsymbol{s}_1, r\boldsymbol{s}_2, \boldsymbol{s}_0; \omega) = \frac{S^{(\mathrm{i})}(\omega)}{r^2} \tilde{C}_{\mathrm{F}}(-\boldsymbol{K}_1, \boldsymbol{K}_2, \omega), \tag{6}$$

where

$$\tilde{C}_{\mathrm{F}}(\boldsymbol{K}_1, \boldsymbol{K}_2, \omega) = \iint C_{\mathrm{F}}(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega) \exp[-\mathrm{i}(\boldsymbol{K}_2 \cdot \boldsymbol{r}_2 + \boldsymbol{K}_1 \cdot \boldsymbol{r}_1)] \mathrm{d}^3 r_1 \mathrm{d}^3 r_2 \tag{7}$$

is the six-dimensional Fourier transform of correlation function $C_{\mathrm{F}}(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega)$, and

$$\boldsymbol{K}_1 = k(\boldsymbol{s}_1 - \boldsymbol{s}_0), \quad \boldsymbol{K}_2 = k(\boldsymbol{s}_2 - \boldsymbol{s}_0). \tag{8}$$

Vectors $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ are analogous to the momentum transfer vectors of quantum mechanical theory of potential scattering. It is clear that by changing the scattering direction and the incidence direction, one can obtain all the Fourier transforms of the correlation function of the scattering medium for $|\boldsymbol{K}_1| \leq 2k$ and $|\boldsymbol{K}_2| \leq 2k$.

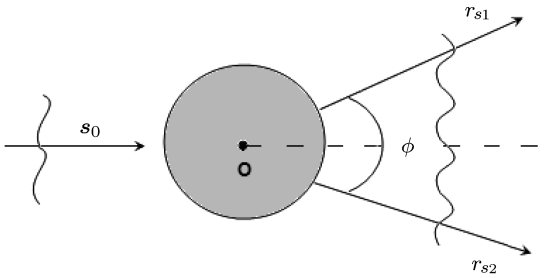


Fig. 2. Observation directions $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$.

Assume that a scalar light wave is scattered from the so-called Gaussian–Schell model media. Analogous to the definition of the radiation source (see Subsection 5.3.1 in Ref. [12], and Subsubsections 5.6.4 in Ref. [14]), the correlation function of the scattering medium can be expressed as

$$C_{\mathrm{F}}(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega) = B \exp\left[-\frac{\boldsymbol{r}_1^2 + \boldsymbol{r}_2^2}{4\sigma^2}\right] \times \exp\left[-\frac{(\boldsymbol{r}_2 - \boldsymbol{r}_1)^2}{2\mu^2}\right], \tag{9}$$

where B is a constant, σ and μ represent the effective radius and the correlation length of the correlation function, respectively.

For simplicity, we make the following variable transforms:

$$\boldsymbol{R}_{\mathrm{S}} = \boldsymbol{r}_2 + \boldsymbol{r}_1, \quad \text{and} \quad \boldsymbol{R}_{\mathrm{D}} = \boldsymbol{r}_2 - \boldsymbol{r}_1. \tag{10}$$

In this case, the six-dimensional spatial frequency vector is naturally represented as

$$\boldsymbol{K}_{\mathrm{S}} = \frac{\boldsymbol{K}_2 + \boldsymbol{K}_1}{2}, \quad \text{and} \quad \boldsymbol{K}_{\mathrm{D}} = \frac{\boldsymbol{K}_2 - \boldsymbol{K}_1}{2}. \tag{11}$$

Substituting Eq. (9) into Eq. (7), and using Eqs. (10) and (11), the six-dimensional Fourier transform of the correlation function can be rewritten as

$$\tilde{C}_{\rm F}(\boldsymbol{K}_1,\boldsymbol{K}_2,\omega)=B\iint\exp\left[-\frac{\boldsymbol{R}_{\rm S}^2+\boldsymbol{R}_{\rm D}^2}{8\sigma^2}\right]\exp\left[-\frac{\boldsymbol{R}_{\rm D}^2}{2\mu^2}\right]\exp[-{\rm i}(\boldsymbol{K}_{\rm S}\cdot\boldsymbol{R}_{\rm S}+\boldsymbol{K}_{\rm D}\cdot\boldsymbol{R}_{\rm D})]{\rm d}^3R_{\rm S}{\rm d}^3R_{\rm D}. \quad (12)$$

Calculating the six-dimensional Fourier transform of Eq. (12) we find

$$\tilde{C}_{\rm F}(\boldsymbol{K}_1,\boldsymbol{K}_2,\omega)=B(2\pi)^3\sigma^3\frac{2^6\sigma^3\mu^3}{(4\sigma^2+\mu^2)^{3/2}}\exp[-2\sigma^2\boldsymbol{K}_{\rm S}^2]\exp\left[-\frac{2\sigma^2\mu^2}{4\sigma^2+\mu^2}\boldsymbol{K}_{\rm D}^2\right]. \quad (13)$$

Substituting Eq. (13) into Eq. (6), and using Eq. (11) again, the cross-spectral density function of the scattering field can be expressed as

$$W^{(\rm S)}(r\boldsymbol{s}_1,r\boldsymbol{s}_2,\boldsymbol{s}_0;\omega)=A\exp\left[-\frac{1}{2}\sigma^2(\boldsymbol{K}_2-\boldsymbol{K}_1)^2\right]\exp\left[-\frac{\sigma^2\mu^2}{2(4\sigma^2+\mu^2)}(\boldsymbol{K}_2+\boldsymbol{K}_1)^2\right], \quad (14)$$

where $A=S^{(\rm i)}(\omega)B(2\pi)^3 2^6\sigma^6\mu^3/[r^2(4\sigma^2+\mu^2)^{3/2}]$.

Let $\boldsymbol{s}_1=\boldsymbol{s}_2=\boldsymbol{s}$, and use Eq. (8), then the cross-spectral density function will be reduced into the spectral density function, which can be expressed as

$$S^{\rm S}(r\boldsymbol{s},\boldsymbol{s}_0;\omega)=A\exp\left[-\frac{2k^2\sigma^2\mu^2}{(4\sigma^2+\mu^2)}(\boldsymbol{s}-\boldsymbol{s}_0)^2\right]. \quad (15)$$

Substituting Eqs. (14) and (15) into Eq. (4), one can readily find that

$$\mu^{\rm S}(r\boldsymbol{s}_1,r\boldsymbol{s}_2,\boldsymbol{s}_0;\omega)=\exp\left[\frac{-2\sigma^4k^2}{(4\sigma^2+\mu^2)}(\boldsymbol{s}_2-\boldsymbol{s}_1)^2\right]. \quad (16)$$

In the next section, we will give some numerical results to show the distribution of the spectral density and the spectral degree of coherence of light field scattered from the Gaussian–Schell model medium.

3. Spectral density and spectral degree of coherence of scattered field

Now we consider the distribution of the spectral density of the scattered field. The variation of the normalized spectral density with θ is discussed. θ is the angle between the incidence direction $\boldsymbol{s}_0$ and the scattering direction $\boldsymbol{s}$, i.e., $\boldsymbol{s}_0\cdot\boldsymbol{s}=\cos\theta$ (see Fig. 1). Figure 3 shows the distributions of the spectral density of the scattered field with three different values of the scaled effective radius $k\sigma$. It is shown that when the scaled effective radius $k\sigma$ increases the effective angular width of the spectral density of the scattered field decreases. Figure 4 exhibits the influence of the scaled correlation length $k\mu$ of the scattering medium on the distribution of the spectral density of the scattering field. It is found that the effective angular width of the spectral density of the scattered field turns smaller with the correlation length of the correlation function increasing.

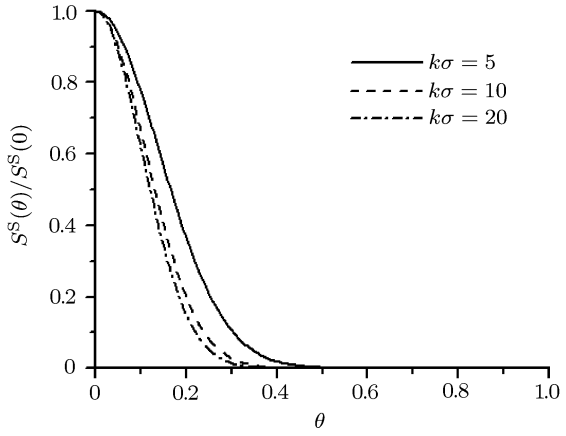


Fig. 3. Normalized spectral densities of scattered field versus angle θ between incidence direction $\boldsymbol{s}_0$ and scattering direction $\boldsymbol{s}$, for three different values of scaled effective radius $k\sigma$. The parameters for calculation are $k=10^7$ and $k\mu=10$.

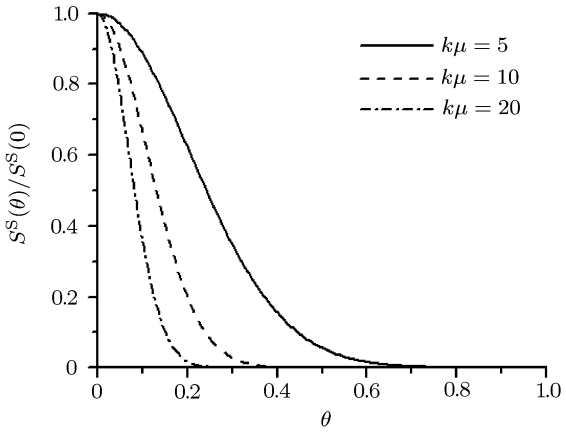


Fig. 4. Normalized spectral densities of scattered field versus angle θ between incidence direction $\boldsymbol{s}_0$ and scattering direction $\boldsymbol{s}$, for three different values of correlation length. The parameters for calculation are $k=10^7$ and $k\sigma=10$.

Now we discuss the behaviours of the spectral degree of coherence of the scattered field. Figures 5 and 6 indicate the variations of the spectral degree of coherence with ϕ, i.e., the angle between a pair of the scattering directions $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$. In the following calculations, we choose $\boldsymbol{s}_1 = \boldsymbol{s}_0$. The influence of scaled effective radius $k\sigma$ on the spectral degree of coherence is presented in Fig. 5. It is found that when scaled effective radius $k\sigma$ increases the effective angular width of the spectral degree of coherence decreases. Figure 6 shows the distributions of the spectral degree of coherence for three different correlation lengths of the correlation function. From Fig. 6 it follows that the effective angular width of the spectral degree of coherence increases with the increase of the correlation length of the correlation function.

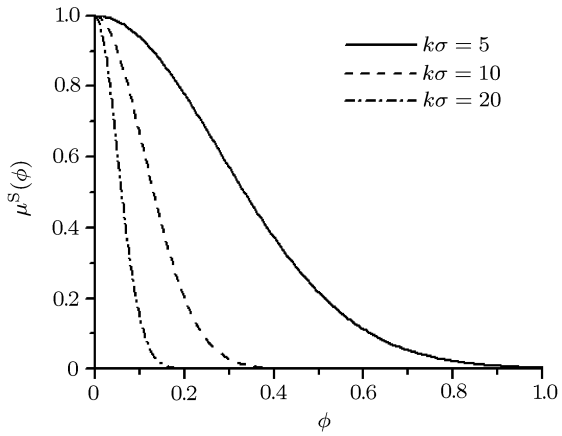


Fig. 5. Spectral degrees of coherence of scattered field versus angle ϕ between a pair of scattering directions $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$, for three different values of scaled effective radius $k\sigma$. The parameters for calculation are $k = 10^7$ and $k\mu = 10$.

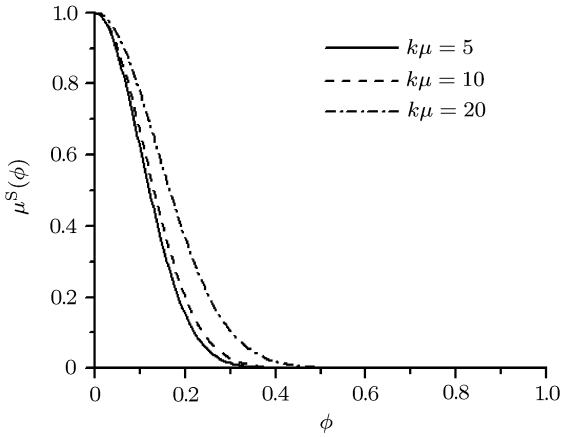


Fig. 6. Spectral degrees of coherence of scattered field versus angle ϕ between a pair of scattering directions $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$, for three different values of correlation length. The parameters for calculation are $k = 10^7$ and $k\sigma = 10$.

4. Conclusion

The scattering of scalar light wave from a Gaussian–Schell model medium is considered. The spectral density and the spectral degree of coherence of the scattered field are discussed. It is shown that properties of the scattered field, i.e., the spectral density and the spectral degree of coherence, are closely related to the properties of the scattering medium, including scaled effective radius $k\sigma$ and scaled correlation length $k\mu$ of the scattering medium. This indicates that one can determine the properties of the scattering medium by considering the spectral density and the spectral degree of coherence of the scattered field.

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