Symmetric bursting behaviour in non-smooth Chua's circuit*

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The dynamics of a non-smooth electric circuit with an order gap between its parameters is investigated in this paper. Different types of symmetric bursting phenomena can be observed in numerical simulations. Their dynamical behaviours are discussed by means of slow-fast analysis. Furthermore, the generalized Jacobian matrix at the non-smooth boundaries is introduced to explore the bifurcation mechanism for the bursting solutions, which can also be used to account for the evolution of the complicated structures of the phase portraits. With the variation of the parameter, the periodic symmetric bursting can evolve into chaotic symmetric bursting via period-doubling bifurcation.

Keywords: non-smooth electric circuit, symmetric bursting, bifurcation mechanism

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1. Introduction

The nonlinear dynamics on multiple time-scale have attracted much attention during recent decades.[1] For the non-dimensional systems, the so-called multiple time-scale implies that the variables or their combination forms can be divided into several groups, which change with an order gap on different time scales. The multiple time-scale phenomena may come from the real slow-fast variation with real time, or the dimensional, the structural, the physical effects in real systems. The type of system can be observed in many practical problems.[2−7] For the two timescale systems, there exist two types of states according to their time scales. The quiescent state corresponding to the subsystem with slow time scale may lead to the spike determined by the subsystem with fast time scale. Bursting phenomena take place when the dynamics alternates between the quiescent states and the spikes. There have been a few approaches to the analysis of bursting behaviour, such as mathematical approach,[8] analog electronic circuits,[9,10] and so on.

However, up to now, most of the theoretical analyses have been restricted to the smooth systems. The bursting phenomena of non-smooth systems are still open problems. In the present paper, we explore a non-smooth circuit, namely, four-order Chua's circuit,[11] which is shown in Fig. 1.

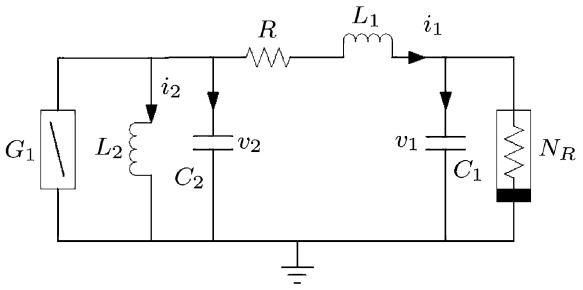


Fig. 1. Four-order Chua's circuit.

The dynamics of the circuit can be described by the following differential equations:[11]

$$\begin{aligned} C_1\dot{v}_1 &= i_1 - g(v_1),\\ C_2\dot{v}_2 &= -G_1 v_2 - i_1 - i_2,\\ L_1\dot{i}_1 &= v_2 - v_1 - Ri_1,\\ L_2\dot{i}_2 &= v_2, \end{aligned} \tag{1}$$

where v_1 and v_2 denote the voltages across capacitors C_1 and C_2 respectively; i_1 and i_2 denote the currents through inductances L_1 and L_2 respectively; $g(v_1)$ represents the current through the nonlinear resistor, and it is expressed as $g(V_1) = G_{\rm b}V_1 + 0.5(G_{\rm a} - G_{\rm b}) \times (|V_1 + B_{\rm P}| - |V_1 - B_{\rm P}|)$.

Here we focus on the non-smooth circuit and consider the dynamics when there exists order gap between the parameters to reveal the bursting phenomena as well as the associated bifurcation mechanism.

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2. Numerical simulation

By employing the transformation[11] $y_1 = v_1/B_{\rm P}$, $y_2 = v_2/B_{\rm P}$, $y_3 = i_1/GB_{\rm P}$, $y_4 = i_2/GB_{\rm P}$, $t = \tau C_2/G$, $\alpha_1 = C_2/C_1$, $\alpha_2 = G_1/G$, $\beta_1 = C_2/L_1G^2$, $\beta_2 = C_2/L_2G^2$, $a = G_{\rm a}/G$, $b = G_{\rm b}/G$, and $G = 1/R$, the dimensionless form of Eq. (1) can be written as

$$\begin{aligned} \dot{y}_1 &= \alpha_1(y_3 - f(y_1)), \\ \dot{y}_2 &= -\alpha_2 y_2 - y_3 - y_4, \\ \dot{y}_3 &= \beta_1(y_2 - y_1 - y_3), \\ \dot{y}_4 &= \beta_2 y_2, \end{aligned} \tag{2}$$

where $f(y_1) = by_1 + 0.5(a-b)(|y_1+1| - |y_1-1|)$.

In order to reveal the dynamical behaviours of this system, we fix the parameters at $\alpha_1 = 2.14$, $\beta_1 = 0.177$, $\beta_2 = 0.003$, $a = -0.42$, and $b = 1.2$ in this paper and take α_2 as a bifurcation parameter. When $\alpha_2 = -0.18$ and $\alpha_2 = -0.28$, the periodic bursting can be obtained by numerical simulation (see Fig. 2). The combination of large-amplitude and small-amplitude oscillation forms L^S solution in the circuit, where L and S correspond to the times of the large-amplitude and the small-amplitude oscillation during one period respectively.

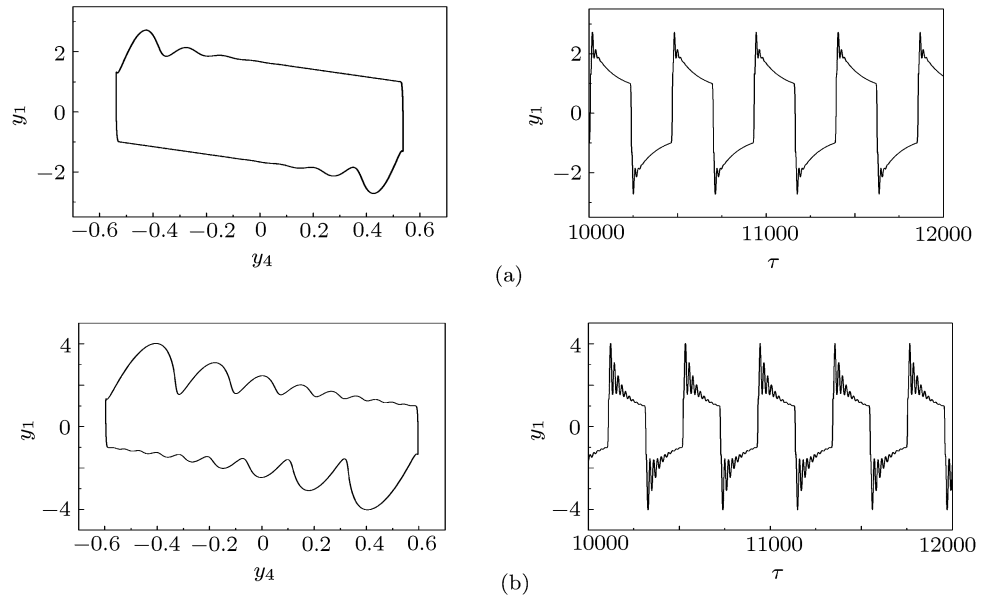


Fig. 2. Phase portraits and time series of system (2) for $\alpha_2 = -0.18$ (a) and -0.28 (b).

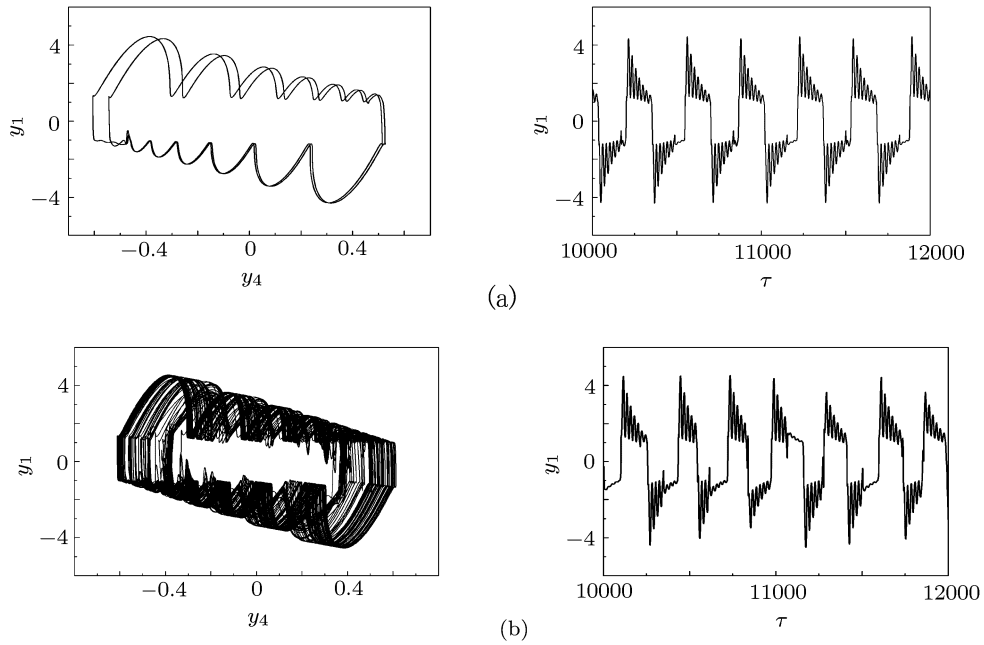


Fig. 3. Phase portraits and time series of system (2) for $\alpha_2 = -0.300$ (a) and -0.304 (b).

From Fig. 2, one can see that period-adding bifurcation occurs with the value of α_2 decreasing, though the period of the movement is almost the same, which implies that S increases quickly for L^S solution.

When α_2 decreases to -0.300, the value of L doubles in each period for L^S oscillation. A cascading of period-doubling bifurcations can be observed with the value of parameter α_2 further decreasing, which leads the system to chaotic bursting at $\alpha_2 = -0.304$ (see Fig. 3).

In the above parameter conditions, β_2 is relatively small, so the dynamics of y_4 changes relatively slowly while the first three equations of Eq. (2) describe the relatively fast change. Then the full system (2) is divided into two subsystems, namely, slow subsystem (y_4) and fast subsystem (y_1, y_2, y_3). Thus the dynamics of system (2) may exhibit a fast-slow effect, which often behaves in a periodic state characterized by mixed mode oscillations. The type of oscillation, which has been observed in the numerical simulations for the circuit, can also be called bursting phenomenon.[8] Revealing the mechanism of the movement is very important for understanding the complicated phenomena of the vector field. In the following, we will investigate the details of typical bifurcation in the evolution of dynamics.

3. Bifurcation mechanism for non-smooth system

According to the slow-fast analysis,[8] in order to analyse the dynamical behaviour of the bursting and classify the burster, slow variable y_4 can be treated as a bifurcation parameter of the fast subsystem, adjusting the behaviour of the system.

The fast subsystem (y_1, y_2, y_3) is a piecewise linear system. Two switching boundaries exist, denoted by $\Sigma_{1,2} : y_1 = \pm 1$. Furthermore, three equilibria, expressed as $E_0 : \{y_{10},\ (1+a)y_{10},\ ay_{10}\}$ for $-1.0 < y_1 < 1.0$, where $y_{10} = -y_4/[a + \alpha_2(1+a)]$; $E_- : \{y_{10},\ (1+b)y_{10} - (a-b),\ by_{10} - (a-b)\}$ for $y_1 \leq -1.0$, where $y_{10} = -[y_4 - (a-b)(\alpha_2+1)]/[b + \alpha_2(1+b)]$; $E_+ : \{y_{10}, (1+b)y_{10}+(a-b), by_{10}+(a-b)\}$ for $y_1 \geq 1.0$, where $y_{10} = -[y_4 + (a-b)(\alpha_2+1)]/[b + \alpha_2(1+b)]$, can be observed. The stability of E_0 is determined by the eigenvalues of J_0, while the stabilities of $E_\pm$ are determined by those of $J_\pm$, where J_0 and $J_\pm$ are expressed as

$$J_0 = \begin{vmatrix} -a\alpha_1 & 0 & \alpha_1 \\ 0 & -\alpha_2 & -1 \\ -\beta_1 & \beta_1 & -\beta_1 \end{vmatrix} \text{ and } J_\pm = \begin{vmatrix} -b\alpha_1 & 0 & \alpha_1 \\ 0 & -\alpha_2 & -1 \\ -\beta_1 & \beta_1 & -\beta_1 \end{vmatrix}.$$

For example, for $\alpha_2 = -0.18$,

$$J_0 = \begin{vmatrix} 0.8988 & 0 & 2.14 \\ 0 & 0.18 & -1 \\ -0.177 & 0.177 & -0.177 \end{vmatrix}$$

with eigenvalues $\lambda_1^0 = 0.5829$, $\lambda_{2,3}^0 = 0.1594 \pm 0.5615\,\mathrm{i}$,

$$J_\pm = \begin{vmatrix} -2.568 & 0 & 2.14 \\ 0 & 0.18 & -1 \\ -0.177 & 0.177 & -0.177 \end{vmatrix}$$

with eigenvalues $\lambda_1^\pm = -2.4029$, $\lambda_{2,3}^\pm = -0.0810 \pm 0.3466\,\mathrm{i}$.

Obviously, the equilibrium $E_\pm$ are asymptotically stable focus points, while E_0 is an unstable focus point.

There exist various definitions of bifurcation in the literature, which are all consistent with each other for smooth dynamical systems, but become inconsistent when applied to non-smooth systems.[12] So we should analyse the non-smooth bifurcation in the fast subsystem especially. Because the vector field is non-smooth but still continuous, we could use the generalized differential of Clarke to set up a generalized Jacobian matrix.[13] The generalized Jacobian of the fast subsystem, expressed as $J(\pm 1) = \{qJ_0 + (1-q)J_\pm, \forall q \in [0,1]\}$, at the bifurcation point is the closed convex hull of the Jacobians on each side of the switching boundary. The eigenvalues of $J(\pm 1)$ are set-valued and form a path in the complex plane with q as the path parameter.

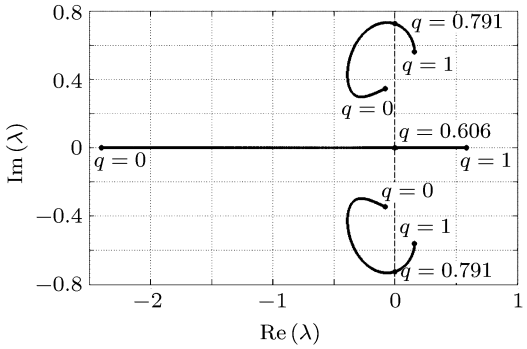


Fig. 4. Set of eigenvalues of the generalized Jacobian for $\alpha_2 = -0.18$.

Now we still take the case for $\alpha_2 = -0.18$ as an example to analyse the bifurcation mechanism. The

path of eigenvalues of the generalized Jacobian at $y_1 = \pm 1.0$ is depicted in Fig. 4, from which one may find that the eigenvalues of $J(q)$ are purely imaginary for $q = 0.791$, and one eigenvalue crosses the origin for $q = 0.606$. The path of the eigenvalues of $J(\pm 1)$ shows that the discontinuous bifurcation is a multiple crossing bifurcation,[13] in the sense that the eigenvalues cross the imaginary axis more than once during their jump.

By treating y_4 as a bifurcation parameter of the fast subsystem for $\alpha_2 = -0.18$, the bifurcation diagram for equilibria of the fast subsystem can be seen in Fig. 5, where the equilibria of the fast subsystem form a Z-shaped curve, and the trajectory of the periodic bursting is also appended. The Z-shaped curve is made up of three branches, i.e. the solid upper branch, the dot middle branch and the solid lower branch. The solid branches represent stable focuses while the dot branch represents unstable focus. A and B are equilibrium points of bifurcation. They are located at the switching boundaries $\Sigma_{1,2} : y_1 = \pm 1$ respectively. The dash lines, y_{n1}, y_{n2}, y_{n3} are the nullclines of the slow variable y_4. Above y_{n1} (below y_{n3}), we have $\dot{y}_4 > 0$ ($\dot{y}_4 < 0$).

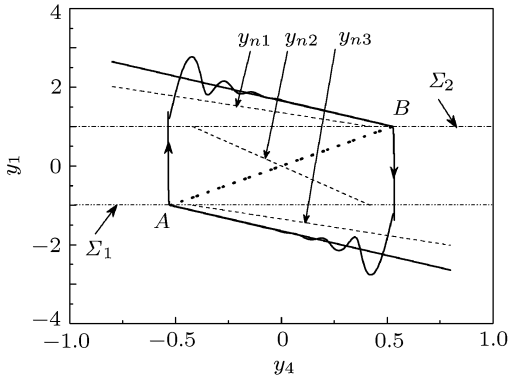


Fig. 5. Symmetric non-smooth fold/fold bursting of focus–focus type as observed in slow-fast circuit (2) for $\alpha_2 = -0.18$.

We will analyse the mechanism of the bursting shown in Fig. 5. The arrows on the phase portrait indicate the directions of the flow. We set A as the starting point, which is located at the switching boundary Σ_1 ($\Sigma_1 : y_1 = -1$). To the left of A, the stable focus on the upper branch is the only attractor of the fast subsystem. Thus, beyond that point, the orbit of the full system switches to the stable upper branch. Then, y_4 increases since $\dot{y}_4 > 0$. That is to say, the orbit follows the stable upper branch to the right and oscillates with a frequency of 0.3466 determined by the imaginary part of the complex eigenvalues for E_+. Since the subsystem, (y_1, y_2, y_3), is a non-smooth system, the discontinuous bifurcation is related to the switching boundaries in the vector field. From Fig. 5, we can see that the orbit meets the switching boundary Σ_2 at point B when the value of y_4 increases to $y_{4B} \approx 0.5268$, which leads the trajectory jump back to the stable lower branch via multiple crossing bifurcation. Two conventional bifurcations, namely a Hopf bifurcation and a turning point bifurcation evolve in the combined bifurcation.[13] Here just the turning point bifurcation leads the instability of the equilibria. Then the value of y_4 decreases since $\dot{y}_4 < 0$ and the orbit follows the stable lower branch to the left to begin the next revolution.

Fold/fold bursting is defined in Ref. [8]. It is of a focus-node type as the corresponding up-state is a stable focus while the down-state is a stable node. The spiking-like oscillations occur only in the up-state because the rate of the convergence to the up-state is relatively small.[8]

It is pointed out that though the bursting shown in Fig. 5 occurs in a non-smooth system, the bifurcation mechanism is similar to that of fold/fold bursting. However, the bursting shown in Fig. 5 is of focus–focus type as the corresponding up-state and down-state are both stable focuses. The rates of the convergence to the up-state and to the down-state are both small, which leads the spiking-like oscillations to occur not only at the up-state but also at the down-state passage. Considering the non-smoothness and the symmetry, we may call it symmetric non-smooth fold/fold bursting of focus–focus type.

We now account for the period-adding phenomena of the phase portraits (plotted in Fig. 6) with the decrease of parameter α_2. For comparison, we list the details of the eigenvalues associated with $E_\pm$ for different values of α_2. When $\alpha_2 = -0.18$, $\lambda_1 = -2.4029$, and $\lambda_{2,3} = -0.0810 \pm 0.3466\,\mathrm{i}$, while $\lambda_1 = -2.4027$ and $\lambda_{2,3} = -0.0311 \pm 0.3018\,\mathrm{i}$ for $\alpha_2 = -0.28$. From the former computation results of the complex eigenvalues, one may find that the imaginary part of the complex eigenvalues keeps almost constant, which implies that the oscillating frequencies of the trajectories from D_i to E_i ($i = 1, 2$) are nearly the same. However the absolute value of real part of the complex eigenvalues decreases quickly with the decrease of α_2 and the distance from D_i to E_i ($i = 1, 2$) still keeps almost constant, resulting in the rapid increase of the oscillating time in the spiking-like oscillation.

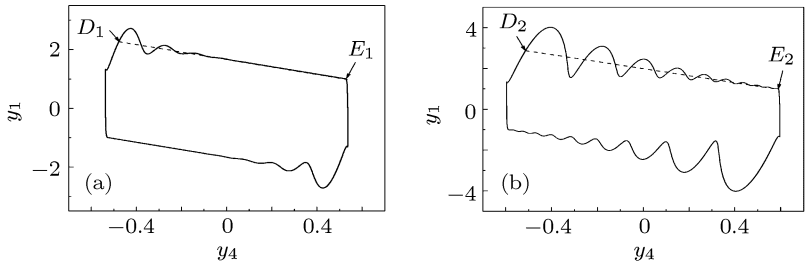


Fig. 6. Phase portraits of system (2) for $\alpha_2 = -0.18$ (a) and -0.28 (b).

4. Conclusions

Symmetric bursting phenomena can be found for a non-smooth electric circuit when there exists an order gap between the parameters. Since its non-smoothness, the bifurcation mechanism of this system is different from that of a smooth system. In particular, the bifurcation is related to the switching boundaries in the vector field. Discontinuous bifurcation in the combination of the Hopf bifurcation and the turning point bifurcation may occur at the non-smooth boundaries. For the special system in this paper, only the turning point bifurcation connects the fast passage (spiking-like oscillation) and the slow passage (quiescent state) to form the bursting behaviour of the system. Besides, with the variation of the parameter, the change rate of the real parts of the eigenvalues for the fast subsystem is much larger than that of the imaginary parts, which may lead to the period-adding phenomena of the portraits.

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