Robust H_∞ control of piecewise-linear chaotic systems with random data loss*

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This paper studies the problem of robust H_∞ control of piecewise-linear chaotic systems with random data loss. The communication links between the plant and the controller are assumed to be imperfect (that is, data loss occurs intermittently, which appears typically in a network environment). The data loss is modelled as a random process which obeys a Bernoulli distribution. In the face of random data loss, a piecewise controller is designed to robustly stabilize the networked system in the sense of mean square and also achieve a prescribed H_∞ disturbance attenuation performance based on a piecewise-quadratic Lyapunov function. The required H_∞ controllers can be designed by solving a set of linear matrix inequalities (LMIs). Chua's system is provided to illustrate the usefulness and applicability of the developed theoretical results.

Keywords: chaos, H_∞ control, piecewise-linear systems, piecewise-quadratic Lyapunov functions, random data loss

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1. Introduction

The theory of piecewise-linear system has been a subject of research in circuits and control system community for some time.[1−5] On one hand, piecewise-linear systems constitute a special class of hybrid systems which arise often in circuits, control systems and biosystems when piecewise-linear components are present. These components include dead-zone, saturation, relays, and hysteresis.[6−9] On the other hand, many other classes of nonlinear systems can be approximated by piecewise-linear systems, thus providing a powerful means to analyse and design more general nonlinear control systems.[10−12]

A number of results have been obtained recently on the analysis and design of piecewise-linear systems. For example, the authors in Ref. [1] studied the well-posedness of piecewise-linear systems and some necessary or sufficient conditions for well-posedness have been developed. The authors in Refs. [2] and [3] presented results on stability and optimal performance analysis for piecewise-linear systems based on a piecewise-continuous Lyapunov function. It has been shown that lower bounds, as well as upper bounds, on the optimal control cost can be obtained by semidefinite programming, and the framework of piecewise-linear systems can be used to analyse smooth nonlinear systems with arbitrary accuracy. The authors in Ref. [8] presented results on the stability analysis of piecewise discrete-time linear systems and robust H_∞ control for continuous-time piecewise-linear systems based on piecewise Lyapunov functions. In recent decades, an increasing interest has been generated in chaotic control and synchronization (see, e.g., Refs. [13]–[26] and references therein). As a special type chaotic systems, the control of piecewise-linear chaotic system attracts more research interest.[26,27]

Most of the aforementioned results concerning piecewise-linear systems are based on an implicit assumption that the communications between the physical plant and controller are perfect. In many practical situations, however, data loss phenomena may often appear, that is, some measurements or control inputs will be lost during their transmission. This is more ob-

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vious and important for the emerging networked control systems (NCSs) where feedback control loops are closed via digital communication channels. Recently, NCSs have received increasing attention because of their many advantages over traditional systems, such as low cost, reduced weight and power requirements, simple installation and maintenance, and high reliability. As a result, the modeling, analysis, and control of NCSs have given rise to problems including network-induced delay, data loss, and quantization, has appeared to be a topic of significant interest to the control community (see, e.g., Refs. [28]–[33] and references therein). In particular, data loss is an important issue to be taken into consideration for the analysis and synthesis of communication-based control systems and many results are available. It is worth while mentioning that most of these results are concerned with linear control systems, and to the best of the authors' knowledge, up to now no effort has been made about how to analyse and synthesize piecewise-linear systems with unreliable communication links. The control of piecewise-linear systems in the presence of data loss is an important problem but still remains open and unsolved.

In this paper, we consider the problem of robust H_∞ control for uncertain piecewise-linear networked systems with both random sensor-to-controller and controller-to-actuator data loss based on a piecewise-quadratic Lyapunov function.[19,34–38] The random data loss is modelled as a stochastic variable satisfying a Bernoulli distribution. A piecewise state feedback controller is designed such that the closed-loop NCS is robustly stochastically stable, and the prescribed H_∞ disturbance attenuation performance is also achieved. It is shown that the controller design problem under consideration is solvable if certain linear matrix inequalities (LMIs) are feasible. An example is exploited to demonstrate the effectiveness of the proposed LMI approach.

The rest of this paper is organized as follows. The problem formulation is presented in Section 2. The H_∞ performance analysis and controller design for such systems are presented in Section 3. Simulation example is shown in Section 4. Finally, conclusions are given in Section 5.

Notations Throughout this paper, for symmetric matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$, the notation $\boldsymbol{X} \geq \boldsymbol{Y}$ (respectively, $\boldsymbol{X} > \boldsymbol{Y}$) means that the matrix $\boldsymbol{X} - \boldsymbol{Y}$ is positive-semidefinite (respectively, positive-definite). I is the identity matrix with appropriate dimensions. $\boldsymbol{M}^{\mathrm{T}}$ represents the transpose of the matrix $\boldsymbol{M}$. $\mathbb{E}\{\cdot\}$ denotes the expectation operator with respect to some probability measure $\mathcal{P}$. The notation $l_2[0,T]$ is used for vector-valued functions, that is, we say that $\boldsymbol{z} : [0,N] \to \boldsymbol{R}^n$ is in $l_2[0,N]$ if $\sum_{k=0}^{N} \boldsymbol{z}^{\mathrm{T}}(k)\boldsymbol{z}(k) < \infty$ and its l_2-norm is defined as $\|\boldsymbol{z}\|_2 = [\sum_{k=0}^{\infty} \|\boldsymbol{z}\|^2]^{1/2}$. Matrices, if their dimensions are not explicitly stated, are assumed to have compatible dimensions for algebraic manipulation.

2. System descriptions and preliminaries

Consider an uncertain piecewise-linear system with unreliable communication links. The physical plant is represented by an uncertain piecewise-linear model, and unreliable communications exist in both the downlink (sensor-to-controller) and uplink (controller-to-actuator). After sampling, we model the physical plant in the following form:

$$\begin{aligned} \boldsymbol{x}(k+1) &= [\boldsymbol{A}_l + \Delta\boldsymbol{A}_l(k)]\boldsymbol{x}(k) + [\boldsymbol{B}_l + \Delta\boldsymbol{B}_l(k)]\boldsymbol{u}(k) \\ &\quad + [\boldsymbol{C}_l + \Delta\boldsymbol{C}_l(k)]\boldsymbol{v}(k) + \boldsymbol{\alpha}_l, \\ \boldsymbol{y}(k) &= \boldsymbol{H}_l\boldsymbol{x}(k) + \boldsymbol{G}_l\boldsymbol{u}(k), \\ &\quad \text{for } \boldsymbol{x} \in X_l,\ l \in L, \end{aligned} \tag{1}$$

where $\{X_l\}_{l\in L} \subseteq \boldsymbol{R}^n$ denotes a partition of the state space into a number of closed polyhedral regions, L is the index set of the regions, $\boldsymbol{x}(k) \in \boldsymbol{R}^n$ is the state variable, $\boldsymbol{u}(k) \in \boldsymbol{R}^p$ is the system input variable, $\boldsymbol{y}(k) \in \boldsymbol{R}^r$ is the controlled output, $\boldsymbol{v}(k) \in \boldsymbol{R}^q$ is the external disturbance which belongs to $l_2[0,\infty)$, $(\boldsymbol{A}_l, \boldsymbol{B}_l, \boldsymbol{C}_l, \boldsymbol{H}_l, \boldsymbol{G}_l, \boldsymbol{\alpha}_l)$ is the l-th nominal local linear model of the system, $\boldsymbol{\alpha}_l$ is the offset term, and $(\Delta\boldsymbol{A}_l(k), \Delta\boldsymbol{B}_l(k), \Delta\boldsymbol{C}_l(k))$ is the uncertainty term of the l-th local model of the system. In this paper, the uncertainty terms are assumed to be of the form

$$\begin{bmatrix} \Delta\boldsymbol{A}_l(k) \\ \Delta\boldsymbol{B}_l(k) \\ \Delta\boldsymbol{C}_l(k) \end{bmatrix} = \boldsymbol{M}_l\boldsymbol{F}(k) \begin{bmatrix} \boldsymbol{N}_{1l} \\ \boldsymbol{N}_{2l} \\ \boldsymbol{N}_{3l} \end{bmatrix}, \tag{2}$$

where $\boldsymbol{M}_l$, $\boldsymbol{N}_{1l}$, $\boldsymbol{N}_{2l}$ and $\boldsymbol{N}_{3l}$ are known real constant matrices and $F(k)$ is an unknown time-varying matrix function satisfying

$$\boldsymbol{F}(k)^{\mathrm{T}}\boldsymbol{F}(k) \leq I \ \ \forall k. \tag{3}$$

It is assumed that all the elements of $\boldsymbol{F}(k)$ are Lebesgue measurable.

Remark 1 It is noted that the system models defined in Eq. (1) are in fact affine instead of linear. The

offset term α_l often exists in the case of linearization around a nonzero operating point.

In this work, the polyhedral region is assumed to be characterized by

$$X_l := \{\boldsymbol{x} | \underline{d}_l \preceq \boldsymbol{c}_l^{\mathrm{T}} \boldsymbol{x} \preceq \overline{d}_l\}$$

with $\preceq$ denoting componentwise inequality. These slabs can be precisely represented by a degenerate ellipsoid,[36]

$$X_l = \{\boldsymbol{x} | \|\boldsymbol{E}_l \boldsymbol{x} + \boldsymbol{f}_l\| \leq 1\},$$

where $\boldsymbol{E}_{lk} = 2\boldsymbol{c}_l^{\mathrm{T}}/(\overline{d}_{lk} - \underline{d}_{lk})$ and $f_{lk} = -(\overline{d}_{lk} + \underline{d}_{lk})/(\overline{d}_{lk} - \underline{d}_{lk})$, and k is the index of column element. Then we have the following LMI description for this ellipsoidal region:

$$\begin{bmatrix} \boldsymbol{x} \\ 1 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \boldsymbol{E}_l^{\mathrm{T}} \boldsymbol{E}_l & \boldsymbol{E}_l^{\mathrm{T}} \boldsymbol{f}_l \\ \boldsymbol{f}_l^{\mathrm{T}} \boldsymbol{E}_l & \boldsymbol{f}_l^{\mathrm{T}} \boldsymbol{f}_l - 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ 1 \end{bmatrix} \leq 0. \tag{4}$$

According to the relationship with the origin, the partitioned regions can be separated into two classes $L = L_0 \bigcup L_1$, where L_0 represents the set of regions with $\boldsymbol{f}_l \boldsymbol{f}_l^{\mathrm{T}} - I \leq 0$ which contains the origin and L_1 the set of regions otherwise.

For the control of the uncertain piecewise-linear system (1), we consider the following piecewise controller as

$$\bar{\boldsymbol{u}}(k) = \boldsymbol{K}_l \bar{\boldsymbol{x}}(k), \quad \boldsymbol{x}(k) \in X_l. \tag{5}$$

Here $\bar{\boldsymbol{x}}(k) \in \boldsymbol{R}^n$ is the input of the controller; $\bar{\boldsymbol{u}}(k) \in \boldsymbol{R}^p$ is the output of the controller; $\boldsymbol{K}_l$ is the controller gain. Due to possible data loss between the physical plant and the controller, the input of the controller is no longer the same as the measurement of the plant (that is, $\bar{\boldsymbol{x}}(k) \neq \boldsymbol{x}(k)$), and the input of the plant is no longer the same as the output of the controller (that is, $\boldsymbol{u}(k) \neq \bar{\boldsymbol{u}}(k)$). In this paper, we model the data loss phenomena via a stochastic approach such that

$$\boldsymbol{u}(k) = e(k) \boldsymbol{K}_l \boldsymbol{x}(k),$$

where $e(k)$ is a Bernoulli distributed process with

$$\begin{aligned} &\mathrm{Prob}\{e(k) = 1\} = \mathbb{E}\{e(k)\} = \bar{e}, \\ &\mathrm{Prob}\{e(k) = 0\} = 1 - \mathbb{E}\{e(k)\} = 1 - \bar{e}. \end{aligned}$$

Then, we have

$$\begin{aligned} \boldsymbol{u}(k) &= e(k) \boldsymbol{K}_l \boldsymbol{x}(k) \\ &= \bar{e} \boldsymbol{K}_l \boldsymbol{x}(k) + \tilde{e}(k) \boldsymbol{K}_l \boldsymbol{x}(k), \end{aligned} \tag{6}$$

where $\tilde{e}(k) = e(k) - \bar{e}$, $\mathbb{E}\{\tilde{e}(k)\} = 0$ and $\mathbb{E}\{\tilde{e}(k)\tilde{e}(k)\} = \bar{e}(1 - \bar{e})$. With control law (5) and by lumping the affine term into the disturbance term of the first equation of Eq. (1), we have the following closed-loop system,

$$\begin{aligned} \boldsymbol{x}(k+1) &= [\bar{\boldsymbol{A}}_l + (\bar{e} + \tilde{e}(k)) \bar{\boldsymbol{B}}_l \boldsymbol{K}_l] \boldsymbol{x}(k) + \bar{\boldsymbol{C}}_l \boldsymbol{v}(k) \\ &\equiv \bar{\boldsymbol{A}}_{cl} \boldsymbol{x}(k) + \bar{\boldsymbol{C}}_l \boldsymbol{v}(k), \\ \boldsymbol{y}(k) &= [\boldsymbol{H}_l + \boldsymbol{G}_l \boldsymbol{K}_l] \boldsymbol{x}(k) \equiv \boldsymbol{H}_{cl} \boldsymbol{x}(k), \\ &\quad \text{for } \boldsymbol{x} \in X_l,\ l \in L, \end{aligned} \tag{7}$$

where $\bar{\boldsymbol{A}}_l = \boldsymbol{A}_l + \Delta \boldsymbol{A}_l(k)$, $\bar{\boldsymbol{B}}_l = \boldsymbol{B}_l + \Delta \boldsymbol{B}_l(k)$, $\bar{\boldsymbol{C}}_l = \boldsymbol{C}_l + \Delta \boldsymbol{C}_l(k)$.

Before formulating the problem to be dealt with in this paper, we first introduce the following definitions.

Definition 1 The closed-loop system of Eq. (7) with $\boldsymbol{v}(t) = 0$ is said to be robustly stochastically stable in the mean square if for any initial condition $\boldsymbol{x}(0)$, there is a $\boldsymbol{W} > 0$ such that

$$\mathbb{E}\left\{\sum_{k=0}^{\infty} \|\boldsymbol{x}(k)\|^2\right\} \leq \boldsymbol{x}(0)^{\mathrm{T}} \boldsymbol{W} \boldsymbol{x}(0).$$

Definition 2 Given a scalar $\gamma > 0$, the closed-loop system (7) is said to be robustly stochastically stable with disturbance attenuation γ if it is robustly stochastically stable and under zero initial condition, the controlled output $\boldsymbol{y}(k)$ satisfies

$$\|\boldsymbol{y}\|_E \leq \gamma \|\boldsymbol{v}\|_2 \tag{8}$$

for all nonzero $\boldsymbol{v} \in l_2[0, \infty)$, where

$$\|\boldsymbol{y}\|_E = \mathbb{E}\left\{\sqrt{\sum_{k=0}^{\infty} \|\boldsymbol{y}(k)\|^2}\right\}.$$

Lemma 1[39] Given real matrices $\boldsymbol{P}$, $\boldsymbol{M}$ and $\boldsymbol{N}$ of appropriate dimensions and with $\boldsymbol{P} = \boldsymbol{P}^{\mathrm{T}}$, then

$$\boldsymbol{P} + \boldsymbol{M} \boldsymbol{F}(k) \boldsymbol{N} + \boldsymbol{N}^{\mathrm{T}} \boldsymbol{F}^{\mathrm{T}}(k) \boldsymbol{M}^{\mathrm{T}} < 0$$

holds for all $\boldsymbol{F}(k)$ satisfying $\boldsymbol{F}^{\mathrm{T}}(k) \boldsymbol{F}(k) \leq \boldsymbol{I}$ if and only if for some $\varepsilon > 0$,

$$\boldsymbol{P} + \varepsilon^{-1} \boldsymbol{M} \boldsymbol{M}^{\mathrm{T}} + \varepsilon \boldsymbol{N}^{\mathrm{T}} \boldsymbol{N} < 0.$$

3. Main results

In this section, we consider the H_∞ control of uncertain piecewise-linear system (1) with random data loss. First, we give a condition guaranteeing the stochastic stability and H_∞ performance for the closed-loop systems in Eq. (7) is guaranteed.

Theorem 1 (H_∞ performance analysis) Given a scalar $\gamma > 0$. Consider the closed-loop piecewise-linear system in Eq. (7) with the given controller gains $\boldsymbol{K}_l$, $l \in L$. If there exist two sets of scalars $\nu_l > 0$, $\varepsilon_l > 0$ and matrices $\boldsymbol{P}_l > 0$, such that the following LMIs are satisfied:

$$\begin{bmatrix} -\boldsymbol{P}_l & 0 & \boldsymbol{\Phi}_1 & \boldsymbol{\Phi}_2 & \boldsymbol{\Phi}_3 \\ * & -\gamma^2\boldsymbol{I} & \boldsymbol{C}_l^{\mathrm{T}}\boldsymbol{P}_j & 0 & 0 \\ * & * & -\boldsymbol{P}_j & 0 & 0 \\ * & * & * & -\boldsymbol{P}_j & 0 \\ * & * & * & * & -\boldsymbol{I} \end{bmatrix} < 0 \tag{9}$$

for all $l \in L_0$, and

$$\begin{bmatrix} -\boldsymbol{P}_l & 0 & \boldsymbol{\Phi}_1 & \boldsymbol{\Phi}_2 & \boldsymbol{\Phi}_3 & \boldsymbol{E}_l^{\mathrm{T}} \\ * & -\gamma^2\boldsymbol{I} & \boldsymbol{C}_l^{\mathrm{T}}\boldsymbol{P}_j & 0 & 0 & 0 \\ * & * & -\boldsymbol{P}_j & 0 & 0 & 0 \\ * & * & * & -\boldsymbol{P}_j & 0 & 0 \\ * & * & * & * & -\boldsymbol{I} & 0 \\ * & * & * & * & * & -\nu_l(\boldsymbol{f}_l\boldsymbol{f}_l^{\mathrm{T}} - \boldsymbol{I}) \end{bmatrix} < 0 \tag{10}$$

for all $l \in L_1$, where $\boldsymbol{\Phi}_1 = (\bar{\boldsymbol{A}}_l + \bar{e}\bar{\boldsymbol{B}}_l\boldsymbol{K}_l)^{\mathrm{T}}\boldsymbol{P}_j$, $\boldsymbol{\Phi}_2 = \sqrt{\bar{e}(1-\bar{e})}\boldsymbol{K}_l^{\mathrm{T}}\bar{\boldsymbol{B}}_l^{\mathrm{T}}\boldsymbol{P}_j$, $\boldsymbol{\Phi}_3 = (\boldsymbol{H}_l + \boldsymbol{G}_l\boldsymbol{K}_l)^{\mathrm{T}}$, then the closed-loop uncertain piecewise-linear system is robustly stochastically stable in the mean-square and the H_∞ norm constraint (8) is achieved.

Proof Here, we first prove the stochastic stability of the piecewise-linear closed-loop system with $\boldsymbol{v}(k) \equiv 0$. To obtain a less conservative stability result, we use the following piecewise quadratic Lyapunov function:

$$\boldsymbol{V}(\boldsymbol{x}(k)) = \boldsymbol{x}(k)^{\mathrm{T}}\boldsymbol{P}_l\boldsymbol{x}(k),\ \boldsymbol{x} \in X_l,\ l \in L, \tag{11}$$

where $\boldsymbol{P}_l > 0$ are matrices to be determined. Without loss of generality, $\boldsymbol{x}(k) \in X_l$ and $\boldsymbol{x}(k+1) \in X_j$, $l \in L$ and $j \in L$. Here, we just consider the proof of the case that $l \in L_1$, from which the proof of the case $l \in L_0$ can be similarly derived. Along the trajectory of the closed-loop system in Eq. (7), we have

$$\begin{aligned} &\Delta\boldsymbol{V}(k) \\ &= \mathbb{E}\{\boldsymbol{V}(k+1)|\boldsymbol{x}(k)\} - \boldsymbol{V}(\boldsymbol{x}(k)) \\ &= \mathbb{E}\{\boldsymbol{x}^{\mathrm{T}}(k)\bar{\boldsymbol{A}}_{cl}^{\mathrm{T}}\boldsymbol{P}_j\bar{\boldsymbol{A}}_{cl}\boldsymbol{x}(k)|\boldsymbol{x}(k)\} - \boldsymbol{x}^{\mathrm{T}}(k)\boldsymbol{P}_l\boldsymbol{x}(k) \\ &= \boldsymbol{x}^{\mathrm{T}}(k)\left[(\bar{\boldsymbol{A}}_l + \bar{e}\bar{\boldsymbol{B}}_l\boldsymbol{K}_l)^{\mathrm{T}}\boldsymbol{P}_j(\bar{\boldsymbol{A}}_l + \bar{e}\bar{\boldsymbol{B}}_l\boldsymbol{K}_l)\right. \\ &\quad \left. + \bar{e}(1-\bar{e})\boldsymbol{K}_l^{\mathrm{T}}\bar{\boldsymbol{B}}_l^{\mathrm{T}}\boldsymbol{P}_j\bar{\boldsymbol{B}}_l\boldsymbol{K}_l - \boldsymbol{P}_l\right]\boldsymbol{x}(k) \\ &\le \begin{bmatrix} \boldsymbol{x}(k) \\ 1 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \tilde{\boldsymbol{\Phi}} - \lambda_l\boldsymbol{E}_l^{\mathrm{T}}\boldsymbol{E}_l & -\lambda_l\boldsymbol{E}_l^{\mathrm{T}}\boldsymbol{f}_l \\ -\lambda_l\boldsymbol{f}_l^{\mathrm{T}}\boldsymbol{E}_l & -\lambda_l(\boldsymbol{f}_l^{\mathrm{T}}\boldsymbol{f}_l - 1) \end{bmatrix} \\ &\quad \times \begin{bmatrix} \boldsymbol{x}(k) \\ 1 \end{bmatrix}, \end{aligned} \tag{12}$$

where λ_l is an arbitrary positive scalar, $\tilde{\boldsymbol{\Phi}} = (\bar{\boldsymbol{A}}_l + \bar{e}\bar{\boldsymbol{B}}_l\boldsymbol{K}_l)^{\mathrm{T}}\boldsymbol{P}_j(\bar{\boldsymbol{A}}_l + \bar{e}\bar{\boldsymbol{B}}_l\boldsymbol{K}_l) + \bar{e}(1-\bar{e})\boldsymbol{K}_l^{\mathrm{T}}\bar{\boldsymbol{B}}_l^{\mathrm{T}}\boldsymbol{P}_j\bar{\boldsymbol{B}}_l\boldsymbol{K}_l - \boldsymbol{P}_l$. With $\boldsymbol{f}_l\boldsymbol{f}_l^{\mathrm{T}} - \boldsymbol{I} > 0$, $l \in L_1$ and Schur complement, inequality (10) implies

$$\tilde{\boldsymbol{\Phi}} + \nu_l^{-1}\boldsymbol{E}_l^{\mathrm{T}}(\boldsymbol{f}_l\boldsymbol{f}_l^{\mathrm{T}} - \boldsymbol{I})^{-1}\boldsymbol{E}_l < 0. \tag{13}$$

Let $\nu_l^{-1} = \lambda_l$ and use the matrix inversion lemma, we obtain

$$\begin{aligned} &\nu_l^{-1}\boldsymbol{E}_l^{\mathrm{T}}(\boldsymbol{f}_l\boldsymbol{f}_l^{\mathrm{T}} - \boldsymbol{I})^{-1}\boldsymbol{E}_l \\ &= -\lambda_l\boldsymbol{E}_l^{\mathrm{T}}(\boldsymbol{I} - \boldsymbol{f}_l\boldsymbol{f}_l^{\mathrm{T}})^{-1}\boldsymbol{E}_l \\ &= -\lambda_l\boldsymbol{E}_l^{\mathrm{T}}\boldsymbol{E}_l - \lambda_l\boldsymbol{E}_l^{\mathrm{T}}\boldsymbol{f}_l(1 - \boldsymbol{f}_l^{\mathrm{T}}\boldsymbol{f}_l)^{-1}\boldsymbol{f}_l^{\mathrm{T}}\boldsymbol{E}_l \\ &= -\lambda_l\boldsymbol{E}_l^{\mathrm{T}}\boldsymbol{E}_l + \lambda_l\boldsymbol{E}_l^{\mathrm{T}}\boldsymbol{f}_l(\boldsymbol{f}_l^{\mathrm{T}}\boldsymbol{f}_l - 1)^{-1}\boldsymbol{f}_l^{\mathrm{T}}\boldsymbol{E}_l. \end{aligned} \tag{14}$$

By substituting Eq. (14) into inequality (13) and using Schur complement, we have

$$\boldsymbol{\Phi} = \begin{bmatrix} \tilde{\boldsymbol{\Phi}} - \lambda_l\boldsymbol{E}_l^{\mathrm{T}}\boldsymbol{E}_l & -\lambda_l\boldsymbol{E}_l^{\mathrm{T}}\boldsymbol{f}_l \\ -\lambda_l\boldsymbol{f}_l^{\mathrm{T}}\boldsymbol{E}_l & -\lambda_l(\boldsymbol{f}_l^{\mathrm{T}}\boldsymbol{f}_l - 1) \end{bmatrix} < 0, \tag{15}$$

and we obtain

$$\begin{aligned} &\mathbb{E}\{\boldsymbol{x}^{\mathrm{T}}(k+1)\boldsymbol{P}_j\boldsymbol{x}(k+1)|\boldsymbol{x}(k)\} - \boldsymbol{x}^{\mathrm{T}}(k)\boldsymbol{P}_l\boldsymbol{x}(k) \\ &\le -\lambda_{\min}(-\boldsymbol{\Phi})\boldsymbol{x}^{\mathrm{T}}(k)\boldsymbol{x}(k). \end{aligned} \tag{16}$$

Taking mathematical expectation on both sides and summing up the inequalities from $k = 0, \ldots, N$, we have

$$\begin{aligned} &\mathbb{E}\{\boldsymbol{x}^{\mathrm{T}}(N+1)\boldsymbol{P}_j\boldsymbol{x}(N+1)|\boldsymbol{x}(N)\} - \boldsymbol{x}^{\mathrm{T}}(0)\boldsymbol{P}_l\boldsymbol{x}(0) \\ &\le -\lambda_{\min}(-\boldsymbol{\Phi})\mathbb{E}\left\{\sum_{k=0}^{N}\|\boldsymbol{x}(k)\|^2\right\}, \end{aligned}$$

which implies

$$\mathbb{E}\left\{\sum_{k=0}^{N}\|\boldsymbol{x}(k)\|^2\right\} \le (\lambda_{\min}(-\boldsymbol{\Phi}))^{-1}\boldsymbol{x}^{\mathrm{T}}(0)\boldsymbol{P}_l\boldsymbol{x}(0).$$

Let $N \to +\infty$, we have

$$\begin{aligned} \mathbb{E}\left\{\sum_{k=0}^{\infty}\|\boldsymbol{x}(k)\|^2|\boldsymbol{x}(0)\right\} &\le (\lambda_{\min}(-\boldsymbol{\Phi}))^{-1}\boldsymbol{x}^{\mathrm{T}}(0)\boldsymbol{P}_l\boldsymbol{x}(0) \\ &= \boldsymbol{x}^{\mathrm{T}}(0)\boldsymbol{W}\boldsymbol{x}(0), \end{aligned}$$

where $\boldsymbol{W} = (\lambda_{\min}(-\boldsymbol{\Phi}))^{-1}\boldsymbol{P}_l > 0$. By Definition 1, this means that the closed-loop system (7) is robustly stochastically stable in the mean square.

Next, we show that the system satisfies (8). Under zero initial condition, we have

$$\begin{aligned}
&\mathbb{E}\{\boldsymbol{x}^{\mathrm{T}}(k+1)\boldsymbol{P}_j\boldsymbol{x}(k+1)|\boldsymbol{x}(k)\} - \boldsymbol{x}^{\mathrm{T}}(k)\boldsymbol{P}_l\boldsymbol{x}(k) + \mathbb{E}\{\boldsymbol{y}^{\mathrm{T}}(k)\boldsymbol{y}(k)|\boldsymbol{x}(k)\} - \gamma^2\boldsymbol{v}^{\mathrm{T}}(k)\boldsymbol{v}(k)\\
&= \mathbb{E}\{\boldsymbol{x}^{\mathrm{T}}(k)(\bar{\boldsymbol{A}}_l + (\bar{e}+\tilde{e}(k))\bar{\boldsymbol{B}}_l\boldsymbol{K}_l)^{\mathrm{T}}\boldsymbol{P}_j(\bar{\boldsymbol{A}}_l + (\bar{e}+\tilde{e}(k))\bar{\boldsymbol{B}}_l\boldsymbol{K}_l)\boldsymbol{x}(k)\} + \mathbb{E}\{2\boldsymbol{x}^{\mathrm{T}}(k)(\bar{\boldsymbol{A}}_l\\
&\quad + (\bar{e}+\tilde{e}(k))\bar{\boldsymbol{B}}_l\boldsymbol{K}_l)^{\mathrm{T}}\boldsymbol{P}_j\bar{\boldsymbol{C}}_l\boldsymbol{v}(k)\} + \boldsymbol{v}^{\mathrm{T}}(k)\bar{\boldsymbol{C}}_l^{\mathrm{T}}\boldsymbol{P}_j\bar{\boldsymbol{C}}_l\boldsymbol{v}(k) + \boldsymbol{x}^{\mathrm{T}}(k)\boldsymbol{H}_{cl}^{\mathrm{T}}\boldsymbol{H}_{cl}\boldsymbol{x}(k) - \gamma^2\boldsymbol{v}^{\mathrm{T}}(k)\boldsymbol{v}(k)\\
&= \tilde{\boldsymbol{x}}(k)^{\mathrm{T}}\begin{bmatrix}\tilde{\boldsymbol{\Phi}} + \boldsymbol{H}_{cl}^{\mathrm{T}}\boldsymbol{H}_{cl} & \boldsymbol{\Phi}_1\boldsymbol{C}_l & 0\\ \boldsymbol{C}_l^{\mathrm{T}}\boldsymbol{\Phi}_1^{\mathrm{T}} & \boldsymbol{C}_l^{\mathrm{T}}\boldsymbol{P}_j\boldsymbol{C}_l - \gamma^2\boldsymbol{I} & 0\\ 0 & 0 & 0\end{bmatrix}\tilde{\boldsymbol{x}}(k)\\
&\le \tilde{\boldsymbol{x}}(k)^{\mathrm{T}}\begin{bmatrix}\hat{\boldsymbol{\Phi}} & \boldsymbol{\Phi}_1 C_l & -\lambda_l\boldsymbol{E}_l^{\mathrm{T}}\boldsymbol{f}_l\\ \boldsymbol{C}_l^{\mathrm{T}}\boldsymbol{\Phi}_1^{\mathrm{T}} & \boldsymbol{C}_l^{\mathrm{T}}\boldsymbol{P}_j\boldsymbol{C}_l - \gamma^2\boldsymbol{I} & 0\\ -\lambda_l\boldsymbol{f}_l^{\mathrm{T}}\boldsymbol{E}_l & 0 & -\lambda_l(\boldsymbol{f}_l^{\mathrm{T}}\boldsymbol{f}_l - 1)\end{bmatrix}\tilde{\boldsymbol{x}}(k), \qquad (17)
\end{aligned}$$

where $\hat{\boldsymbol{\Phi}} = \tilde{\boldsymbol{\Phi}} + \boldsymbol{H}_{cl}^{\mathrm{T}}\boldsymbol{H}_{cl} - \lambda_l\boldsymbol{E}_l^{\mathrm{T}}\boldsymbol{E}_l$ and $\tilde{\boldsymbol{x}}(k) = [\boldsymbol{x}^{\mathrm{T}}(k), \boldsymbol{v}^{\mathrm{T}}(k), 1]^{\mathrm{T}}$. Using the Schur complement and the matrix inversion lemma, we know that inequality (10) implies the following inequality

$$\begin{bmatrix}\hat{\boldsymbol{\Phi}} & \boldsymbol{\Phi}_1\boldsymbol{C}_l & -\lambda_l\boldsymbol{E}_l^{\mathrm{T}}\boldsymbol{f}_l\\ \boldsymbol{C}_l^{\mathrm{T}}\boldsymbol{\Phi}_1^{\mathrm{T}} & \boldsymbol{C}_l^{\mathrm{T}}\boldsymbol{P}_j\boldsymbol{C}_l - \gamma^2\boldsymbol{I} & 0\\ -\lambda_l\boldsymbol{f}_l^{\mathrm{T}}\boldsymbol{E}_l & 0 & -\lambda_l(\boldsymbol{f}_l^{\mathrm{T}}\boldsymbol{f}_l - 1)\end{bmatrix} < 0. \qquad (18)$$

Then it follows that

$$\mathbb{E}\{\boldsymbol{x}^{\mathrm{T}}(k+1)\boldsymbol{P}_j\boldsymbol{x}(k+1)|\boldsymbol{x}(k)\} - \boldsymbol{x}^{\mathrm{T}}(k)\boldsymbol{P}_l\boldsymbol{x}(k) + \mathbb{E}\{\boldsymbol{y}^{\mathrm{T}}(k)\boldsymbol{y}(k)|\boldsymbol{x}(k)\} - \gamma^2\boldsymbol{v}^{\mathrm{T}}(k)\boldsymbol{v}(k) < 0.$$

Taking mathematical expectation on both sides and summing up the above inequality under zero initial condition form $k = 0, 1, 2, \ldots, +\infty$, we have inequality

$$\mathbb{E}\left\{\sum_0^\infty \boldsymbol{y}^{\mathrm{T}}(k)\boldsymbol{y}(k)\right\} < \gamma^2\sum_0^\infty \boldsymbol{v}^{\mathrm{T}}(k)\boldsymbol{v}(k). \quad \square$$

Remark 2 Theorem 1 is obtained for $\bar{e} \in [0, 1]$. When $\bar{e} = 1$ (that is, perfect communication links exist between the plant and controller), Theorem 1 reduces to robust H_∞ performance criterion for the uncertain discrete-time closed-loop piecewise-linear system. Similar work for the discrete-time piecewise-linear system without uncertainties can be found in Ref. [5].

Remark 3 To reduce the conservatism, we use S-procedure in inequality (12) and then use the Schur complement in inequality (13) which needs $|f_l| > 1$, that is, $l \in L_1$. When $l \in L_0$, $|f_l| \le 1$, the S-procedure cannot be used. Here, LMIs (9) are used instead of the LMIs (10).

Remark 4 It should be noted that when $\boldsymbol{P}_l = \boldsymbol{P}_j = \boldsymbol{P}$, then Theorem 1 reduces to the common quadratic Lyapunov function-based stability condition as in Ref. [6]. In fact, the common quadratic Lyapunov function is a special case of the more general piecewise quadratic Lyapunov function.

Remark 5 Due to the discrete nature of the system, the transitions could occur between nonadjacent polyhedral regions in one step. Thus, every polyhedral region pair (l, j) has to be computed in inequalities (9) and (10).

In Theorem 1, the stability and H_∞ performance of the closed-loop system have been derived based on piecewise quadratic Lyapunov function and the S-procedure. In the following, we will consider the piecewise controller design problem and derive a controller design method based on LMIs. We have the following H_∞ controller design result.

Theorem 2 (H_∞ controller design) Consider the closed-loop uncertain piecewise-linear system (7). If there exist two sets of scalars $\nu_l > 0$, $\varepsilon_l > 0$, matrices $\boldsymbol{R}_l$, $\boldsymbol{Q}_l$, $\boldsymbol{Z}_l > 0$ such that the following LMIs are satisfied,

$$\begin{bmatrix} \boldsymbol{\Psi}_1 & 0 & \boldsymbol{\Psi}_2 & \boldsymbol{\Psi}_3 & \boldsymbol{\Psi}_4 & 0 & \boldsymbol{\Psi}_5 & 0 & \boldsymbol{\Psi}_6 \\ * & -\gamma^2\boldsymbol{I} & \boldsymbol{C}_l^{\mathrm{T}} & 0 & 0 & 0 & \boldsymbol{N}_{3l}^{\mathrm{T}} & 0 & 0 \\ * & * & -\boldsymbol{Z}_j & 0 & 0 & \varepsilon_l\boldsymbol{M}_l & 0 & 0 & 0 \\ * & * & * & -\boldsymbol{Z}_j & 0 & 0 & 0 & \varepsilon_l\boldsymbol{M}_l & 0 \\ * & * & * & * & -\boldsymbol{I} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\varepsilon_l\boldsymbol{I} & 0 & 0 & 0 \\ * & * & * & * & * & * & -\varepsilon_l\boldsymbol{I} & 0 & 0 \\ * & * & * & * & * & * & * & -\varepsilon_l\boldsymbol{I} & 0 \\ * & * & * & * & * & * & * & * & -\varepsilon_l\boldsymbol{I} \end{bmatrix} < 0 \tag{19}$$

for all $l \in L_0$, and

$$\begin{bmatrix} \boldsymbol{\Psi}_1 & 0 & \boldsymbol{\Psi}_2 & \boldsymbol{\Psi}_3 & \boldsymbol{\Psi}_4 & \boldsymbol{R}_l^{\mathrm{T}}\boldsymbol{E}_l^{\mathrm{T}} & 0 & \boldsymbol{\Psi}_5 & 0 & \boldsymbol{\Psi}_6 \\ * & -\gamma^2\boldsymbol{I} & \boldsymbol{C}_l^{\mathrm{T}} & 0 & 0 & 0 & 0 & \boldsymbol{N}_{3l}^{\mathrm{T}} & 0 & 0 \\ * & * & -\boldsymbol{Z}_j & 0 & 0 & 0 & \varepsilon_l\boldsymbol{M}_l & 0 & 0 & 0 \\ * & * & * & -\boldsymbol{Z}_j & 0 & 0 & 0 & 0 & \varepsilon_l\boldsymbol{M}_l & 0 \\ * & * & * & * & -\boldsymbol{I} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \boldsymbol{\Psi}_7 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -\varepsilon_l\boldsymbol{I} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -\varepsilon_l\boldsymbol{I} & 0 & 0 \\ * & * & * & * & * & * & * & * & -\varepsilon_l\boldsymbol{I} & 0 \\ * & * & * & * & * & * & * & * & ** & -\varepsilon_l\boldsymbol{I} \end{bmatrix} < 0 \tag{20}$$

for all $l \in L_1$, where $\boldsymbol{\Psi}_1 = \boldsymbol{Z}_l - \boldsymbol{R}_l - \boldsymbol{R}_l^{\mathrm{T}}$, $\boldsymbol{\Psi}_2 = \boldsymbol{R}_l^{\mathrm{T}}\boldsymbol{A}_l^{\mathrm{T}} + \bar{e}\boldsymbol{Q}_l^{\mathrm{T}}\boldsymbol{B}_l^{\mathrm{T}}$, $\boldsymbol{\Psi}_3 = \sqrt{\bar{e}(1-\bar{e})}\boldsymbol{Q}_l^{\mathrm{T}}\boldsymbol{B}_l^{\mathrm{T}}$, $\boldsymbol{\Psi}_4 = \boldsymbol{Q}_l^{\mathrm{T}}\boldsymbol{G}_l^{\mathrm{T}} + \boldsymbol{R}_l^{\mathrm{T}}\boldsymbol{H}_l^{\mathrm{T}}$, $\boldsymbol{\Psi}_5 = \boldsymbol{R}_l^{\mathrm{T}}\boldsymbol{N}_{1l}^{\mathrm{T}} + \bar{e}\boldsymbol{Q}_l^{\mathrm{T}}\boldsymbol{N}_{2l}^{\mathrm{T}}$, $\boldsymbol{\Psi}_6 = \sqrt{\bar{e}(1-\bar{e})}\boldsymbol{Q}_l^{\mathrm{T}}\boldsymbol{N}_{2l}^{\mathrm{T}}$, $\boldsymbol{\Psi}_7 = -\nu_l(\boldsymbol{f}_l\boldsymbol{f}_l^{\mathrm{T}} - I)$, then the closed-loop uncertain piecewise-linear system (7) is mean-square asymptotically stable and the H_∞ norm constraint (8) is achieved. Moreover, the controller gain for each local system is given by $\boldsymbol{K}_l = \boldsymbol{Q}_l\boldsymbol{R}_l^{-1}$, $l \in L$.

Proof Applying the Schur complement to inequality (20) yields

$$\begin{bmatrix} \boldsymbol{\Psi}_1 & 0 & \boldsymbol{\Psi}_2 & \boldsymbol{\Psi}_3 & \boldsymbol{\Psi}_4 & \boldsymbol{R}_l^{\mathrm{T}}\boldsymbol{E}_l^{\mathrm{T}} \\ * & -\gamma^2\boldsymbol{I} & \boldsymbol{C}_l^{\mathrm{T}} & 0 & 0 & 0 \\ * & * & -\boldsymbol{Z}_j & 0 & 0 & 0 \\ * & * & * & -\boldsymbol{Z}_j & 0 & 0 \\ * & * & * & * & -\boldsymbol{I} & 0 \\ * & * & * & * & * & \boldsymbol{\Psi}_7 \end{bmatrix}$$

$$+ \varepsilon_l \begin{bmatrix} 0 \\ 0 \\ \boldsymbol{M}_l \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \boldsymbol{M}_l \\ 0 \\ 0 \\ 0 \end{bmatrix}^{\mathrm{T}} + \varepsilon_l^{-1} \begin{bmatrix} \boldsymbol{\Psi}_5 \\ \boldsymbol{N}_{3l}^{\mathrm{T}} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Psi}_5 \\ \boldsymbol{N}_{3l}^{\mathrm{T}} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^{\mathrm{T}}$$

$$+ \varepsilon_l \begin{bmatrix} 0 \\ 0 \\ 0 \\ \boldsymbol{M}_l \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \boldsymbol{M}_l \\ 0 \\ 0 \end{bmatrix}^{\mathrm{T}} + \varepsilon_l^{-1} \begin{bmatrix} \boldsymbol{\Psi}_6 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Psi}_6 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^{\mathrm{T}} < 0. \tag{21}$$

Using Lemma 1 and notice that

$$(\boldsymbol{Z}_l - \boldsymbol{R}_l)^{\mathrm{T}}\boldsymbol{Z}_l^{-1}(\boldsymbol{Z}_l - \boldsymbol{R}_l) = \boldsymbol{Z}_l - \boldsymbol{R}_l - \boldsymbol{R}_l^{\mathrm{T}} + \boldsymbol{R}_l^{\mathrm{T}}\boldsymbol{Z}_l^{-1}\boldsymbol{R}_l \geq 0\,,$$

we have

$$\begin{bmatrix} -\boldsymbol{R}_l^{\mathrm{T}}\boldsymbol{Z}_l^{-1}\boldsymbol{R}_l & 0 & \bar{\boldsymbol{\Psi}}_2 & \bar{\boldsymbol{\Psi}}_3 & \boldsymbol{\Psi}_4 & \boldsymbol{R}_l^{\mathrm{T}}\boldsymbol{E}_l^{\mathrm{T}} \\ * & -\gamma^2\boldsymbol{I} & \bar{\boldsymbol{C}}_l^{\mathrm{T}} & 0 & 0 & 0 \\ * & * & -\boldsymbol{Z}_j & 0 & 0 & 0 \\ * & * & * & -\boldsymbol{Z}_j & 0 & 0 \\ * & * & * & * & -\boldsymbol{I} & 0 \\ * & * & * & * & * & \boldsymbol{\Psi}_7 \end{bmatrix} < 0, \tag{22}$$

where $\bar{\boldsymbol{\Psi}}_2 = \boldsymbol{R}_l^{\mathrm{T}}\bar{\boldsymbol{A}}_l^{\mathrm{T}} + \bar{e}\boldsymbol{Q}_l^{\mathrm{T}}\bar{\boldsymbol{B}}_l^{\mathrm{T}}$, $\bar{\boldsymbol{\Psi}}_3 = \sqrt{\bar{e}(1-\bar{e})}\boldsymbol{Q}_l^{\mathrm{T}}\bar{\boldsymbol{B}}_l^{\mathrm{T}}$. Multiply $\mathrm{diag}\{\boldsymbol{Z}_l\boldsymbol{R}_l^{-T}, \boldsymbol{I}, \boldsymbol{I}, \boldsymbol{I}, \boldsymbol{I}, \boldsymbol{I}\}$ and its transpose on both sides of inequality (22) yields

$$\begin{bmatrix} -\boldsymbol{Z}_l & 0 & \tilde{\boldsymbol{\Psi}}_2 & \tilde{\boldsymbol{\Psi}}_3 & \tilde{\boldsymbol{\Psi}}_4 & \boldsymbol{Z}_l^{\mathrm{T}}\boldsymbol{E}_l^{\mathrm{T}} \\ * & -\gamma^2\boldsymbol{I} & \bar{\boldsymbol{C}}_l^{\mathrm{T}} & 0 & 0 & 0 \\ * & * & -\boldsymbol{Z}_j & 0 & 0 & 0 \\ * & * & * & -\boldsymbol{Z}_j & 0 & 0 \\ * & * & * & * & -\boldsymbol{I} & 0 \\ * & * & * & * & * & \boldsymbol{\Psi}_7 \end{bmatrix} < 0, \tag{23}$$

where $\tilde{\boldsymbol{\Psi}}_2 = \boldsymbol{Z}_l\bar{\boldsymbol{A}}_l^{\mathrm{T}} + \bar{e}\boldsymbol{Y}_l^{\mathrm{T}}\bar{\boldsymbol{B}}_l^{\mathrm{T}}$, $\tilde{\boldsymbol{\Psi}}_3 = \sqrt{\bar{e}(1-\bar{e})}\boldsymbol{Y}_l^{\mathrm{T}}\bar{\boldsymbol{B}}_l^{\mathrm{T}}$, $\tilde{\boldsymbol{\Psi}}_4 = \boldsymbol{Y}_l^{\mathrm{T}}\boldsymbol{G}_l^{\mathrm{T}} + \boldsymbol{Z}_l\boldsymbol{H}_l^{\mathrm{T}}$, $\boldsymbol{Y}_l = \boldsymbol{Q}_l\boldsymbol{R}_l^{-1}\boldsymbol{Z}_l$. Multiply $\mathrm{diag}\{\boldsymbol{P}_l, \boldsymbol{I}, \boldsymbol{P}_j, \boldsymbol{P}_j, \boldsymbol{I}, \boldsymbol{I}\}$ and its transpose on both sides of inequality (23) with $\boldsymbol{P}_l = \boldsymbol{Z}_l^{-1}$ and let $\boldsymbol{K}_l = \boldsymbol{Y}_l\boldsymbol{P}_l$, we obtain inequality (10). Similarly, we know that inequality (19) implies inequality (9). According to Theorem 1, it can be concluded that the closed-loop piecewise-linear system is stochastically stable with a guaranteed H_∞ performance and thus the proof is completed. □

Note that the conditions in inequalities (19) and (20) are LMIs not only over the matrix variables but also over the scalar γ. Then one can obtain the minimum H_∞ attenuation level of the uncertain discrete-time piecewise-linear system by solving a convex optimization problem as follows:

$$\min_{\boldsymbol{Z}_l>0,\nu_l>0,\varepsilon_l>0,\boldsymbol{R}_l,\boldsymbol{Q}_l} \gamma \text{ subject to} \\ \text{inequalities (19) and (20).} \tag{24}$$

In the next section, we will illustrate how to solve the optimization problem addressed above.

From the proof of Theorem 2, it is easy to know that if there are no uncertainties in the piecewise-linear system (1), we have the following corollary.

Corollary 1 (H_∞ controller design) Consider the closed-loop piecewise-linear system (7) without uncertain. If there exist two sets of scalars $\nu_l > 0$, matrices $\boldsymbol{R}_l$, $\boldsymbol{Q}_l$, $\boldsymbol{Z}_l > 0$ such that the following LMIs are satisfied,

$$\begin{bmatrix} \boldsymbol{\Psi}_1 & 0 & \boldsymbol{\Psi}_2 & \boldsymbol{\Psi}_3 & \boldsymbol{\Psi}_4 \\ * & -\gamma^2\boldsymbol{I} & \boldsymbol{C}_l^{\mathrm{T}} & 0 & 0 \\ * & * & -\boldsymbol{Z}_j & 0 & 0 \\ * & * & * & -\boldsymbol{Z}_j & 0 \\ * & * & * & * & -\boldsymbol{I} \end{bmatrix} < 0 \tag{25}$$

for all $l \in L_0$, and

$$\begin{bmatrix} \boldsymbol{\Psi}_1 & 0 & \boldsymbol{\Psi}_2 & \boldsymbol{\Psi}_3 & \boldsymbol{\Psi}_4 & \boldsymbol{R}_l^{\mathrm{T}}\boldsymbol{E}_l^{\mathrm{T}} \\ * & -\gamma^2\boldsymbol{I} & \boldsymbol{C}_l^{\mathrm{T}} & 0 & 0 & 0 \\ * & * & -\boldsymbol{Z}_j & 0 & 0 & 0 \\ * & * & * & -\boldsymbol{Z}_j & 0 & 0 \\ * & * & * & * & -\boldsymbol{I} & 0 \\ * & * & * & * & * & \boldsymbol{\Psi}_7 \end{bmatrix} < 0 \tag{26}$$

for all $l \in L_1$, then the closed-loop uncertain piecewise-linear system (7) without uncertain is mean-square asymptotically stable and the H_∞ norm constraint (8) is achieved. Moreover, the controller gain for each local system is given by $\boldsymbol{K}_l = \boldsymbol{Q}_l\boldsymbol{R}_l^{-1}$, $l \in L$.

4. Simulation examples

To illustrate the results of H_∞ controller synthesis, a piecewise-linear chaotic system will be considered.

Example 1 Consider the well-known Chua's circuit.[40] The dynamics of Chua's circuit is governed by

$$\begin{aligned} C_1\frac{\mathrm{d}v_1}{\mathrm{d}\tau} &= \frac{v_2 - v_1}{R} - g(v_1) - \tilde{u}(\tau), \\ C_2\frac{\mathrm{d}v_2}{\mathrm{d}\tau} &= \frac{v_1 - v_2}{R} + i_L, \\ L\frac{\mathrm{d}i_L}{\mathrm{d}\tau} &= -v_2 - R_0 i_L - \tilde{v}(\tau), \end{aligned} \tag{27}$$

where v_1 and v_2 are the voltages at the ends of C_1 and C_2, respectively, i_L is the current in the inductance L, and $\tilde{u}(\tau)$ is the current from generator as active control action of the circuit, $\tilde{v}(\tau) = 5\,\mathrm{e}^{-0.05\tau}\cos(0.5\pi\tau)$ is a deterministic voltage disturbance. Chua's circuit contains only one nonlinear component $g(v_1)$ which is a 3-segment piecewise-linear resistor and its v–i characteristic function is given as follows:

$$g(v_1) = \begin{cases} m_0 v_1 + (m_1 - m_0)B_p, & v_1 \geq B_p, \\ m_1 v_1, & |v_1| < B_p, \\ m_0 v_1 - (m_1 - m_0)B_p, & v_1 \leq -B_p. \end{cases}$$

Thus the nonlinear Chua's circuit is a piecewise-linear system which often exhibits a chaotic behaviour.[41] In our case, we choose $R = 10/7$, $R_0 = 0$, $C_1 = 1$, $C_2 = 19/2$, $L = 19/14$, $m_0 = -0.25$, $m_1 = -0.53$, $B_p = 1$. With the sampling period $T_s = 0.01$ seconds, one has the following discretized piecewise linear model of the controlled Chua's circuit:

$$\boldsymbol{x}(k+1) = \boldsymbol{A}_l x(k) + \boldsymbol{B}_l \boldsymbol{u}(k) + \boldsymbol{C}_l \boldsymbol{v}(k) + \boldsymbol{\alpha}_l,$$
$$\boldsymbol{y}(k) = \boldsymbol{H}_l \boldsymbol{x}(k), \quad \text{for } \boldsymbol{x} \in X_l,\ l = 1, 2, 3,$$

where the three regions are $x_1 = (-5, -1], (-1, 1)$, $[1, +5)$ which denoted by X_1, X_2 and X_3, respectively. The system matrices are given by

$$\boldsymbol{A}_1 = \boldsymbol{A}_3 = \begin{bmatrix} 0.9955 & 0.0070 & 0 \\ 0.0007 & 0.9993 & -0.0011 \\ 0.0000 & 0.0074 & 1.0000 \end{bmatrix},$$

$$\boldsymbol{A}_2 = \begin{bmatrix} 0.9983 & 0.0070 & 0 \\ 0.0007 & 0.9993 & -0.0011 \\ 0.0000 & 0.0074 & 1.0000 \end{bmatrix},$$

$$\boldsymbol{B}_1 = \boldsymbol{B}_2 = \boldsymbol{B}_3 = \begin{bmatrix} -0.01 \\ 0 \\ 0 \end{bmatrix},$$

$$\boldsymbol{C}_1 = \boldsymbol{C}_2 = \boldsymbol{C}_3 = \begin{bmatrix} 0 \\ 0 \\ -0.01 \end{bmatrix},$$

$$\boldsymbol{H}_1 = \boldsymbol{H}_2 = \boldsymbol{H}_3 = [\ 1 \quad 1 \quad 0\ ],$$

$$\boldsymbol{\alpha}_1 = -\boldsymbol{\alpha}_3 = \begin{bmatrix} 0.0028 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{\alpha}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The three ellipsoidal regions can be exactly represented by

$$\varepsilon_l = \{\boldsymbol{x} | \|\boldsymbol{E}_l \boldsymbol{x} + \boldsymbol{f}_l\| \le 1\},\ l = 1, 2, 3$$

with $\boldsymbol{E}_l = [1, 0, 0] g_l$, $g_1 = 0.5$, $g_2 = 1$, $g_3 = 0.5$ and $f_1 = 1.5$, $f_2 = 0$, $f_3 = -1.5$, thus we have $L_0 = 2$, $L_1 = 1, 3$. The control objective is to stabilize the circuit to the origin with given disturbance attenuation γ. Suppose $\bar{e} = 0.8$, this corresponds to the case that 20 percent of data are lost during the communication. Let $\gamma = 1$ and use Corollary 1, we obtain the piecewise controller gains K_l as follows:

$$\boldsymbol{K}_1 = \boldsymbol{K}_3 = [\ 100.0946 \quad 97.0460 \quad 0.0192\ ],$$

$$\boldsymbol{K}_2 = [\ 100.0997 \quad 97.4072 \quad 0.0133\ ].$$

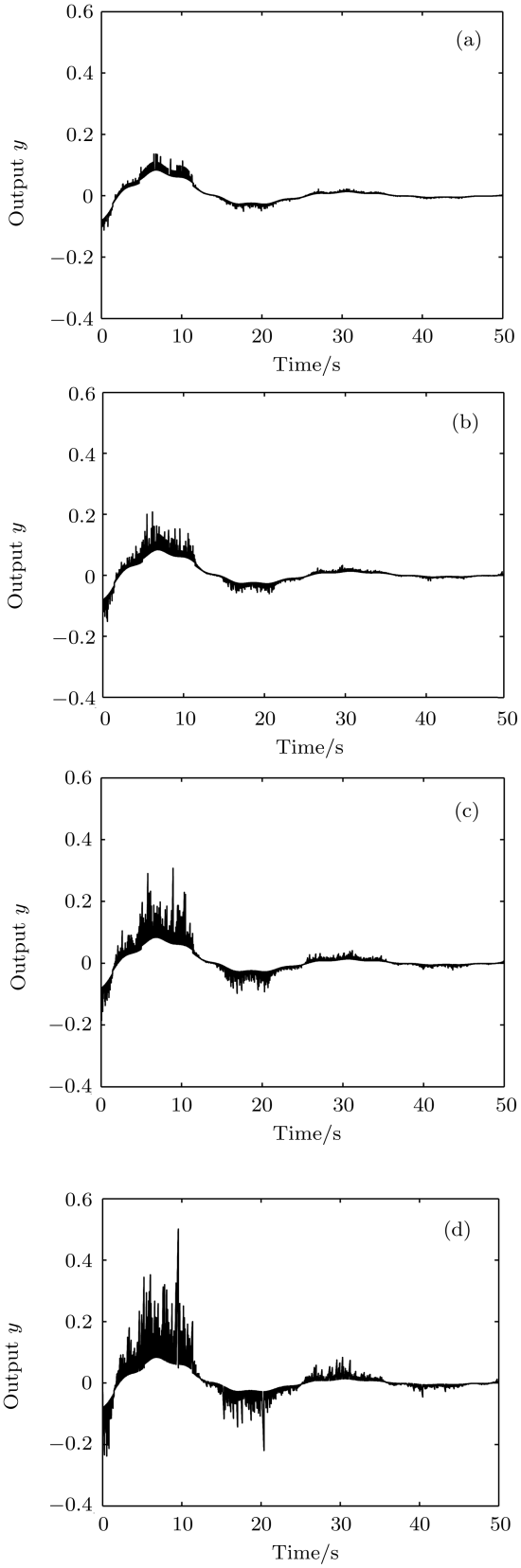


Fig. 1. Output of controlled Chua's circuit with different data loss cases.

With the obtained controller gains, simulation has been carried out. Results with initial condition $x(0) =$

$[2, -2, -1.5]^{\mathrm{T}}$ are recorded in Fig. 1(a). It can be observed that the controller design approach proposed in this paper not only stabilizes the piecewise-linear Chua's circuit but also effectively attenuates the disturbances as expected under unreliable communication links. With the same controller, the outputs of the data loss cases $\bar{e} = 0.6$, $\bar{e} = 0.4$ and $\bar{e} = 0.2$ are recorded in Figs. 1(b), 1(c) and 1(d), respectively. It is shown that more severe data loss deteriorates the control performance.

5. Conclusion

In this paper, we have considered the robust H_∞ control design for uncertain piecewise-linear chaotic systems with random data loss based on a piecewise quadratic Lyapunov function. A piecewise H_∞ controller design method is proposed to stabilize the system and also attenuate the effect of external disturbances. It is shown the H_∞ control problem can be solved by linear matrix inequality techniques. One simulation example has been presented to demonstrate our results.

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