Knotted picture of the whole complete quantum measurement process of quantum teleportation

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Based on the previous work about the knotted pictures of quantum states, quantum logic gates and unitary transformations, this paper further gives the whole complete quantum measurement process of quantum teleportation from the viewpoint of knot theory.

Keywords: Bell bases, single particle quantum state, knotted picture, flip operation, inversion operation, composition operation

PACC: 0367, 0240

1. Introduction

Quantum entanglement[1−12] plays an essential role in quantum teleportation, quantum computation and the understanding of quantum phenomena. Because of its importance we try to find its mathematical structure and physical background. Knot theory is a powerful mathematical tool for researching physical phenomena.[13−16] As to the mathematical structure of quantum entanglement, we have found that there is an intimate relationship between quantum entanglement and knot theory,[17−19] and we have successfully proved that there is a one-to-one correspondence between four Bell bases and four oriented links of the linkage 4_1 in knot theory,[17,18] in general, a one-to-one correspondence between GHZ states and the oriented links.[19] As to the physical background we try to use quantum network theory[20] to determine the state of entanglement for a given quantum state.[21,22] We have carried out a series of studies of quantum information from the point of view of knot theory,[23−28] such as the knotted pictures of some quantum states, some quantum logic gates and some unitary transformations. It is well known that Bennett *et al.*[1] suggested a scheme for teleporting a quantum state from one system to another, i.e. the process of quantum teleportation. To the best of our knowledge, up to now no researcher has successfully given the knotted picture of the process of quantum teleportation. In fact, our previous studies mentioned above have prepared all the prerequisites for the description of quantum teleportation in knot theory. Based on these studies, we find that this problem can be easily solved at this time. Hence in this paper we will discuss the knotted picture of the whole complete quantum measurement process of quantum teleportation. To perform quantum teleportation of the initial quantum state carried by particle A, the receiver Bob and the sender Alice share an entangled pair of particles 1 and 2, which they have obtained from some source of entangled particles in, say, state $|\Phi^+\rangle_{12}$, one of the well-known Bell bases. Before, Alice cannot say anything about the state of particle 1 on its own. Nor does she know the state of particle 1. She only knows that these particles are orthogonal to each other. First, particle A, which carries the state to be sent to Bob, is given to Alice. Alice now measures particles A and 2 together, by projecting them onto the Bell-state bases. After projecting the two particles into an entangled state, Alice only knows the correlations between the two particles, she does not know anything about the individual states of them. In the process of quantum teleportation, Bell basis is chosen to be the quantum channel. It is well known that for the system consisting of two spin-$\frac{1}{2}$ particles 1 and 2, there are four well-known entangled Bell bases, which are[18]

$$|\Phi^+\rangle_{12} = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle_{12} + |\downarrow\downarrow\rangle_{12}),$$
$$|\Phi^-\rangle_{12} = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle_{12} - |\downarrow\downarrow\rangle_{12}),$$
$$|\Psi^+\rangle_{12} = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle_{12} + |\downarrow\uparrow\rangle_{12}),$$

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$$|\Psi^-\rangle_{12} = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle_{12} - |\downarrow\uparrow\rangle_{12}). \quad (1)$$

Previously we have introduced a unified expression $|\text{Bell}\rangle^i_{12}$ to express the above four Bell bases:[25]

$$\begin{aligned} i = 1 &\longleftrightarrow |\Phi^+\rangle_{12}, \\ i = 2 &\longleftrightarrow |\Phi^-\rangle_{12}, \\ i = 3 &\longleftrightarrow |\Psi^+\rangle_{12}, \\ i = 4 &\longleftrightarrow |\Psi^-\rangle_{12}. \end{aligned} \quad (2)$$

In the process of quantum teleportation, supposing that the quantum channel state is chosen to be $|\Phi^+\rangle_{12}$, the unknown quantum state of particle A is $|\phi\rangle_{\rm A}$, then before any measurement, the entire system, comprising Alice's unknown particle A and the EPR pair (particle 1 and particle 2), is in a pure product state $|\phi\rangle_{\rm A}$ $|\Phi^+\rangle_{12}$. The process of teleportation is composed of two parts: Alice's measurement on particle A and particle 2, abbreviated as M(A,2); Bob's measurement on particle 1, abbreviated as M(1). Actually this process is a process of re-entanglement, it is equivalent to the expansion of product state $|\phi\rangle_{\rm A}$ $|\Phi^+\rangle_{12}$ in terms of the four Bell bases $|\text{Bell}\rangle^i_{\rm A2}$, for which $|\text{Bell}\rangle^1_{\rm A2} = |\Phi^+\rangle_{\rm A2}$, $|\text{Bell}\rangle^2_{\rm A2} = |\Phi^-\rangle_{\rm A2}$, $|\text{Bell}\rangle^3_{\rm A2} = |\Psi^+\rangle_{\rm A2}$, $|\text{Bell}\rangle^4_{\rm A2} = |\Psi^-\rangle_{\rm A2}$, i.e.

$$|\phi\rangle_{\rm A}|\Phi^+\rangle_{12} = \sum_{i=1}^{4} |\text{Bell}\rangle^i_{\rm A2}|\phi\rangle^i_1. \quad (3)$$

The corresponding knotted equation is

$$\{|\phi\rangle_{\rm A}\}\{|\Phi^+\rangle_{12}\} = \sum_{i=1}^{4}\{|\text{Bell}\rangle^i_{\rm A2}\}\{|\phi\rangle^i_1\}, \quad (4)$$

where $\{|\phi\rangle_{\rm A}\}$, $\{|\Phi^+\rangle_{12}\}$,$\{|\text{Bell}\rangle^i_{\rm A2}\}$, and $\{|\phi\rangle^i_1\}$ represent the knotted pictures of the corresponding quantum states respectively. The rest of the present paper is organized as follows. In Section 2 we shall discuss the knotted picture of M(A,2), i.e. $\{|\text{Bell}\rangle^i_{\rm A2}\}$. In Section 3 we shall discuss knotted picture of M(1), i.e. $\{|\phi\rangle^i_1\}$. In Section 4 we shall point out the one-one correspondence between the knotted pictures of the unitary transformations involved in the two measurements M(A,2) and M(1). In Section 5 we shall discuss the completion of the process of quantum teleportation.

2. Knotted picture of measurement M(A,2)

Introducing four unitary operators based on the three Pauli operators,[25] namely U_i $(i = 1, 2, 3, 4)$, where

$$U_1 = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad U_2 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$U_3 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad U_4 = \mathrm{i}\sigma_y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (5)$$

and using Eqs. (1) and (2), then we can easily obtain the expression of $|\text{Bell}\rangle^i_{12}$ in terms of single particle states $\begin{pmatrix} |\uparrow\rangle_1 \\ |\downarrow\rangle_1 \end{pmatrix}$ and $\begin{pmatrix} |\uparrow\rangle_2 \\ |\downarrow\rangle_2 \end{pmatrix}$ as follows:[25]

$$|\text{Bell}\rangle^i_{12} = \frac{1}{\sqrt{2}}(|\uparrow\rangle_1, |\downarrow\rangle_1)U_i \begin{pmatrix} |\uparrow\rangle_2 \\ |\downarrow\rangle_2 \end{pmatrix}. \quad (6)$$

From Eq. (6) we can obtain the corresponding knotted equation

$$\{|\text{Bell}\rangle^i_{12}\} = \{(|\uparrow\rangle_1, |\downarrow\rangle_1)\}\tilde{U}_i \left\{ \begin{pmatrix} |\uparrow\rangle_2 \\ |\downarrow\rangle_2 \end{pmatrix} \right\}, \quad (7)$$

where $\tilde{U}_i$ is the corresponding knotted operator of unitary operator U_i.

The links corresponding to $\{|\text{Bell}\rangle^i_{12}\}$ are shown in Fig. 1.[18]

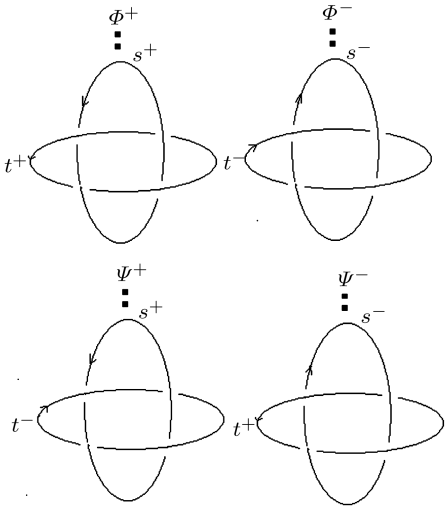


Fig. 1. Knotted pictures of the four Bell bases $\{|\text{Bell}\rangle^i_{12}\}$.

The knotted picture of quantum states $(|\uparrow\rangle_1, |\downarrow\rangle_1)$ and $\begin{pmatrix} |\uparrow\rangle_2 \\ |\downarrow\rangle_2 \end{pmatrix}$ can be given by the oriented knots shown in Fig. 2, which are represented by $\{(|\uparrow\rangle_1, |\downarrow\rangle_1)\}$ and $\left\{ \begin{pmatrix} |\uparrow\rangle_2 \\ |\downarrow\rangle_2 \end{pmatrix} \right\}$ respectively.[25]

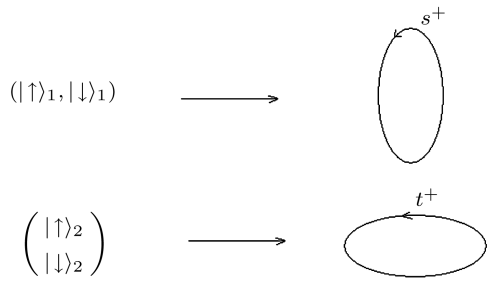


Fig. 2. Knotted pictures of quantum states $(|\uparrow\rangle_1, |\downarrow\rangle_1)$ and $\begin{pmatrix} |\uparrow\rangle_2 \\ |\downarrow\rangle_2 \end{pmatrix}$.

Similarly, we have

$$\{|\text{Bell}\rangle^i_{\text{A}2}\} = \{(|\uparrow\rangle_{\text{A}}, |\downarrow\rangle_{\text{A}})\}\tilde{U}_i \left\{ \begin{pmatrix} |\uparrow\rangle_2 \\ |\downarrow\rangle_2 \end{pmatrix} \right\}. \quad (8)$$

In order to obtain the knotted operators $\tilde{U}_i$ of the four unitary operators U_i, we introduce the composition operator $\tilde{C}$, which is composed of the following three surgical operations.[25]

1) Cut

Cut knot $\left\{ \begin{pmatrix} |\uparrow\rangle_2 \\ |\downarrow\rangle_2 \end{pmatrix} \right\}$, the point being cut c becomes two unconnected points c' and c'', then the resulted knot will become an open cycle which can pass through knot $\{(|\uparrow\rangle_1, |\downarrow\rangle_1)\}$.

2) Pass through

Let the being cut knot $\left\{ \begin{pmatrix} |\uparrow\rangle_2 \\ |\downarrow\rangle_2 \end{pmatrix} \right\}$ pass through knot $\{(|\uparrow\rangle_1, |\downarrow\rangle_1)\}$, reach the required position.

3) Glue

Glue two unconnected points c' and c'' of knot $\left\{ \begin{pmatrix} |\uparrow\rangle_2 \\ |\downarrow\rangle_2 \end{pmatrix} \right\}$ into one point c, thus obtain the required link.

These three steps are shown in Figs. (3a), (3b) and (3c) respectively.

Using composition operator $\tilde{C}$ we can easily obtain[25]

$$\{(|\uparrow\rangle_1, |\downarrow\rangle_1)\}\tilde{C} \left\{ \begin{pmatrix} |\uparrow\rangle_2 \\ |\downarrow\rangle_2 \end{pmatrix} \right\} = \{|\Phi^+\rangle_{12}\}. \quad (9)$$

Obviously if we change the direction of orientation of knot $\{(|\uparrow\rangle_1, |\downarrow\rangle_1)\}$, from counterclockwise to clockwise, i.e. from positive direction to negative direction, then we can use the resulting inversed knot $\{(|\uparrow\rangle_1, |\downarrow\rangle_1)\}_{\text{inv}}$ to obtain the following equation:[25]

$$\{(|\uparrow\rangle_1, |\downarrow\rangle_1)\}_{\text{inv}}\tilde{C} \left\{ \begin{pmatrix} |\uparrow\rangle_2 \\ |\downarrow\rangle_2 \end{pmatrix} \right\} = \{|\Psi^-\rangle_{12}\}. \quad (10)$$

Similarly, we have

$$\{(|\uparrow\rangle_1, |\downarrow\rangle_1)\}\tilde{C} \left\{ \begin{pmatrix} |\uparrow\rangle_2 \\ |\downarrow\rangle_2 \end{pmatrix} \right\}_{\text{inv}} = \{|\Psi^+\rangle_{12}\}, \quad (11)$$

and

$$\{(|\uparrow\rangle_1, |\downarrow\rangle_1)\}_{\text{inv}}\tilde{C} \left\{ \begin{pmatrix} |\uparrow\rangle_2 \\ |\downarrow\rangle_2 \end{pmatrix} \right\}_{\text{inv}} = \{|\Phi^-\rangle_{12}\}. \quad (12)$$

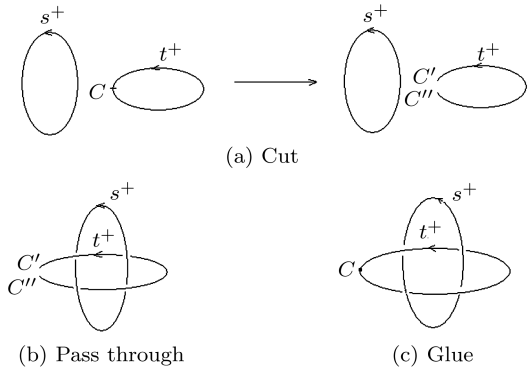


Fig. 3. Three steps of composition operation.

In order to give a unified form of Eqs. (9)–(12) we can write $\tilde{U}_i$ as

$$\tilde{U}_i = \tilde{l}_i\tilde{C}\tilde{r}_i, \quad (13)$$

where $\tilde{l}_i$ is the left operator operating on the left knot, and $\tilde{r}_i$ is the right operator operating on the right knot. There are only two kinds of left (right) operators which are identity operator I and inversion operator τ, I keeps the orientation of the left (right) knot invariant, whereas τ changes the orientation of the left (right) knot from positive direction to negative direction and *vice versa*. Obviously $\tau^2 = \tau\tau = I$. Left and right operations are shown in Figs. 4 and 5 respectively.[25]

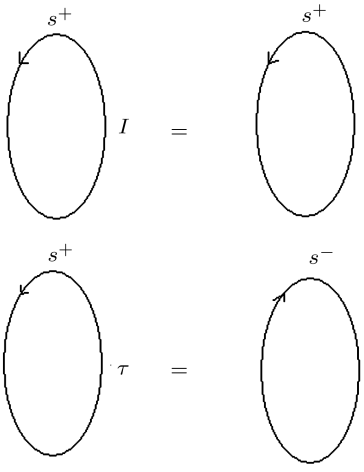


Fig. 4. Left operations of the operators I and τ.

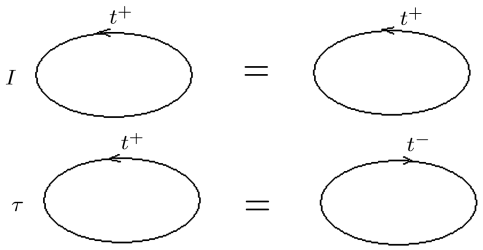


Fig. 5. Right operations of the operators I and τ.

From Eq. (13) we readily obtain

$$\tilde{U}_1 = I\tilde{C}I, \quad \tilde{U}_2 = \tau\tilde{C}\tau, \quad \tilde{U}_3 = I\tilde{C}\tau,$$

and

$$\tilde{U}_4 = \tau\tilde{C}I. \tag{14}$$

3. Knotted picture of M(1) on the quantum channel

Now we discuss the knotted picture of M(1), i.e. $\{|\phi\rangle_1^i\}$, where $|\phi\rangle_1^i$ are the following quantum states:

$$\begin{aligned}
|\phi\rangle_1^1 &= \begin{pmatrix} a \\ b \end{pmatrix}_1 = a|\uparrow\rangle_1 + b|\downarrow\rangle_1, \\
|\phi\rangle_1^2 &= \begin{pmatrix} a \\ -b \end{pmatrix}_1 = a|\uparrow\rangle_1 - b|\downarrow\rangle_1, \\
|\phi\rangle_1^3 &= \begin{pmatrix} b \\ a \end{pmatrix}_1 = a|\downarrow\rangle_1 + b|\uparrow\rangle_1, \\
|\phi\rangle_1^4 &= \begin{pmatrix} -b \\ a \end{pmatrix}_1 = a|\downarrow\rangle_1 - b|\uparrow\rangle_1.
\end{aligned} \tag{15}$$

Quantum states $|\phi\rangle_1^i$ shown in Eq. (15) are superposed states of two eigenstates $|\uparrow\rangle_1$ and $|\downarrow\rangle_1$, there are three characteristics which must be shown in their corresponding knotted pictures. The first characteristic is the two eigenstates $|\uparrow\rangle_1$ and $|\downarrow\rangle_1$ involved in the quantum states $|\phi\rangle_1^i$, which can be represented by two orientated circles: $|\uparrow\rangle_1$ corresponding to the circle with counterclockwise orientation, and $|\downarrow\rangle_1$ corresponding to the circle with clockwise orientation. Since $|\uparrow\rangle_1$ and $|\downarrow\rangle_1$ are the eigenstates of the same particle 1, they must connect each other through the surgical operations cut and glue described above. If we locate the oriented circle representing $|\uparrow\rangle_1$ on the left side, and arrange, the oriented circle representing $|\downarrow\rangle_1$ on the right side, then by the surgical operations, cut and glue, we can connect the two circles into one twisted knot as shown in Fig. 6. There are two possible ways of connection, one corresponds to positive cross connecting two oriented circles (left part of Fig. 6), the other corresponds to a negative cross connecting two oriented circles (right part of Fig. 6).[26]

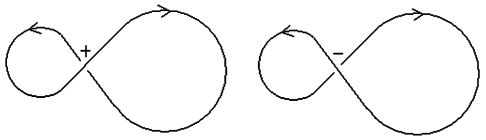


Fig. 6. Two possible ways of connecting two oppositely oriented circles into one twisted knot.

The second characteristic is the plus or minus sign between the two composed eigenstates. Naturally we may use the positive cross to represent the positive sign, and the negative cross to denote the negative sign. The third characteristic is the value of two coefficients before the two eigenstates $|\uparrow\rangle_1$ and $|\downarrow\rangle_1$. For the purpose of representation of these coefficients we must use non-topological method, let the area of the orientated circle equal the square of the coefficient before each of corresponding eigenstates separately, i.e., for the quantum state $|\phi\rangle_1^1 = \begin{pmatrix} a \\ b \end{pmatrix}_1 = a|\uparrow\rangle_1 + b|\downarrow\rangle_1$, we let $\pi R_a^2 = a^2$, and $\pi R_b^2 = b^2$, where R_a and R_b are the radii of the corresponding circles. Using this method we can successfully obtain the four quasi-knots for quantum states $|\phi\rangle_1^i$ as shown in Fig. 7.[26]

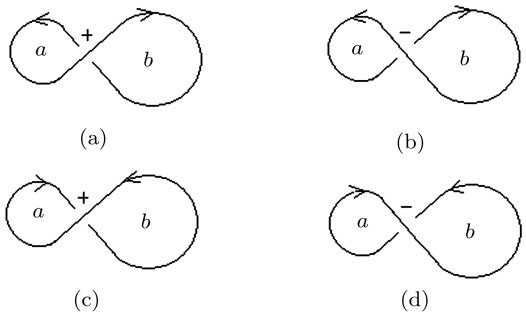


Fig. 7. Four kinds of quasi-knots for the knotted picture of single particle quantum state.

where figure 7(a) shows the knotted picture of $|\phi\rangle_1^1$, i.e.,

$$\{|\phi\rangle_1^1\} = \{a|\uparrow\rangle_1 + b|\downarrow\rangle_1\} = \left\{\begin{pmatrix} a \\ b \end{pmatrix}_1\right\},$$

figure 7(b) shows the knotted picture of $|\phi\rangle_1^2$, i.e.,

$$\{|\phi\rangle_1^2\} = \{a|\uparrow\rangle_1 - b|\downarrow\rangle_1\} = \left\{\begin{pmatrix} a \\ -b \end{pmatrix}_1\right\},$$

figure 7(c) shows the knotted picture of $|\phi\rangle_1^3$, i.e.,

$$\{|\phi\rangle_1^3\} = \{a|\downarrow\rangle_1 + b|\uparrow\rangle_1\} = \left\{\begin{pmatrix} b \\ a \end{pmatrix}_1\right\},$$

and figure 7(d) shows the knotted picture of $|\phi\rangle_1^4$, i.e.,

$$\{|\phi\rangle_1^4\} = \{a|\downarrow\rangle_1 - b|\uparrow\rangle_1\} = \left\{\begin{pmatrix} -b \\ a \end{pmatrix}_1\right\}.$$

It can be easily verified that the four quantum states $|\phi\rangle_1^i$ are interrelated to each other through the four unitary operators shown in Eq. (5),[26] i.e.

$$\begin{aligned}
\begin{pmatrix} a \\ b \end{pmatrix}_1 &= U_1 \begin{pmatrix} a \\ b \end{pmatrix}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}_1, \\
\begin{pmatrix} a \\ -b \end{pmatrix}_1 &= U_2 \begin{pmatrix} a \\ b \end{pmatrix}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}_1, \\
\begin{pmatrix} b \\ a \end{pmatrix}_1 &= U_3 \begin{pmatrix} a \\ b \end{pmatrix}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}_1, \\
\begin{pmatrix} -b \\ a \end{pmatrix}_1 &= U_4 \begin{pmatrix} a \\ b \end{pmatrix}_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}_1 . \quad (16)
\end{aligned}$$

Equation (16) can be abbreviated to

$$|\phi\rangle_1^i = U_i|\phi\rangle_1^1 . \quad (17)$$

The corresponding knotted equation of Eq. (17) is

$$\{|\phi\rangle_1^i\} = \tilde{U}_i\{|\phi\rangle_1^1\} , \quad (18)$$

where $\{|\phi\rangle_1^i\}$ denotes the knotted picture of $|\phi\rangle_1^i$, and $\tilde{U}_i$ represents the knotted operation of U_i. We can express $\tilde{U}_i$ by introducing the following three surgical operations: I, f and τ, which are shown in Fig. 8.

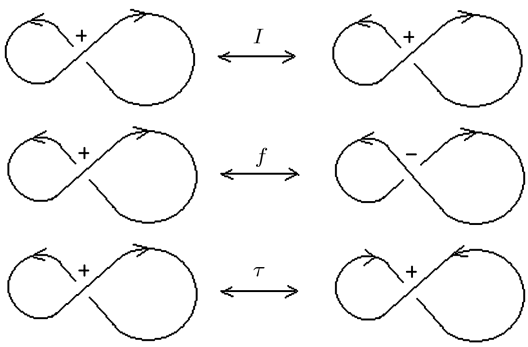


Fig. 8. Three surgical operations: identity, flip and inversion.

Identity operation I does not induce any change of the knot, flip operation f changes the sign of the cross, inversion operation τ changes the orientation of the knot. In terms of these three surgical operations, we have

$$\tilde{U}_1 = I\ ,\ \tilde{U}_2 = f,\ \ \tilde{U}_3 = \tau,\ \ \tilde{U}_4 = f\tau. \quad (19)$$

Thus we have found the surgical knotted operations $\tilde{U}_i$ of unitary operators U_i. The knotted pictures $\{|\phi\rangle_1^i\}$ of replicas $|\phi\rangle_1^i$ appearing in measurement M(1) in the process of quantum teleportation are already shown in Fig. 7.

In order to differentiate the two-side two-qubit operator $\tilde{U}_i$ in Eq. (14) and the one-side one-qubit operator $\tilde{U}_i$ in Eq. (19) for the same unitary operator U_i, we use symbol $\tilde{U}_2^i$ to replace $\tilde{U}_i$ in Eq. (14), and use symbol $\tilde{U}_1^i$ to replace $\tilde{U}_i$ in Eq. (19), hence we have

$$\tilde{U}_2^1 = I\tilde{C}I,\ \tilde{U}_2^2 = \tau\tilde{C}\tau,\ \tilde{U}_2^3 = I\tilde{C}\tau,\ \tilde{U}_2^4 = \tau\tilde{C}I, \quad (20)$$

and

$$\tilde{U}_1^1 = I\ ,\ \tilde{U}_1^2 = f,\ \tilde{U}_1^3 = \tau,\ \tilde{U}_1^4 = f\tau. \quad (21)$$

4. One-one correspondence between knotted pictures of unitary transformations involved in two measurements M(A,2) and M(1)

During measurements M(A,2) and M(1), pure product state $|\phi\rangle_{\rm A}\ |\Phi^+\rangle_{12}$ collapses stochastically into one of the four product states $|{\rm Bell}\rangle_{\rm A2}^i|\phi\rangle_1^i$ $(i = 1,2,3,4)$. For any one product state, entangled state $|{\rm Bell}\rangle_{\rm A2}^i$ and single particle state $|\phi\rangle_1^i$ must correspond to the same number i. From Eq. (8) and the replacement of $\tilde{U}_i$ by $\tilde{U}_2^i$, we have

$$\{|{\rm Bell}\rangle_{\rm A2}^i\} = \{(|\uparrow\rangle_{\rm A}, |\downarrow\rangle_{\rm A})\}\tilde{U}_2^i \left\{\begin{pmatrix} |\uparrow\rangle_2 \\ |\downarrow\rangle_2 \end{pmatrix}\right\}.$$

Similarly from Eq. (18) and the replacement of $\tilde{U}_i$ by $\tilde{U}_1^i$, we have $\{|\phi\rangle_1^i\} = \tilde{U}_1^i\{|\phi\rangle_1^1\}$. Since the results of measurements M(A,2) and M(1) correspond to the same value of number i, we obtain one-one correspondence $\tilde{U}_2^i \longleftrightarrow \tilde{U}_1^i$. For the case $i = 1$, $|{\rm Bell}\rangle_{\rm A2}^1 = |\Phi^+\rangle_{\rm A2}$ and $\tilde{U}_2^1 = I\tilde{C}I$, the corresponding one-side one-qubit operator is $\tilde{U}_1^1 = I$; for the case $i = 2$, $|{\rm Bell}\rangle_{\rm A2}^2 = |\Phi^-\rangle_{\rm A2}$ and $\tilde{U}_2^2 = \tau\tilde{C}\tau$, the corresponding one-side one-qubit operator is $\tilde{U}_1^2 = f$; for the case $i = 3$, $|{\rm Bell}\rangle_{\rm A2}^3 = |\Psi^+\rangle_{\rm A2}$ and $\tilde{U}_2^3 = I\tilde{C}\tau$, the corresponding one-side one-qubit operator is $\tilde{U}_1^3 = \tau$; for the case $i = 4$, $|{\rm Bell}\rangle_{\rm A2}^4 = |\Psi^-\rangle_{\rm A2}$ and $\tilde{U}_2^4 = \tau\tilde{C}I$, the corresponding one-side one-qubit operator is $\tilde{U}_1^4 = f\tau$.

5. Completion of the process of quantum teleportation

After Alice completed her measurement M(A,2) she knew the entangled state $|\text{Bell}\rangle^i_{\text{A}2}$ and thus the value of number i, then she told this value to Bob through the classical channel. After Bob completed his measurement M(1) and received the message about the value of i through the classical channel, he readily knew that his measured result on particle 1 was the quantum state $|\phi\rangle^i_1$. Then from knotted equation (18), i.e., $\{|\phi\rangle^i_1\} = \tilde{U}^i_1\{|\phi\rangle^1_1\}$, he readily knew the knotted picture of the unknown quantum state $\{|\phi\rangle^1_1\} = (\tilde{U}^i_1)^{-1}\{|\phi\rangle^i_1\}$, and hence the unknown quantum state $|\phi\rangle^1_1 = \binom{a}{b}_1 = a|\uparrow\rangle_1 + b|\downarrow\rangle_1$ brought by particle 1. This state is just the teleported state sent by Alice, originally brought by particle A, i.e. $|\phi\rangle_{\text{A}} = |\phi\rangle^1_{\text{A}} = a|\uparrow\rangle_{\text{A}} + b|\downarrow\rangle_{\text{A}}$. Hence, from the measurements M(A,2), M(1) and the requirement of the same value of number i, Bob can easily know the teleported unknown state sent by Alice, thus the quantum teleportation process is completed. For example, if the result of measurement M(A,2) is $|\text{Bell}\rangle^2_{\text{A}2} = |\Phi^-\rangle_{\text{A}2}$, i.e. $i = 2$, after Alice told this result to Bob, Bob readily knew that his measured result was $\{|\phi\rangle^2_1\}$, from the relation $\{|\phi\rangle^2_1\} = \tilde{U}^2_1\{|\phi\rangle^1_1\} = f\{|\phi\rangle^1_1\}$ he easily knew $\{|\phi\rangle^1_1\} = f^{-1}\{|\phi\rangle^2_1\} = f\{|\phi\rangle^2_1\}$, since $f^{-1} = f$. This process is schematically shown in Fig. 9.

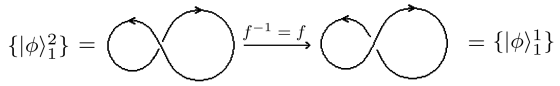


Fig. 9. Finding the unknown quantum state in the process of quantum teleportation for case $i = 2$.

Though from quantum channel, Bob receives only the replica of the knotted picture of the unknown quantum state to be sent by Alice, i.e. $\{|\phi\rangle^i_1\}$, from the classical channel he knows the knotted picture of the unitary transformation which is needed for him, i.e. $\tilde{U}^i_2$, then by one-one correspondence $\tilde{U}^i_2 \longleftrightarrow \tilde{U}^i_1$ Bob can readily know the original quantum state which Alice wants to send to him. Therefore we have completed the whole process of the quantum teleportation from the viewpoint of knot theory.

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