Interactions between defects and propagating hydrodynamic solitons*

Pan Jun-Ting(潘军廷), Chen Wei-Zhong(陈伟中)†, and Zheng Lu-Jie(郑鹭杰)

The Key Laboratory of Modern Acoustics, Ministry of Education, and Institute of Acoustics, Nanjing University, Nanjing 210093, *China*

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This paper studies the hydrodynamic solitons propagating along a long trough with a defective bed. The slight deviation from the plane in the bed serves as the depth defects. Based on the perturbation method, it finds that the free surface wave is governed by a Korteweg-de Vries (KdV) equation with a defect term (KdVD). The numerical calculations show that, for a single-convexity localized defect, the propagating soliton is decelerated as it comes into the defect region, and it is accelerated back to its initial velocity as it leaves, which has a dipole effect. As a result, its displacement is lagged in contrast to the uninfluenced one. And an up-step defect makes the propagating soliton decelerate simply. The opposite influence will occur for a single-concavity localized defect and a down-step one. The defect-induced influence on propagating hydrodynamic solitons depends on the polarity of defects, which agrees with that on non-propagating ones. However, the involved dipole effect of the single localized defect is not displayed in non-propagating cases.

Keywords: propagating hydrodynamic soliton, defect, interaction
PACC: 0340K, 0290

1. Introduction

At present, the soliton has been widely accepted in various fields as a structural basis for viewing and understanding the dynamic behaviour of complex nonlinear systems. In order to successfully interpret kinds of soliton phenomena, one mainly refers to kinds of nonlinear models. In this respect, many nonlinear integrable models have been proposed, such as the well-known Korteweg-de Vries (KdV) equation, the nonlinear Schrödinger (NLS) equation, the sine-Gordon equation, and so on. And many studies based on such models have been widely performed, which mainly concentrate on the propagating property of solitons,[1,2] the soliton–soliton interactions,[3−9] and so on. However, for the integrability of the models, these classical equations are idealized regardless of the existence of impurities or defects in physics. An increasing amount of interest has been, consequently, spent on the solitons in inhomogeneous media or soliton perturbations.[10−21] Since the 1990s, in particular, much attention has been paid to the dynamic behaviour of solitons influenced by impurities in the Frenkel–Kontorova (FK) model.[22,23] Up to now, some interesting impurity–soliton interactions (ISI) have been revealed by a number of theoretical[24,25] and experimental[26,27] investigations in the FK model. These studies found that an impurity can attract or repel non-propagating solitons depending on its polarity. And now more and more special attention has been devoted in good time to further study the ISI in different physical systems.[28,29]

Recently, the studies of ISI have been extended to the hydrodynamic system.[30] In Ref. [30], a slightly defective bed or sidewall of a water layer was served as a single impurity, making the layer depth or breadth slightly different from the plane. The results showed that the non-propagating hydrodynamic solitons can be attracted or repelled by the impurities, which are the same as the non-propagating solitons in the FK model.[24−27] However, how about the case of propagating hydrodynamic solitons? To our knowledge, the analysis in this respect has not been performed yet. In this paper, as a new improvement to Ref. [30], we will use the perturbation method[13,15] to study the interactions between depth defects and propagating hydrodynamic solitons.

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†Corresponding author. E-mail: wzchen@nju.edu.cn
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2. The propagating hydrodynamic soliton equation with depth defects

We now consider a shallow layer of water filled in a long trough with a defective bed $z = h(x, x_{\rm d})$ which is slightly different from the uniform bed $z = -h_0$ in the defect region centred at $x_{\rm d}$ (see Fig. 1). The water is assumed to be inviscid and incompressible.

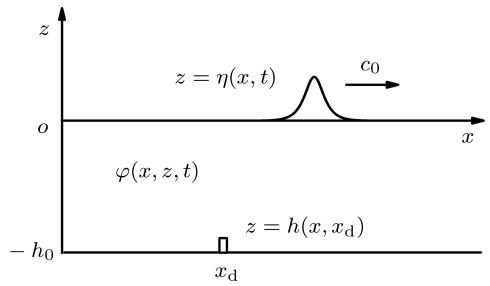


Fig. 1. The sketch of the propagating soliton along the trough with a defective bed centred at $x_{\rm d}$.

Furthermore, the motion of the water is simplified to be irrotatoional and potential. It can be governed by a 2-dimensional Laplace equation for its velocity potential $\varphi(x, z, t)$,

$$\varphi_{xx} + \varphi_{zz} = 0, \quad \text{at} \quad h(x, x_{\rm d}) \le z \le \eta(x, t), \tag{1}$$

where x and z are horizontal and vertical coordinates, and t is time, respectively. The subscripts denote the derivatives with respect to them. The upper surface of the water is free so that $\varphi(x, z, t)$ and $\eta(x, t)$ satisfy

$$\eta_t - \varphi_z + \varphi_x \eta_x = 0, \quad \text{at} \quad z = \eta(x, t), \tag{2}$$

$$\varphi_t + \frac{1}{2}(\varphi_x^2 + \varphi_z^2) + g\eta = 0, \quad \text{at} \quad z = \eta(x, t), \tag{3}$$

where g is the gravitational acceleration. And the bed is rigid, so that

$$\varphi_z - h_x \varphi_x = 0, \quad \text{at} \quad z = h(x, x_{\rm d}). \tag{4}$$

Both the upper and bottom boundaries of the water are irregular, so equations (2)–(4) are defined at nonconstat surfaces. We need to regularize them by Taylor expansions at their corresponding regular planes, that is,

$$\eta_t - \varphi_z + \varphi_x \eta_x - \varphi_{zz}\eta + \varphi_{xz}\eta\eta_x - \frac{1}{2}\varphi_{zzz}\eta^2 + \cdots = 0, \quad \text{at} \quad z = 0, \tag{5}$$

$$\varphi_t + \frac{1}{2}(\varphi_x^2 + \varphi_z^2) + g\eta + \varphi_{tz}\eta + (\varphi_x\varphi_{xz} + \varphi_z\varphi_{zz})\eta + \cdots = 0, \quad \text{at} \quad z = 0, \tag{6}$$

and

$$\varphi_z - h_x\varphi_x + (\varphi_{zz} - h_x\varphi_{xz})(h + h_0) + \frac{1}{2}\varphi_{zzz}(h + h_0)^2 + \cdots = 0, \ \text{at} \ z = -h_0. \tag{7}$$

We are interested in the case of a slight variation centred at $x_{\rm d}$ from the perfect bed, namely,

$$h(x, x_{\rm d}) = -h_0[1 - F(x, x_{\rm d})], \tag{8}$$

where the function $F(x, x_{\rm d})$ describes the shape of the bed with a defect centred at $x_{\rm d}$. Substituting Eq. (8) into Eq. (7), we obtain

$$\varphi_z - h_0 F_x\varphi_x + h_0(\varphi_{zz} - h_0 F_x\varphi_{xz})F + \frac{1}{2}h_0^2\varphi_{zzz}F^2 + \cdots = 0, \quad \text{at} \quad z = -h_0. \tag{9}$$

For small wavy motion, we can assume that $\varphi(x, z, t)$ and $\eta(x, t)$ are small. In order to study the propagating soliton along a trough with a defective bed, we expand $\varphi(x, z, t)$ and $\eta(x, t)$ in the powers of a small parameter ϵ, and determine the coefficients in the perturbation expansions

$$\varphi = \epsilon\varphi_1 + \epsilon^2\varphi_2 + \epsilon^3\varphi_3 + \epsilon^4\varphi_4 + \epsilon^5\varphi_5 + \cdots, \tag{10}$$

and

$$\eta = \epsilon^2\eta_2 + \epsilon^3\eta_3 + \epsilon^4\eta_4 + \epsilon^5\eta_5 + \cdots. \tag{11}$$

At the same time, we introduce the following scale transformations of variables

$$\xi = \epsilon(x - c_0 t), \quad \tau = \epsilon^3 t, \quad \text{and} \quad z = z, \tag{12}$$

where $c_0 = \sqrt{gh_0}$ is the wave velocity. Furthermore, we can assume

$$F(x, x_{\rm d}) = \epsilon^2 f(x, x_{\rm d}) \tag{13}$$

for the small defect.

After the multiple-scaling expansion of Eqs. (10)–(12), the hydrodynamic equations (1), (5), (6), and (9) are transformed into a chain of equations in different orders of the small parameter ϵ. Finally, after a long series of complex computations, we obtain a KdV equation with a defect term (KdVD) at order ϵ^5,

$$\eta_t + c_0\eta_x + \frac{3g}{2c_0}\eta\eta_x + \frac{gh_0^3}{6c_0}\eta_{xxx} - \frac{gh_0}{2c_0}(F\eta)_x = 0, \tag{14}$$

where x, t, η, and F have been recovered from the slow variables ξ, τ, η_2, and f by the inverse transformations $x = \epsilon^{-1}\xi + \epsilon^{-3}c_0\tau$, $t = \epsilon^{-3}\tau$, $\eta = \epsilon^2\eta_2$, and $F = \epsilon^2 f$, respectively, so as to have everything written in physical dimensions.

From Eq. (14), we can see that the defect term $(F\eta)_x$ depends on not only the external force F, but also the wave amplitude η. Furthermore, this kind of defect term is related to not only the defect F, but also the value of its derivative F_x with respect to x. It is different from the one discussed for the non-propagating hydrodynamic NLS solitons[30] which only depends on the defect function itself, the form of $F\eta$. Moreover, it is clear that for a localized defect, such as a rectangular defect (see Fig. 2(a)), F_x will lead to a dipole effect (see Fig. 2(b)). That is, for a localized defect, the term $(F\eta)_x$ includes a dipole effect on propagating hydrodynamic solitons, a type of defect-induced influence which is different from the ISI found for non-propagating solitons in Ref. [30].

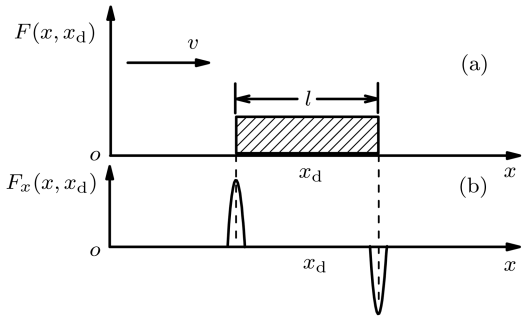


Fig. 2. A single-convexity rectangular defect with a finite width. (a) The sketch of a single-convexity rectangular defect with a width l, where the arrow "v" denotes the propagating direction of the soliton. (b) Its gradient.

3. Interactions between propagating hydrodynamic solitons and depth defects

The influence of defects on propagating hydrodynamic solitons, based on Eq. (14), depends on the details of the depth defect function $F(x, x_\mathrm{d})$. We will study a single localized defect in the following. Moreover, as its limiting case, a step defect will be investigated in order to further interpret the single localized defect effects.

3.1. A single localized defect

As we know, the single localized defects in ISI for NLS solitons are usually expressed as a δ-function,[22−27,30] because the NLS model with a δ-defect is still integrable in mathematics. By integrating the NLS equation with a δ-defect, the exact solutions with impurity-induced localized mode have been explicitly obtained,[23,24] and validated in experiments.[30] Motivated by this, we start our investigation by introducing a δ-defect in depth, that is, $F(x, x_\mathrm{d}) = q\delta(x - x_\mathrm{d})$, where $F(x, x_\mathrm{d}) > 0$ stands for a convex defect and $F(x, x_\mathrm{d}) < 0$ for a concave one. We integrate the KdVD equation (14) with respect to x once in a small interval from $x_\mathrm{d} - \varepsilon$ to $x_\mathrm{d} + \varepsilon$. When $\varepsilon \to 0$, we have

$$\frac{gh_0^3}{6c_0}\left(\eta_{xx}|_{x=x_\mathrm{d}+\varepsilon} - \eta_{xx}|_{x=x_\mathrm{d}-\varepsilon}\right) - \frac{gh_0}{2c_0}\left[(F\eta)|_{x=x_\mathrm{d}+\varepsilon} - (F\eta)|_{x=x_\mathrm{d}-\varepsilon}\right] = 0. \quad (15)$$

From Eq. (15), we can obtain that, when $F(x, x_\mathrm{d}) = q\delta(x - x_\mathrm{d})$, the terms containing $F(x, x_\mathrm{d})$ in Eq. (15) will be vanished because the function $\eta(x, t)$ is continuous and the $\delta(x - x_\mathrm{d})$ function is symmetrical. So the δ-defect does not make the η_x loss its smoothness at $x = x_\mathrm{d}$, which is substantially different from that in the NLS solitons.[22−27,30] The defect term in the KdVD equation (14), $(F\eta)_x$, unlike that in the NLS equation which is in the form of $F\eta$, leads to this difference. In physics, of course, the single localized defects always occupy a finite width instead of the infinite narrow width in δ-function. As a result, some effects of the term $(F\eta)_x$ in Eq. (14) will be left and they will influence the propagation of the KdV soliton. For simplicity, we will choose a single localized defect with a finite width l (see, e.g., Fig. 2(a)) to study these kinds of effect with the aid of numerical calculations.

In order to simulate the KdVD equation (14), we use the centred finite-difference scheme for the spatial discretization, and the equation will be transformed to a set of N (the number of the grid meshes) ordinary differential equations (ODEs). We then integrate the set of the ODEs by using the fifth-order Runge–Kutta algorithm as described in Ref. [31]. The strength of the single localized defect is described by its width l and height, and the former is fixed in our simulation while the height is adjustable. Here, we choose

$g = 9.8$ m/s^2, $h_0 = 1$ m, $N = 3500$, and the integral duration is 70 s. The spatial step is set to be 0.1 m, so that the space investigated is 350 m. Moreover, we choose 100 grid meshes (i.e., the $l = 10$ m in Fig. 2(a)) to be defective. Specifically, we set $F(x, x_\mathrm{d}) = \pm 0.12$ m when $2000 \leq N \leq 2100$ (i.e., 200 m $\leq x \leq$ 210 m, $x_\mathrm{d} = 205$ m) and $F(x, x_\mathrm{d}) = 0$ m for other regions.

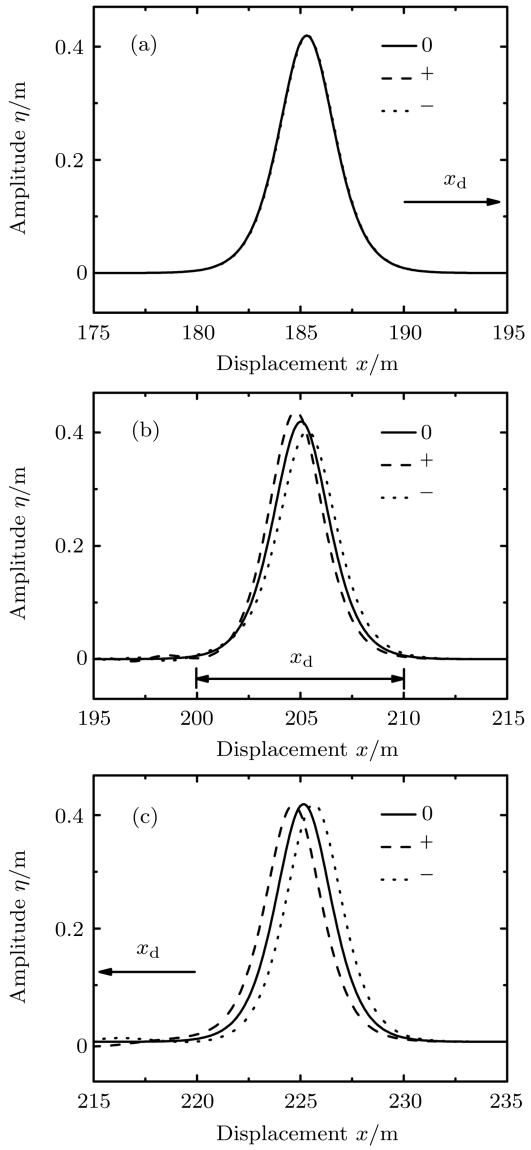


Fig. 3. The interactions between a KdV soliton and single localized defects. The KdV soliton displacement is lagged by the single-convexity localized defect and led by the single-concavity one. Profiles (a), (b) and (c) show the propagating soliton nears, reaches and overreaches the defects, respectively. The signs "0","+", and "–" represent the cases of no defect, the single-convexity localized defect, and the single-concavity localized defect, respectively.

The numerical results show that when a KdV soliton is influenced by a single localized defect, its displacement will be lagged by a single-convexity localized defect (see the dashed curve in Fig. 3), but led by a single-concavity one (see the dotted curve in Fig. 3). The specific variations of the soliton velocity are shown in Fig. 4. From Fig. 4, we can see that, when a KdV soliton propagates near a single-convexity localized defect, it will be decelerated by the defect at first. When the soliton reaches the defect region, its velocity will be gradually reduced to a constant. When the soliton nearly passes the defect region, it will be gradually accelerated by the defect. And at last, the velocity will reach its initial uninfluenced value and then keep this velocity for the further propagations. The opposite influence will occur for a single-concavity localized defect.

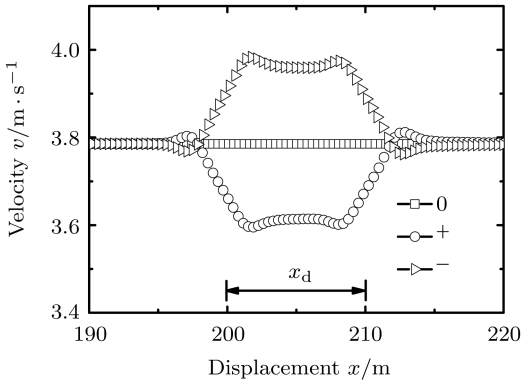


Fig. 4. The variations of a KdV soliton velocity influenced by single localized defects.

3.2. A step defect

In order to well analyse the resulting defect effects of the above single localized defect, we assume that the defect is infinitely long in the trough bed, namely, we suppose $l \to +\infty$ in Fig. 2(a). In this case, the single localized defect will evolve into a step defect. Therefore, we introduce a step defect function $F(x, x_\mathrm{d}) = q\theta(x - x_\mathrm{d})$ into Eq. (15), where $\theta(x - x_\mathrm{d})$ is a step function and satisfies

$$\theta(x - x_\mathrm{d}) = \begin{cases} 1, & x \geq x_\mathrm{d}, \\ 0, & x < x_\mathrm{d}. \end{cases} \tag{16}$$

Here, $F(x, x_\mathrm{d}) > 0$ stands for an up-step defect and $F(x, x_\mathrm{d}) < 0$ for a down-step one.

Using Eq. (16), we can obtain $F(x, x_\mathrm{d})|_{x=x_\mathrm{d}+\varepsilon} = q$ and $F(x, x_\mathrm{d})|_{x=x_\mathrm{d}-\varepsilon} = 0$ in Eq. (15), so that,

$$\frac{gh_0^3}{6c_0}\left(\eta_{xx}|_{x=x_\mathrm{d}+\varepsilon} - \eta_{xx}|_{x=x_\mathrm{d}-\varepsilon}\right) - \frac{gh_0}{2c_0} q\eta \Big|_{x=x_\mathrm{d}} = 0. \tag{17}$$

From Eq. (17), we can obtain that, when $F(x, x_{\rm d}) = q\theta(x - x_{\rm d})$, the terms containing $F(x, x_{\rm d})$ in Eq. (15) will not be vanished and the η_x will loss its smoothness at $x = x_{\rm d}$. Therefore, the effects of the term $(F\eta)_x$ in Eq. (14) will influence the propagation of the KdV soliton.

Additionally, using Eq. (16), we can obtain from Eq. (14) the following equation:

$$\eta_t + c_0\eta_x + \frac{3g}{2c_0}\eta\eta_x + \frac{gh_0^3}{6c_0}\eta_{xxx} - \frac{gh_0}{2c_0}\left[q(\delta(x - x_{\rm d})\eta + \theta(x - x_{\rm d})\eta_x)\right] = 0. \quad (18)$$

Thus, specially, beyond the small interval, $x_{\rm d} - \varepsilon \sim x_{\rm d} + \varepsilon$, from Eq. (18) we can have the following standard KdV equation:

$$\eta_t + c\eta_x + \frac{3g}{2c_0}\eta\eta_x + \frac{gh_0^3}{6c_0}\eta_{xxx} = 0, \quad (19)$$

where $c = c_0$ when $\theta(x - x_{\rm d}) \equiv 0$, and $c = c_0 - (gh_0/2c_0)q$ for $\theta(x - x_{\rm d}) \equiv 1$.

In the following, we will numerically calculate the step defect effects based on the KdVD equation (14). Here, we set $F(x, x_{\rm d}) = \pm 0.12$ m when $x \geq 200$ m (i.e., $x_{\rm d} = 200$ m), and $F(x, x_{\rm d}) = 0$ m for other regions. Besides, we use the same algorithm and its parameters as discussed for the case of the single localized defect.

The numerical results show that when a KdV soliton is influenced by a step defect, its displacement will be lagged by an up-step defect (see the dashed curve in Fig. 5) while led by a down-step one (see the dotted curve in Fig. 5). The specific variations of the soliton velocity are shown in Fig. 6. As is shown in Fig. 6, when a KdV soliton propagates near a step defect, it will be gradually decelerated by an up-step defect while gradually accelerated by a down-step one firstly. When the soltion reaches the step fields, its velocity will gradually reach a constant which has been changed from the initial velocity, and then keep this constant value on in other propagating regions.

Consequently, from Fig. 6 we can obtain that, within the small defect interval $x_{\rm d} - \varepsilon \sim x_{\rm d} + \varepsilon$, the defect term in Eq. (18) results in deceleration effects on the KdV soliton for an up-step defect, and acceleration effects for a down-step one. In other defect regions, equation (18) has degenerated to Eq. (19). Therefore, figure 6 displays that the soliton velocity will keep a constant value, which has already been decelerated by an up-step defect or accelerated by a down-step one.

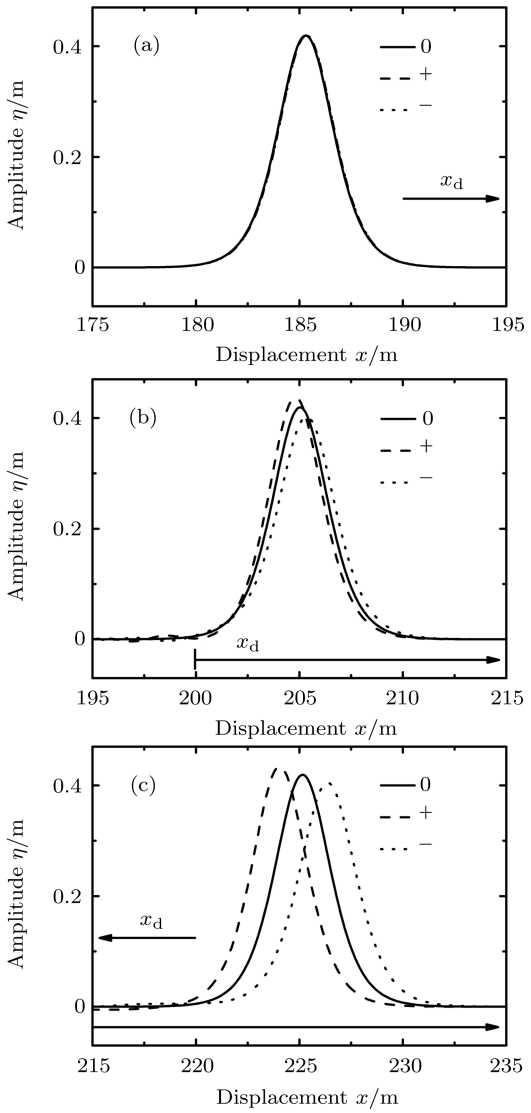


Fig. 5. The interactions between a KdV soliton and step defects. The KdV soliton displacement is lagged by the up-step defect and led by the down-step one. Profiles (a), (b) and (c) show the propagating soliton nears, reaches and overreaches the defects, respectively. The signs "0","+", and"–" represent the case of no defect, the up-step defect, and the down-step defect, respectively.

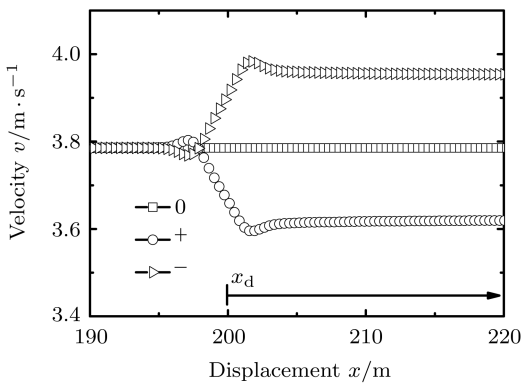


Fig. 6. The variations of a KdV soliton velocity influenced by step defects.

Based on these effects of step defects, we can in-

terpret the defect-induced influence on the single localized defect. For example, a single-convexity localized defect centred at $x_{\rm d}$ with a separation l (see, Fig. 2(a)) can be regarded as a couple of step defects, an up-step defect at $x_{\rm d} - l/2$ and a down-step one at $x_{\rm d} + l/2$. Thus, its influence on the propagating hydrodynamic soliton can be understood as follows. At first the soliton comes near the up-step defect at $x_{\rm d} - l/2$ and it decelerates the soliton. Then the soliton comes near the down-step defect at $x_{\rm d} + l/2$ and it accelerates the soliton. So the dipole effect makes the soliton speed down and up as shown in Fig. 4. Similarly we can also discuss a single-concavity localized defect. Clearly, as for hydrodynamic solitons, there is also a polarity influence of defect for propagating solitons, which is consistent with the previous ISI of the non-propagating ones.[30] While, the kind of dipole defect-induced effect for the propagating solitons is not found for the non-propagating ones as indicated in Ref. [30].

4. Conclusion and discussion

In summary, we have investigated the propagation of hydrodynamic solitons along a trough with a defective bed. We have addressed ourselves to the influence on the propagating solitons from two types of the defects in the bed, including a single localized defect and a step defect. Our study is based on the KdVD equation derived from the hydrodynamics by means of the perturbation method. The numerical investigations show that, an up-step defect, one of the step defects, will decelerate the KdV soliton while a down-step one will accelerate it. As a result, the displacement of the KdV soliton is lagged by the up-step defect and led by the down-step one. A single-convexity defect, one of the single localized defects, will decelerate the KdV soliton when the soliton comes near the defect, and will accelerate it until the velocity recovers its initial value when the soliton leaves, leading to a constant lag from the uninfluenced soliton, which have a dipole effect. And a single-concavity localized defect leads to opposite influence. Moreover, the effects of the single localized defect can be understood as the action of a couple of step defects.

In short, the defect-induced influence on propagating hydrodynamic solitons includes positive and negative effects, namely, deceleration and acceleration. Furthermore, it depends on the polarity of the defects, such as convexity or concavity. Clearly this is consistent with the case for the non-propagating solitons. However, from the present study, we find that a dipole defect-induced effect is involved on propagating hydrodynamic solitons, which is not displayed in non-propagating cases.

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