New application to Riccati equation*

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To seek new infinite sequence of exact solutions to nonlinear evolution equations, this paper gives the formula of nonlinear superposition of the solutions and Bäcklund transformation of Riccati equation. Based on the tanh-function expansion method and homogenous balance method, new infinite sequence of exact solutions to Zakharov–Kuznetsov equation, Karamoto–Sivashinsky equation and the set of (2+1)-dimensional asymmetric Nizhnik–Novikov–Veselov equations are obtained with the aid of symbolic computation system Mathematica. The method is of significance to construct infinite sequence exact solutions to other nonlinear evolution equations.

Keywords: Riccati equation, formula of nonlinear superposition, nonlinear evolution equation, exact solution

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1. Introduction

In the 1960s', with computer development, differential equations had a new period. In this period, soliton was one of the important discoveries in science and mathematic fields. Discussing differential equations is to construct the solutions and explain the significance of the solutions. And a number of direct methods for constructing solutions to nonlinear evolution equations are presented, such as, the homogeneous method,[1] the tanh-function expansion method,[2] the auxiliary equation method,[3−42] and so on. In Ref. [3], $\tanh(\xi)$ is replaced by the solution $z(\xi)$ of Riccati equation (1) and many exact solutions to the nonlinear evolution equations are constructed. In Refs. [4]–[12], many exact solutions to the nonlinear evolution equations are searched for by combing Eq. (1) with some formal solutions.

$$\frac{\mathrm{d}z(\xi)}{\mathrm{d}\xi} = z'(\xi) = R + z^2(\xi). \tag{1}$$

In Refs. [13]–[15], the authors obtained many exact solutions to Karamoto–Sivashinsky (KS) equation[15] and other nonlinear evolution equations with Eq. (2) and some formal solutions

$$\frac{\mathrm{d}z(\xi)}{\mathrm{d}\xi} = z'(\xi) = a + bz(\xi) + cz^2(\xi). \tag{2}$$

In Refs. [16]–[22], the authors found a number of exact solutions to combined Korteweg-de Vries (KdV) equation with variable coefficients and forcible term, discrete KdV equation and Zakharov–Kuznetsov (ZK) equation with arbitrary nonlinear term by the following projective Riccati equations:

$$f'(\xi) = pf(\xi)g(\xi), \tag{3}$$

$$g'(\xi) = q + pg^2(\xi) - rf(\xi). \tag{4}$$

In Refs. [23]–[30], the exact solutions to ZK equation and two kinds of KdV equation with variable coefficients were constructed by the following first kind of elliptic auxiliary equation (5) and the second kind of elliptic auxiliary equation (6):

$$\left[\frac{\mathrm{d}z(\xi)}{\mathrm{d}\xi}\right]^2 = a + bz^2(\xi) + cz^4(\xi), \tag{5}$$

$$\left[\frac{\mathrm{d}z(\xi)}{\mathrm{d}\xi}\right]^2 = az(\xi) + bz^2(\xi) + cz^3(\xi). \tag{6}$$

In Refs. [31]–[41], many soliton-like and other exact solutions to nonlinear discrete system, a set of (2+1)-dimensional dispersive long wave equations were derived by auxiliary equation method of triangular function type and hyperbolic function type. In Refs. [3]–[41], many exact solutions to nonlinear evolution equations were obtained by the auxiliary equation method.

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These literatures have the following characters: obtaining the new solutions of the auxiliary equation as soon as possible or seeking new formal solutions of nonlinear evolution equations. As a matter of fact, we can obtain the finite exact solutions of nonlinear evolution equations by using the above auxiliary equation methods. Therefore, to seek infinite sequence exact solutions to nonlinear evolution equations, at first, we give some new solutions of the Riccati equation (2), Bäcklund transformation and formula of nonlinear superposition of the solutions, then construct an infinite sequence of exact solutions to the ZK equation, KS equation and a set of $(2+1)$-dimensional asymmetric Nizhnik–Novikov–Veselov (ANNV) equations, which include an infinite sequence of complexiton solutions, hyperbolic function and triangular function solutions. The method is of significance to construct an infinite sequence of exact solutions to other nonlinear evolution equations.

2. Application of the method and the formula of nonlinear superposition and Bäcklund transformation of solutions to Riccati equation

2.1. Application of the method

Suppose the given nonlinear evolution equation

$$H(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots) = 0. \tag{7}$$

Taking traveling transformation $u(x,t) = u(\xi), \xi = px + \omega t$, where p and ω are constants to be determined, we obtain the following ordinary differential equation:

$$G(u, u_\xi, u_{\xi\xi}, u_{\xi\xi\xi}, \ldots) = 0. \tag{8}$$

Set the formal solution of Eq. (8) as follows:

$$u(x,t) = u(\xi) = \sum_{s=0}^{r} g_s z^s(\xi), \tag{9}$$

where g_s $(s = 0, 1, 2, \ldots, r)$ are constants to be determined, r is a natural number to be determined by balancing nonlinear term and highest order derivative, $z(\xi)$ is the solution of the Riccati equation (2). Substituting Eqs. (2) and (9) into Eq. (8), and setting the coefficients of all powers of $z(\xi)$ to zero, we obtain a set of nonlinear algebraic equations about g_s $(s = 0, 1, 2, \ldots, r)$, p, ω with the help of symbolic computation system Mathematica, solving the set of nonlinear algebraic equations, and taking each solution with an arbitrary solution of Riccati equation (2) into Eq. (9), we obtain the exact solutions of Eq. (7).

2.2. Bäcklund transformation of the solution

When $z_{k-1}(\xi)$ is the solution of Eq. (2), the following $z_k(\xi)$ are also the solutions of Eq. (2):

$$z_k(\xi) = \frac{-aB + cnz_{k-1}(\xi)}{bB + cn + cBz_{k-1}(\xi)}, \quad (k = 1, 2, 3, \cdots), \tag{10}$$

$$z_k(\xi) = -\frac{b}{c} + \frac{bA}{cA + cmz_{k-1}(\xi) - bBz_{k-1}^2(\xi)}, \quad (k = 1, 2, 3, \ldots). \tag{11}$$

2.3. The formula of nonlinear superposition of the solution

When $z_{k-1}(\xi)$ and $z_{k-2}(\xi)$ are the solutions of Eq. (2), the following $z_k(\xi)$ $(k = 2, 3, 4, \ldots)$ are also the solutions of Eq. (2):

$$z_k(\xi) = \frac{N_1\left[\mp 2a^2A + \left[\mp 2a^2C + N_3 z_{k-1}(\xi)\right] z_{k-2}(\xi)\right]}{N_2 \pm N_4\left[Abz_{n-2}(\xi) + [Ab - aC + Acz_{k-2}(\xi)]z_{k-1}(\xi)\right]}, \tag{12}$$

where $N_1 = (b \pm \sqrt{b^2 - 4ac})^2$, $N_2 = 4a\left[\pm 2Ab^3 \mp 5aAbc \mp ab^2C \pm 2a^2cC - \sqrt{b^2 - 4ac}(-2Ab^2 + aAc + abC)\right]$, $N_3 = [\pm Ab^2 \mp 2aAc \mp abC + \sqrt{b^2 - 4ac}(Ab - aC)]$, $N_4 = [\pm Ab^2 \mp 2aAc \mp abC + \sqrt{b^2 - 4ac}(Ab - aC)]$.

When $z_{k-1}(\xi)$, $z_{k-2}(\xi)$ and $z_{k-3}(\xi)$ are the solutions of Eq. (2), the following $z_k(\xi)$ $(k = 3, 4, \ldots)$ are also the solutions of Eq. (2):

$$z_k(\xi) = \frac{aNz_{k-3}(\xi) - [P_2(\xi) + (al + bL)z_{k-3}(\xi)]z_{k-1}(\xi) + P_1(\xi)z_{k-2}(\xi)}{anz_{k-3}(\xi) + [P_3(\xi) + cLz_{k-3}(\xi)]z_{k-1}(\xi) + [al - c(L + N)z_{k-3}(\xi)]z_{k-2}(\xi)}, \tag{13}$$

where $P_1(\xi) = aL + [a(l + n) + b(L + N)]z_{k-3}(\xi)$, $P_2(\xi) = a(L + N) + (an + bN)z_{k-2}(\xi)$, $P_3(\xi) = -a(l + n) + cNz_{k-2}(\xi)$, a, b, c, N, L, m, n, l, A, B and C are arbitrary constants.

2.4. The solutions of Riccati equation (2)

We derive the solutions of Eq. (2) as follows:

$$z_0(\xi) = -\frac{1}{2c}\left[b + \sqrt{b^2-4ac}\tanh\left[\frac{1}{2}(\sqrt{b^2-4ac}\xi)\right]\right],\ (b^2-4ac>0), \tag{14}$$

$$z_0(\xi) = -\frac{1}{2c}\left[b + \sqrt{b^2-4ac}\coth\left[\frac{1}{2}(\sqrt{b^2-4ac}\xi)\right]\right],\ (b^2-4ac>0), \tag{15}$$

$$z_0(\xi) = \frac{1}{2c}\left[-b + \sqrt{-b^2+4ac}\tan\left[\frac{1}{2}(\sqrt{-b^2+4ac}\xi)\right]\right],\ (b^2-4ac<0), \tag{16}$$

$$z_0(\xi) = -\frac{1}{2c}\left[b + \sqrt{-b^2+4ac}\cot\left[\frac{1}{2}(\sqrt{-b^2+4ac}\xi)\right]\right],\ (b^2-4ac<0), \tag{17}$$

$$z_0(\xi) = \frac{2a\left(d_1\exp(\sqrt{b^2-4ac}\xi)+d_2\right)}{(-b+\sqrt{b^2-4ac})\left(d_1\exp(\sqrt{b^2-4ac}\xi)+d_2\right)-2d_2\sqrt{b^2-4ac}},\ (b^2-4ac>0), \tag{18}$$

$$z_0(\xi) = -\frac{2a\left[d_1\cos\left(\frac{1}{2}\sqrt{-b^2+4ac}\xi\right)+d_2\sin\left(\frac{1}{2}\sqrt{-b^2+4ac}\xi\right)\right]}{D_1\cos\left(\frac{1}{2}\sqrt{-b^2+4ac}\xi\right)+D_2\sin\left(\frac{1}{2}\sqrt{-b^2+4ac}\xi\right)},\ (b^2-4ac<0), \tag{19}$$

$$z_0(\xi) = -\frac{2a(d_1+d_2\xi)}{bd_1+d_2(-2+b\xi)},\ (b^2-4ac=0), \tag{20}$$

$$z_0(\xi) = -\frac{1}{d_1+c\xi},\ (a=b=0), \tag{21}$$

where $D_1 = bd_1 - d_2\sqrt{-b^2+4ac}$, $D_2 = d_1\sqrt{-b^2+4ac} + bd_2$, d_1, d_2 are arbitrary constants.

Based on the solutions (14)–(21), Bäcklund transformations (10), (11) and the formula of nonlinear superposition of Eqs. (12), (13), we can seek the infinite sequence of new solutions of Eq. (2). Several formulas are given as follows:

$$z_k(\xi) = \frac{-aB+cnz_{k-1}(\xi)}{bB+cn+cBz_{k-1}(\xi)},\quad (k=1,2,3,\ldots),$$

$$z_0(\xi) = -\frac{1}{2c}\left[b + \sqrt{b^2-4ac}\tanh\left[\frac{1}{2}(\sqrt{b^2-4ac}\xi)\right]\right],\ (b^2-4ac>0). \tag{22}$$

The solutions of Eq. (2) are obtained by the first iteration of the formula of nonlinear superposition Eq. (22), we obtain the solutions of Eq. (2) as follows:

$$z_1(\xi) = \frac{2aB+bn+n\sqrt{b^2-4ac}\tanh\left[\frac{1}{2}(\sqrt{b^2-4ac}\xi)\right]}{-bB-2cn+B\sqrt{b^2-4ac}\tanh\left[\frac{1}{2}(\sqrt{b^2-4ac}\xi)\right]},\quad (b^2-4ac>0), \tag{23}$$

$$z_k(\xi) = \frac{-aB+cnz_{k-1}(\xi)}{bB+cn+cBz_{k-1}(\xi)},\ (k=1,2,3,\ldots),$$

$$z_0(\xi) = \frac{1}{2c}\left[-b + \sqrt{-b^2+4ac}\tan\left[\frac{1}{2}(\sqrt{-b^2+4ac}\xi)\right]\right],\ (b^2-4ac<0). \tag{24}$$

Applying the first iteration of the formula of nonlinear superposition Eq. (24), we derive the following solutions of Eq. (2):

$$z_1(\xi) = \frac{-2aB-bn+n\sqrt{-b^2+4ac}\tan\left[\frac{1}{2}(\sqrt{-b^2+4ac}\xi)\right]}{bB+2cn+B\sqrt{-b^2+4ac}\tan\left[\frac{1}{2}(\sqrt{-b^2+4ac}\xi)\right]},\ (b^2-4ac<0), \tag{25}$$

$$z_k(\xi) = \frac{aNz_{k-3}(\xi)-[P_2(\xi)+(al+bL)z_{k-3}(\xi)]z_{k-1}(\xi)+P_1(\xi)z_{k-2}(\xi)}{anz_{k-3}(\xi)+[P_3(\xi)+cLz_{k-3}(\xi)]z_{k-1}(\xi)+[al-c(L+N)z_{k-3}(\xi)]z_{k-2}(\xi)},\quad (k=3,4,5,\ldots),$$

$$z_2(\xi)=\frac{2aB+bn+n\sqrt{b^2-4ac}\tanh\left[\frac{1}{2}(\sqrt{b^2-4ac}\xi)\right]}{-bB-2cn+B\sqrt{b^2-4ac}\tanh\left[\frac{1}{2}(\sqrt{b^2-4ac}\xi)\right]},\quad z_1(\xi)=-\frac{1}{d_1+c\xi},$$
$$z_0(\xi)=-\frac{1}{2c}\left[b+\sqrt{b^2-4ac}\tanh\left[\frac{1}{2}(\sqrt{b^2-4ac}\xi)\right]\right], \tag{26}$$

where $P_1(\xi)=aL+[a(l+n)+b(L+N)]z_{k-3}(\xi)$, $P_2(\xi)=a(L+N)+(an+bN)z_{k-2}(\xi)$, $P_3(\xi)=-a(l+n)+cNz_{k-2}(\xi)$.

With the first iteration of the formula of nonlinear superposition (26), we obtain the following complexiton solutions of Eq. (2):

$$z_3(\xi)=-\frac{\Phi_1(\xi)+\Phi_2(\xi)\tanh\left(\frac{1}{2}\sqrt{b^2-4ac}\xi\right)+M_3\phi_8(\xi)\tanh^2\left(\frac{1}{2}\sqrt{b^2-4ac}\xi\right)}{\Psi_1(\xi)+\Psi_2(\xi)\tanh\left(\frac{1}{2}\sqrt{b^2-4ac}\xi\right)+M_5\varphi_8(\xi)\tanh^2\left(\frac{1}{2}\sqrt{b^2-4ac}\xi\right)}, \tag{27}$$

$$z_k(\xi)=-\frac{b}{c}+\frac{bA}{cA+cmz_{k-1}(\xi)-bBz_{k-1}^2(\xi)},\quad (k=1,2,3,\ldots),$$
$$z_0(\xi)=-\frac{\Phi_1(\xi)+\Phi_2(\xi)\tanh\left(\frac{1}{2}\sqrt{b^2-4ac}\xi\right)+M_3\phi_8(\xi)\tanh^2\left(\frac{1}{2}\sqrt{b^2-4ac}\xi\right)}{\Psi_1(\xi)+\Psi_2(\xi)\tanh\left(\frac{1}{2}\sqrt{b^2-4ac}\xi\right)+M_5\varphi_8(\xi)\tanh^2\left(\frac{1}{2}\sqrt{b^2-4ac}\xi\right)},\quad (b^2-4ac>0), \tag{28}$$

where $\Phi_1(\xi)=M_1[\phi_1(\xi)+\phi_2(\xi)]$, $\Phi_2(\xi)=2\sqrt{b^2-4ac}\left[\phi_3(\xi)+a^2B\left[\phi_4(\xi)+M_2\right]+an[\phi_5(\xi)+c^2\phi_6(\xi)+\phi_7(\xi)]\right]$, $\Psi_1(\xi)=M_4[\varphi_1(\xi)+2c\varphi_2(\xi)]$, $\Psi_2(\xi)=2c\sqrt{b^2-4ac}\left[\varphi_3(\xi)+\varphi_4(\xi)+a\left[\varphi_5(\xi)+c[\varphi_6(\xi)+\varphi_7(\xi)]\right]\right]$, $\phi_1(\xi)=bc[-B(L+N)+(aB+cL)n\xi]$, $\phi_2(\xi)=b(aB+cL)nd_1+2c\left[-cLn+aB[-n+(L+N)(c\xi+d_1)]\right]$, $\phi_3(\xi)=a^3B^2n^2(c\xi+d_1)+bcLn^2N[c(-1+b\xi)+bd_1]$, $\phi_4(\xi)=c\left[BLn+BN[bn+c(L+N)]\xi+n^2(-n+2cL\xi+bn\xi)\right]$, $\phi_5(\xi)=-B^3LnN\xi+b^2BnNd_1$, $\phi_6(\xi)=[BN^2+L(n^2+2BN)](-1+b\xi)-LnNd_1$, $\phi_7(\xi)=bc[Ln^2d_1+BN[n(-1+b\xi)+(2L+N)d_1]]$, $\phi_8(\xi)=c[cLn\xi+B(L+N+an\xi)]+(aB+cL)nd_1$, $\varphi_1(\xi)=bc[-B(L+N)+(aB+cL)n\xi]+b(aB+cL)nd_1$, $\varphi_2(\xi)=-cLn+aB[-n+(L+N)(c\xi+d_1)]$, $\varphi_3(\xi)=cLn^2N[c(-1+b\xi)bd_1]$, $\varphi_4(\xi)=a^2Bn\left[n^2(c\xi+d_1)+B[n-L(c\xi+d_1)]\right]$, $\varphi_5(\xi)=bBn[B(L+N)+nN(c\xi+d_1)]$, $\varphi_6(\xi)=B^2N(L+N)+Ln^3(c\xi+d_1)$, $\varphi_7(\xi)=Bn[N^2(c\xi+d_1)+2L(n+cN\xi+Nd_1)]$, $\varphi_8(\xi)=c[cLn\xi+B(L+N+an\xi)]+(aB+cL)nd_1$, $M_1=an(2aB+bn)+(abB+b^2n-2ac)N$, $M_2=[bn(n^2+BN)+c(2Ln^2+BLN+BN^2)]d_1$, $M_3=(b^2-4ac)[bnN+a(n^2+BN)]$, $M_4=bcnN+a[bBn+2c(n^2+BN)]$, $M_5=(b^2-4ac)n(-aB+cN)$, $b^2-4ac>0$.

$$z_k(\xi)=\frac{aNz_{k-3}(\xi)-[P_2(\xi)+(al+bL)z_{k-3}(\xi)]z_{k-1}(\xi)+P_1(\xi)z_{k-2}(\xi)}{anz_{k-3}(\xi)+[P_3(\xi)+cLz_{k-3}(\xi)]z_{k-1}(\xi)+[al-c(L+N)z_{k-3}(\xi)]z_{k-2}(\xi)},\quad (k=3,4,5,\ldots),$$
$$z_2(\xi)=\frac{-2aB-bn+n\sqrt{-b^2+4ac}\tan\left[\frac{1}{2}(\sqrt{-b^2+4ac}\xi)\right]}{bB+2cn+B\sqrt{-b^2+4ac}\tan\left[\frac{1}{2}(\sqrt{-b^2+4ac}\xi)\right]},\quad z_1(\xi)=-\frac{1}{d_1+c\xi},$$
$$z_0(\xi)=\frac{1}{2c}\left[-b+\sqrt{-b^2+4ac}\tan\left[\frac{1}{2}(\sqrt{-b^2+4ac}\xi)\right]\right], \tag{29}$$

where $P_1(\xi)=aL+[a(l+n)+b(L+N)]z_{k-3}(\xi)$, $P_2(\xi)=a(L+N)+(an+bN)z_{k-2}(\xi)$, $P_3(\xi)=-a(l+n)+cNz_{k-2}(\xi)$.

Applying the first iteration of the formula of nonlinear superposition (29), we seek the following complexiton solutions of Eq. (2):

$$z_3(\xi)=-\frac{\Upsilon_1(\xi)+\Upsilon_2(\xi)\tan\left(\frac{1}{2}\sqrt{-b^2+4ac}\xi\right)-Q_2\chi_9(\xi)\tan^2\left(\frac{1}{2}\sqrt{-b^2+4ac}\xi\right)}{\Gamma_1(\xi)+\Gamma_2(\xi)\tan\left(\frac{1}{2}\sqrt{-b^2+4ac}\xi\right)-Q_4\psi_8(\xi)\tan^2\left(\frac{1}{2}\sqrt{-b^2+4ac}\xi\right)}, \tag{30}$$

$$z_k(\xi) = -\frac{b}{c} + \frac{bA}{cA + cmz_{k-1}(\xi) - bBz_{k-1}^2(\xi)}, \quad (k = 1, 2, 3, \ldots),$$
$$z_0(\xi) = -\frac{\Upsilon_1(\xi) + \Upsilon_2(\xi)\tan\left(\frac{1}{2}\sqrt{-b^2+4ac}\xi\right) - Q_2\chi_9(\xi)\tan^2\left(\frac{1}{2}\sqrt{-b^2+4ac}\xi\right)}{\Gamma_1(\xi) + \Gamma_2(\xi)\tan\left(\frac{1}{2}\sqrt{-b^2+4ac}\xi\right) - Q_4\psi_8(\xi)\tan^2\left(\frac{1}{2}\sqrt{-b^2+4ac}\xi\right)}, \quad (b^2-4ac<0), \quad (31)$$

where $\Gamma_1(\xi) = Q_3\left[\psi_1(\xi) + \psi_2(\xi)\right]$, $\Gamma_2(\xi) = 2c\sqrt{-b^2+4ac}\left[\psi_3(\xi) + \psi_4(\xi) + a\left[\psi_5(\xi) + c[\psi_6(\xi) + \psi_7(\xi)]\right]\right]$, $\Upsilon_1(\xi) = Q_1[\chi_1(\xi) + \chi_2(\xi)]$, $\Upsilon_2(\xi) = -2\sqrt{-b^2+4ac}\left[\chi_3(\xi) + a^2B[\chi_4(\xi) + c\chi_5(\xi)] + an[\chi_6(\xi) + \chi_7(\xi) + \chi_8(\xi)]\right]$, $\chi_1(\xi) = b\left[c^2Ln\xi + aBnd_1 + c[-B(L+N-an\xi) + Lnd_1\right]$, $\chi_2(\xi) = 2c\left[-cLn + aB[-n + (L+N)(c\xi+d_1)]\right]$, $\chi_3(\xi) = a^3C^2n^2(c\xi+d_1) + bcLn^2N[c(-1+b\xi) + bd_1]$, $\chi_4(\xi) = c^2(2Ln^2 + BLN + BN^2)\xi + bn(n^2+BN)d_1$, $\chi_5(\xi) = n^2(-n+bn\xi+2Ld_1) + B(Ln+bnN\xi+LNd_1+N^2d_1)$, $\chi_6(\xi) = -c^3LnN\xi + b^2BnNd_1$, $\chi_7(\xi) = c^2[BN^2(-1+b\xi) + L[(n^2+2BN)(-1+b\xi) - nNd_1]]$, $\chi_8(\xi) = bc\left[Ln^2d_1 + BN[n(-1+b\xi) + (2L+N)d_1]\right]$, $\chi_9(\xi) = c^2Ln\xi + aBnd_1 + c[B(L+N+an\xi) + Lnd_1]$, $\psi_1(\xi) = b\left[c^2Ln\xi + aBnd_1 + c[-B(L+N-an\xi) + Lnd_1]\right]$, $\psi_2(\xi) = 2c\left[-cLn + aB[-n + (L+N)(c\xi+d_1)]\right]$, $\psi_3(\xi) = cLn^2N[c(-1+b\xi) + bd_1]$, $\psi_4(\xi) = a^2Bn[n^2(c\xi+d_1) + B[n - L(c\xi+d_1)]]$, $\psi_5(\xi) = bBn[B(L+N) + nN(c\xi+d_1)]$, $\psi_6(\xi) = B^2N(L+N) + Ln^3(c\xi+d_1)$, $\psi_7(\xi) = Bn[N^2(c\xi+d_1) + 2L(n+cN\xi+Nd_1)]$, $\psi_8(\xi) = c^2Ln\xi + aBnd_1 + c[B(L+n+an\xi) + Lnd_1]$, $Q_1 = 2a^2Bn + b^2nN + a[-2cnN + b(n^2+BN)]$, $Q_2 = (b^2-4ac)[bnN + a(n^2+BN)]$, $Q_3 = -bcnN - a[bBn + 2c(n^2+BN)]$, $Q_4 = (b^2-4ac)n(aB-cN)$, $b^2 - 4ac < 0$.

$$z_k(\xi) = -\frac{b}{c} + \frac{bA}{cA + cmz_{k-1}(\xi) - bBz_{k-1}^2(\xi)}, \quad (k = 1, 2, 3, \ldots),$$
$$z_0(\xi) = -\frac{1}{2c}\left[b + \sqrt{b^2-4ac}\tanh\left[\frac{1}{2}(\sqrt{b^2-4ac}\xi)\right]\right], \quad (b^2-4ac>0); \quad (32)$$
$$z_k(\xi) = -\frac{b}{c} + \frac{bA}{cA + cmz_{k-1}(\xi) - bBz_{k-1}^2(\xi)}, \quad (k = 1, 2, 3, \ldots),$$
$$z_0(\xi) = \frac{1}{2c}\left[-b + \sqrt{-b^2+4ac}\tan\left[\frac{1}{2}(\sqrt{-b^2+4ac}\xi)\right]\right], \quad (b^2-4ac<0). \quad (33)$$

By the solutions of Eq. (2), Bäcklund transformations (10), (11) and the formula of nonlinear superposition of Eqs. (12), (13), we obtain the infinite sequence of new solutions of Eq. (2).

3. Application of the method

Example 1 The exact solutions of the ZK equation[23]

$$u_t + \delta u_x + \alpha u u_x + \beta u_{xxx} + \gamma u_{xyy} = 0. \quad (34)$$

Substituting $u(x, y, t) = u(\xi), \xi = px + qy + \omega t$ into Eq. (34), and integrating once about ξ, we obtain the following ordinary differential equation

$$(\omega + p\delta)u + \frac{1}{2}p\alpha u^2 + (p^3\beta + \gamma pq^2)u'' = 0. \quad (35)$$

Balancing u^2 and u'', we obtain $r = 2$. Set the solution of Eq. (35) as follows:

$$u(\xi) = g_0 + g_1 z(\xi) + g_2 z^2(\xi), \quad (36)$$

where g_0, g_1, g_2, p, q and ω are constants to be determined. Substituting Eqs. (36) and (2) into Eq. (35), and setting the coefficients of all powers of $z^j(\xi)$ $(j = 0, 1, 2, 3, 4)$ to zero, we obtain a set of nonlinear algebraic equations. Solving the set of equations with the aid of Mathematica, we obtain the following solutions:

$$\omega = p[(b^2-4ac)(p^2\beta + q^2\gamma) - \delta],$$
$$g_0 = -\frac{2}{\alpha}(b^2+2ac)(p^2\beta + q^2\gamma),$$
$$g_2 = -\frac{12c^2}{\alpha}(p^2\beta + q^2\gamma),$$
$$g_1 = -\frac{12cb}{\alpha}(p^2\beta + q^2\gamma); \quad (37)$$
$$\omega = -p[(b^2-4ac)(p^2\beta + q^2\gamma) + \delta],$$
$$g_0 = -\frac{12ac}{\alpha}(p^2\beta + q^2\gamma),$$
$$g_2 = -\frac{12c^2}{\alpha}(p^2\beta + q^2\gamma),$$
$$g_1 = -\frac{12cb}{\alpha}(p^2\beta + q^2\gamma). \quad (38)$$

Inserting Eqs. (37) and (38) into Eq. (36), we can de-

rive exact solutions of Eq. (34) as follows:

$$u_1(x,y,t) = -\frac{2(p^2\beta+q^2\gamma)}{\alpha} \times[b^2+2ac+6bcz(\xi)+6c^2z^2(\xi)], \quad (39)$$

$$u_2(x,y,t) = -\frac{12c(p^2\beta+q^2\gamma)}{\alpha}[a+bz(\xi)+cz^2(\xi)] = -\frac{12c(p^2\beta+q^2\gamma)}{\alpha}z'(\xi), \quad (40)$$

where $\xi = px+qy+p[(b^2-4ac)(p^2\beta+q^2\gamma)-\delta]t$ and $z(\xi)$ is the solution of Riccati equation (2).

Applying the formula of superposition (28) to construct solutions, and taking the solutions into Eqs. (39), (40), we can search for new infinite sequence of comlexiton solutions of hyperbolic function and rational function to the ZK equation (34). Using the formula of superposition (31) to seek solutions, and substituting the solutions into Eqs. (39), (40), we can obtain a new infinite sequence of comlexiton solutions of triangular function and rational function to the ZK equation (34). By the formula of superposition (32) to search for solutions, and inserting the solutions into Eqs. (39), (40), we can obtain a new infinite sequence of hyperbolic function exact solutions to the ZK equation. With the formula of superposition (33) to look for solutions, and taking it into Eqs. (39), (40), we can obtain new infinite sequence of triangular function exact solutions to the ZK equation.

Example 2 The exact solutions of the KS equation[15]

$$u_t + uu_x + \alpha u_{xx} + \beta u_{xxxx} = 0. \quad (41)$$

Substituting $u(x,t) = u(\xi)$, $\xi = px+\omega t$ into Eq. (41), and integrating once about ξ, we obtain the following ordinary differential equation:

$$\omega u + \frac{1}{2}pu^2 + \alpha p^2 u' + \beta p^4 u''' = 0. \quad (42)$$

Balancing u^2 and u''', we obtain $r=3$. Suppose that the solution of Eq. (42) is as follows:

$$u(\xi) = g_0 + g_1 z(\xi) + g_2 z^2(\xi) + g_3 z^3(\xi), \quad (43)$$

where g_0, g_1, g_2, g_3, p and ω are constants to be determined. Substituting Eqs. (43) and (2) into Eq. (42), and equating the coefficients of all powers of z^j (ξ) $(j=0,1,2,\ldots,6)$ to zero, we obtain a set of nonlinear algebraic equations, solving the set of nonlinear algebraic equations by Mathematica, we derive the following solutions:

$$\omega = \mp\frac{30\sqrt{11\alpha^3}p}{19\sqrt{19\beta}}, \quad g_0 = \pm\frac{30\sqrt{11\alpha^3}}{19\sqrt{19\beta}},$$
$$g_1 = \frac{270cp\alpha}{19}, \quad g_3 = -120c^3p^3\beta, \quad g_2 = 0,$$
$$a = -\frac{11\alpha}{76cp^2\beta}, \quad b = 0, \quad (\alpha\beta>0); \quad (44)$$

$$\omega = -\frac{30\sqrt{11\alpha^3}p}{19\sqrt{19\beta}}, \quad g_0 = \frac{10[\mp589+27\sqrt{209}]\sqrt{\alpha^3}}{3249\sqrt{\beta}},$$
$$g_1 = \frac{80cp\alpha}{19}, \ g_3 = -120c^3p^3\beta,$$
$$g_2 = \pm60c^2p^2\sqrt{\alpha\beta}, \quad a = -\frac{20\alpha}{171cp^2\beta},$$
$$b = \mp\frac{\sqrt{\alpha}}{3p\sqrt{\beta}}, \quad (\alpha\beta>0); \quad (45)$$

$$\omega = \frac{30\sqrt{11\alpha^3}p}{19\sqrt{19\beta}}, \quad g_0 = \frac{10[\pm589-27\sqrt{209}]\sqrt{\alpha^3}}{3249\sqrt{\beta}},$$
$$g_1 = \frac{80cp\alpha}{19}, \quad g_3 = -120c^3p^3\beta,$$
$$g_2 = \mp60c^2p^2\sqrt{\alpha\beta}, \quad a = -\frac{20\alpha}{171cp^2\beta},$$
$$b = \pm\frac{\sqrt{\alpha}}{3p\sqrt{\beta}}, \quad (\alpha\beta>0); \quad (46)$$

$$\omega = \mp\frac{30\mathrm{i}\sqrt{\alpha^3}p}{19\sqrt{19\beta}}, \quad g_0 = \pm\frac{30\mathrm{i}\sqrt{\alpha^3}}{19\sqrt{19\beta}},$$
$$g_1 = -\frac{90cp\alpha}{19}, \ g_3 = -120c^3p^3\beta, \ g_2 = 0,$$
$$a = \frac{\alpha}{76cp^2\beta}, \ b = 0, \quad (\alpha\beta<0); \quad (47)$$

$$\omega = -\frac{30\mathrm{i}\sqrt{\alpha^3}p}{19\sqrt{19\beta}}, \ g_0 = \frac{10[\pm437+27\mathrm{i}\sqrt{19}]\sqrt{\alpha^3}}{3249\sqrt{\beta}},$$
$$g_1 = -\frac{280cp\alpha}{19}, \ g_3 = -120c^3p^3\beta,$$
$$g_2 = \pm60c^2p^2\sqrt{\alpha\beta}, \quad a = \frac{7\alpha}{171cp^2\beta},$$
$$b = \mp\frac{\sqrt{\alpha}}{3p\sqrt{\beta}}; \quad (48)$$

$$\omega = \frac{30\mathrm{i}\sqrt{\alpha^3}p}{19\sqrt{19\beta}}, \ g_0 = \frac{10[\pm437-27\mathrm{i}\sqrt{19}]\sqrt{\alpha^3}}{3249\sqrt{\beta}},$$
$$g_1 = -\frac{280cp\alpha}{19}, \quad g_3 = -120c^3p^3\beta,$$
$$g_2 = \pm60c^2p^2\sqrt{\alpha\beta}, \quad a = \frac{7\alpha}{171cp^2\beta},$$
$$b = \mp\frac{\sqrt{\alpha}}{3p\sqrt{\beta}}; \quad (49)$$

$$\omega = \mp\frac{30\sqrt{11\alpha^3}p}{19\sqrt{19\beta}},$$
$$g_0 = \frac{15}{361}\left[171bp\alpha \pm \frac{2\sqrt{209\alpha^3}}{\sqrt{\beta}} - 361b^3p^3\beta\right],$$
$$g_1 = c\left[\frac{270p\alpha}{19} - 90b^2p^3\beta\right],$$
$$g_3 = -120c^3p^3\beta, \ g_2 = -180bc^2p^3\beta,$$
$$a = \frac{1}{76c}\left[19b^2 - \frac{11\alpha}{p^2\beta}\right], \quad (\alpha\beta>0); \quad (50)$$

$$\omega = \mp\frac{30\mathrm{i}\sqrt{\alpha^3}p}{19\sqrt{19\beta}},$$
$$g_0 = -\frac{45}{19}bp\alpha \pm \frac{30\mathrm{i}\sqrt{\alpha^3}}{19\sqrt{19\beta}} - 15b^3p^3\beta,$$
$$g_1 = -\frac{90cp}{19}[\alpha + 19b^2p^2\beta],\ g_3 = -120c^3p^3\beta,$$
$$g_2 = -180bc^2p^3\beta,\ a = \frac{\alpha + 19b^2p^2\beta}{76cp^2\beta},\ (\alpha\beta < 0). \quad (51)$$

Inserting Eqs. (44)–(51) and (2) into Eq. (43), we can obtain a new infinite sequence of exact solutions of Eq. (41). For example, substituting Eq. (45) into Eq. (43), we obtain exact solutions of Eq. (41) as follows:

$$\begin{aligned} & u_3(x,t) \\ = & \frac{10[\mp 589 + 27\sqrt{209}]\sqrt{\alpha^3}}{3249\sqrt{\beta}} \\ & + \frac{80cp\alpha}{19}z\left(px - \frac{30\sqrt{11\alpha^3}p}{19\sqrt{19\beta}}t\right) \\ & \pm 60c^2p^2\sqrt{\alpha\beta}z^2\left(px - \frac{30\sqrt{11\alpha^3}p}{19\sqrt{19\beta}}t\right) \\ & - 120c^3p^3\beta z^3\left(px - \frac{30\sqrt{11\alpha^3}p}{19\sqrt{19\beta}}t\right), \end{aligned} \quad (52)$$

where we construct a new infinite sequence of exact solutions of the KS equation (41) by the method of example 1 (not given).

Example 3 The exact solutions of the (2+1)-dimensional ANNV equation[42]

$$u_t - u_{xxx} + \alpha(uv)_x = 0, \quad (53)$$
$$u_x + \alpha v_y = 0. \quad (54)$$

Substituting $u(x,y,t) = u(\xi)$, $v(x,y,t) = v(\xi)$, $\xi = px + qy + \omega t$ into Eqs. (53) and (54), and integrating once about ξ, we obtain the following ordinary differential equation

$$\omega u - p^3u'' + \alpha puv = 0, \quad (55)$$
$$pu + \beta qv = 0. \quad (56)$$

Balancing u'' and uv, u and v, we obtain $r = 2$. Let the solution of Eqs. (55) and (56) be

$$u(\xi) = g_0 + g_1z(\xi) + g_2z^2(\xi), \quad (57)$$
$$v(\xi) = f_0 + f_1z(\xi) + f_2z^2(\xi), \quad (58)$$

where g_0, g_1, g_2, f_0, f_1, f_2, p, q and ω are constants to be determined. Substituting Eqs. (57), (58) and (2) into Eqs. (55), (56), and equating the coefficients of all powers of $z(\xi)$ to zero, we obtain a set of nonlinear algebraic equations, solving the set of nonlinear algebraic equations by Mathematica, we obtain the following solutions

$$\begin{aligned} & \omega = b^2p^3 - 4acp^3,\ f_2 = \frac{6c^2p^2}{\alpha},\ f_1 = \frac{6bcp^2}{\alpha}, \\ & f_0 = \frac{6acp^2}{\alpha},\ g_0 = -\frac{6acpq\beta}{\alpha}, \\ & g_2 = -\frac{6c^2pq\beta}{\alpha},\ g_1 = -\frac{6bcpq\beta}{\alpha}; \end{aligned} \quad (59)$$

$$\begin{aligned} & \omega = -b^2p^3 + 4acp^3,\ f_2 = \frac{6c^2p^2}{\alpha},\ f_1 = \frac{6bcp^2}{\alpha}, \\ & f_0 = \frac{b^2p^2 + 2acp^2}{\alpha},\ g_0 = -\frac{b^2pq\beta + 2acpq\beta}{\alpha}, \\ & g_2 = -\frac{6c^2pq\beta}{\alpha},\ g_1 = -\frac{6bcpq\beta}{\alpha}. \end{aligned} \quad (60)$$

Inserting Eqs. (59), (60) into Eqs. (57), (58), we obtain exact solutions of Eqs. (53) and (54) as follows:

$$\begin{aligned} v_4(x,y,t) &= \frac{6cp^2}{\alpha}[a + bz(\xi) + cz^2(\xi)] \\ &= \frac{6cp^2}{\alpha}z'(\xi), \end{aligned} \quad (61)$$

$$\begin{aligned} u_4(x,y,t) &= -\frac{6cpq\beta}{\alpha}[a + bz(\xi) + cz^2(\xi)] \\ &= -\frac{6cp^2}{\alpha}z'(\xi), \end{aligned} \quad (62)$$

where $\xi = px + qy + (b^2p^3 - 4acp^3)t$.

$$\begin{aligned} & v_5(x,y,t) \\ = & \frac{p^2}{\alpha}[b^2 + 2ac + 6bcz(\xi) + 6c^2z^2(\xi)], \end{aligned} \quad (63)$$

$$\begin{aligned} & u_5(x,y,t) \\ = & -\frac{pq\beta}{\alpha}[b^2 + 2ac + 6bcz(\xi) + 6c^2z^2(\xi)], \end{aligned} \quad (64)$$

where $\xi = px + qy + (-b^2p^3 + 4acp^3)t$ and $z(\xi)$ is the solution of Riccati equation (2). We construct a new infinite sequence of exact solutions of the (2+1)-dimensional ANNV equation by the method of example 1 (not given).

4. Conclusion

In theory, nonlinear evolution equations have infinite solutions. In Refs. [1]–[42], the exact solutions to nonlinear evolution equations by the auxiliary equation method are finite. Because the method for combining finite solutions of the auxiliary equation with the formal solutions of nonlinear evolution equations cannot obtain all infinite sequence solutions of the auxiliary equation. In this paper, to seek infinite sequence exact solutions to nonlinear evolution equations, we gave some new solutions of the Riccati

equation (2), Bäcklund transformations (10) and (11) and formula of nonlinear superposition of the solutions (12) and (13), then construct infinite sequence of exact solutions to ZK equation, KS equation and a set of (2+1)-dimensional ANNV equations. And the solutions include infinite sequence of complexiton solutions, hyperbolic function and triangular function solutions. The method is of significance to construct infinite sequence of exact solutions to other nonlinear evolution equations.

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