Stability for manifolds of equilibrium states of generalized Birkhoff system*

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Stability for the manifolds of equilibrium states of a generalized Birkhoff system is studied. A theorem for the stability of the manifolds of equilibrium states of the general autonomous system is used to the generalized Birkhoffian system and two propositions on the stability of the manifolds of equilibrium states of the system are obtained. An example is given to illustrate the application of the results.

Keywords: generalized Birkhoff system, manifold of equilibrium states, stability
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1. Introduction

In 1983, American physicist Santilli R M pointed out that Birkhoff mechanics is the most general form of mechanics, which is constructed from Hamilton mechanics by the transformation theory.[1] In addition to the generalization of Galilei's relativity, Santilli studied the Birkhoff equations, the transformation theory of Birkhoff equations, and so on.[2] After that, the frame of Birkhoff dynamics was constructed first in Ref. [3]. In recent years, the study of Birkhoff dynamics has become a popular subject and many important results have been obtained.[4−17] The Birkhoffian mechanics is the generalization of the Hamiltonian mechanics, and it is applicable to atomic and molecular physics and hadron physics.[12] We advanced a generalized Birkhoffian system in 1993, which is superior to traditional classical mechanical systems.[18] Recently, the study of the generalized Birkhoff system made a series of contributions such as the basic frame,[19] the inverse problem of dynamics[20] the integral invariant,[21] the time integral theorem[22] the conformal invariance[23] as well as the symmetry.[18]

Mei has done much research on the stability of the Birkhoff system.[24−26] Fu *et al.* further studied the stability of the relativistic Birkhoff system.[27−29] When the equilibrium positions of the system are not isolated, we need to investgate the stability of the manifolds of equilibrium states. In this paper, the stability of the manifolds of equilibrium states of the generalized Birkhoff system is studied. First, a theorem for the stability of the manifolds of equilibrium states of the general autonomous system is introduced. Secondly, two propositions on the stability of the manifolds of equilibrium states of the generalized Birkhoff system are obtained. Finally an example is given to illustrate the application of the results.

2. Equations of motion of generalized Birkhoff system

The generalized Birkhoff equations are[18,19]

$$\Omega_{\mu\nu}\dot{a}^{\nu} - \frac{\partial B}{\partial a^{\mu}} - \frac{\partial R_{\mu}}{\partial t} = -\Lambda_{\mu}, \quad (\mu,\ \nu = 1,\ 2, \ldots,\ 2n), \tag{1}$$

where $B = B(t,\ \boldsymbol{a})$ is called the Birkhoffian, $R_{\mu} = R_{\mu}(t,\ \boldsymbol{a})$ are the Birkhoff functions, $\Lambda_{\mu} = \Lambda_{\mu}(t,\ \boldsymbol{a})$ are called additional items and

$$(\Omega_{\mu\nu}) = \left(\frac{\partial R_{\nu}}{\partial a^{\mu}} - \frac{\partial R_{\mu}}{\partial a^{\nu}}\right). \tag{2}$$

When $\Lambda_{\mu} = 0$ ($\mu = 1,\ 2,\ \ldots,\ 2n$), the generalized Birkhoff equations become the Birkhoff equations.

Suppose that the system is nonsingular, i.e.

$$\det(\Omega_{\mu\nu}) \neq 0,$$

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then all of $\dot{a}^\mu$ can be solved by Eqs. (1), and we obtain

$$\dot{a}^\mu = F_\mu(t,\ \boldsymbol{a}), \tag{3}$$

where

$$F_\mu = \Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t} - \Lambda_\mu\right),$$
$$\Omega^{\mu\nu}\Omega_{\nu\rho} = \delta^\mu_\rho.$$

3. Stability of the manifolds of equilibrium states of generalized Birkhoff system

A mechanical system with steady constraints is

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}), \tag{4}$$

where $f(\boldsymbol{x}) \in C(R^n,\ R^m)$, and it satisfies the uniqueness condition. Suppose that system (4) has a first integral

$$g(\boldsymbol{x}) = C_0, \quad g(\boldsymbol{x}) \in C(R^n,\ \ R^m),\ \ m < n. \tag{5}$$

Let

$$L = \{\, \boldsymbol{x} |\, g(\boldsymbol{x}) = C_0 = \text{const}\}\,, \tag{6}$$

obviously, L is a manifold in R^n, and it is an invariant set of system (4). Let initial value $\boldsymbol{x}_0 \in L$, then the solutions satisfying the initial value will also belong to L, i.e.

$$\boldsymbol{x} = \boldsymbol{x}(t_0,\ \boldsymbol{x}_0,\ t) \in L. \tag{7}$$

Suppose that the equilibrium position of system (4) is not isolated, it turns into a submanofold of L and is called the manifold of equilibrium states as follows:

$$\varepsilon = \{\, \boldsymbol{x} |\, f(\boldsymbol{x}) = 0\}\,. \tag{8}$$

For the stability of the manifolds of equilibrium states of general systems, there exists the following theorem.[3]

Theorem For system (4) with steady constraints, if there exists fixed sign function $V(x)$ with respect to ε in L, whose complete derivative $\dot{V}$ is a fixed sign function that is different from the sign of $V(x)$ with respect to ε in L or identically equal to zero, then the manifold of equilibrium states ε is stable in L.

For the generalized Birkhoff system with steady constraints, equations (1) become

$$\left(\frac{\partial R_v}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu - \frac{\partial B}{\partial a^\mu} + \Lambda_\mu = 0. \tag{9}$$

So we can obtain the equilibrium equations

$$\frac{\partial B}{\partial a^\mu} - \Lambda_\mu = 0, \quad (\mu = 1, 2, \ldots, 2n). \tag{10}$$

If we can obtain equilibrium positions $a^\mu = a^\mu_0$ $(\mu = 1, 2, \ldots, 2n)$ from Eq. (10), then they are called isolated equilibrium states, otherwise they are called the manifolds of equilibrium states and denoted as

$$\varepsilon = \left\{ \boldsymbol{a} |\, \frac{\partial B}{\partial a^\mu} - \Lambda_\mu = 0 \right\}. \tag{11}$$

The above theorem is applied to the generalized Birkhoff system, we have the following propositions.

Proposition 1 For the disturbed equations of motion of the generalized Birkhoff system with steady constraints, if there exists fixed sign function $V(\boldsymbol{a})$ with respect to ε in L, whose complete derivative $\dot{V}$ with respect to time t is a fixed sign function that is different from the sign of $V(\boldsymbol{a})$ with respect to ε in L or identically equal to zero, then the manifold of equilibrium states ε is stable in L.

Proposition 2 For the disturbed equations of motion of the generalized Birkhoff system with steady constraints, if there exists a function $V(\boldsymbol{a})$ that is not always positive (negative) with respect to ε in L, and its complete derivative $\dot{V}$ is always positive (negative) with respect to ε in L, then the manifold of equilibrium states ε is unstable in L.

The Birkhoff system is a special case of genaralized Birkhoff system, so the results of the present paper are applicable to the Birkhoff system.

4. Illustrative example

An example is given to illustrate the application of the above results to the generalized Birkhoff system.

The fourth order generalized Birkhoff system is

$$\begin{aligned}
&R_1 = R_2 = 0, \quad R_3 = a^1, \quad R_4 = a^2,\\
&B = \frac{1}{2}\left\{b_{11}(a^1 + a^2)^2 + b_{22}(a^3 + a^4)^2\right\},\\
&\Lambda_1 = \Lambda_2 = b_{12}(a^3 + a^4),\\
&\Lambda_3 = \Lambda_4 = -b_{12}(a^1 + a^2),
\end{aligned} \tag{12}$$

where b_{11}, b_{12}, and b_{22} are constants. Now we come to study the stability of the manifolds of equilibrium states.

The equilibrium equation (10) gives

$$b_{11}(a^1+a^2)+b_{12}(a^3+a^4)=0,$$
$$b_{12}(a^1+a^2)+b_{22}(a^3+a^4)=0. \tag{13}$$

Let

$$\begin{vmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{vmatrix} \neq 0, \tag{14}$$

then we will have

$$a^1+a^2=0, \quad \text{and} \quad a^3+a^4=0. \tag{15}$$

So we obtain the manifold of equilibrium states of the system as follows:

$$\varepsilon=\left\{ a^\mu \middle| \, a^1+a^2=0, a^3+a^4=0 \right\}. \tag{16}$$

Choosing the following Lyapunov function:

$$V=\frac{1}{2}\{b_{11}(a^1+a^2)^2+2b_{12}(a^1+a^2)(a^3+a^4) + b_{22}(a^3+a^4)^2\}, \tag{17}$$

if it satisfies the following conditions:

$$b_{11}>0, \quad \begin{vmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{vmatrix} > 0, \tag{18}$$

then V is positive-definite for $a^1+a^2=0$ and $a^3+a^4=0$. If the following conditions hold:

$$-b_{11}>0, \quad \begin{vmatrix} -b_{11} & -b_{12} \\ -b_{12} & -b_{22} \end{vmatrix} > 0, \tag{19}$$

then V is negative-definite for $a^1+a^2=0$ and $a^3+a^4=0$. We can obtain

$$\dot{V}=0. \tag{20}$$

From Proposition 1 it follows that the manifold of equilibrium states (16) is stable when b_{11}, b_{12} and b_{22} satisfy conditions (18) and (19).

5. Conclusion

The theorem for the stability of the manifolds of equilibrium states of the general system with steady constraint is applied to the generalized Birkhoff system, and some conditions of the stability or unstability of the manifolds of equilibrium states of the system are obtained. The Birkhoff system is a special case of generalized Birkhoff system, so the results of the paper are applicable to the Birkhoff system.

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