Poisson theory and integration method of Birkhoffian systems in the event space*

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This paper focuses on studying the Poisson theory and the integration method of a Birkhoffian system in the event space. The Birkhoff's equations in the event space are given. The Poisson theory of the Birkhoffian system in the event space is established. The definition of the Jacobi last multiplier of the system is given, and the relation between the Jacobi last multiplier and the first integrals of the system is discussed. The researches show that for a Birkhoffian system in the event space, whose configuration is determined by $(2n+1)$ Birkhoff's variables, the solution of the system can be found by the Jacobi last multiplier if $2n$ first integrals are known. An example is given to illustrate the application of the results.

Keywords: Birkhoffian system, event space, method of integration, Jacobi last multiplier

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1. Introduction

The study of dynamics in the event space has important significance not only geometrically but also mechanically. Since the generalized coordinates and the time are on an equal footing in the event space, one can choose parameters elastically to establish the equations in a simpler form, and next by the parametric equations in the event space not only the dynamical equations in the usual configuration space can be obtained, but also the energy integral can be obtained directly. In 1960, Synge[1] studied the dynamics of holonomic conservative systems in the event space. In 1983, Rumjantsev[2] established the parametric equations with multipliers in the event space for nonholonomic conservative systems. Mei [3,4] studied the variational principle, the parametric equations and the method of their integration for nonholonomic nonconservative systems. In the last 20 years, much attention has been paid to the study of the dynamics in the event space, and a series of important results have been made.[5−28] But the researches on the integration method of mechanical systems in the event space are not enough. In this paper, the integration method of Birkhoffian systems in the event space is further studied. The Poisson theory of the systems is established, and the method of Jacobi last multiplier for integrating Birkhoff's equations in the event space is presented.

2. Differential equations of motion of Birkhoffian systems in event space

We consider a Birkhoffian system which is determined by the Birkhoff's variables a^μ $(\mu = 1, \ldots, 2n)$. Now we construct an event space of $(2n+1)$ dimensions. The coordinates of space points are a^μ $(\mu = 1, \ldots, 2n)$ and t, which may be regarded as totally independent variables. Introducing the notation

$$x_1 = t, \quad x_{\mu+1} = a^\mu, \quad (\mu = 1, \ldots, 2n), \tag{1}$$

then, all the variables x_α $(\alpha = 1, \ldots, 2n+1)$ may be given as functions of some parameter τ. Let $x_\alpha = x_\alpha(\tau)$ be some curves of class C^2 such that

$$\frac{\mathrm{d}x_\alpha}{\mathrm{d}\tau} = x'_\alpha \tag{2}$$

are not all zero at the same time, and we have

$$\dot{x}_\alpha = \frac{\mathrm{d}x_\alpha}{\mathrm{d}t} = \frac{x'_\alpha}{x'_1}. \tag{3}$$

For a given Birkhoffian $B = B(t, a^\nu)$ and given Birkhoff's functions $R_\mu = R_\mu(t, a^\nu)$ $(\mu = 1, \ldots, 2n)$, Birkhoff's functions $B_\beta(x_\alpha)$ $(\beta = 1, \ldots, 2n+1)$ of

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parametric form in the event space are defined by the following equalities

$$B_1(x_\alpha) = -B(x_1, x_2, \ldots, x_{2n+1}),$$
$$B_{\mu+1}(x_\alpha) = R_\mu(x_1, x_2, \ldots, x_{2n+1}),$$
$$(\mu = 1, \ldots, 2n). \tag{4}$$

The differential equations of motion for the Birkhoffian system in the event space can be written as follows:

$$\Big(\frac{\partial B_\beta}{\partial x_\alpha} - \frac{\partial B_\alpha}{\partial x_\beta}\Big)x'_\beta = 0, \ (\alpha = 1, \ldots, 2n+1). \tag{5}$$

For the Birkhoffian system in the event space, it is easy to verify[24] that the parametric equations (5) are not independent, in which the first equation of Eqs. (5) is the result of the last $2n$ equations of Eqs. (5). Suppose that one may solve $x'_{\mu+1}$ from the last $2n$ equations of Eqs. (5), that is

$$x'_{\mu+1} = \Omega^{\mu+1,\nu+1}\Omega_{\nu+1,1}x'_1 \ (\mu = 1, \ldots, 2n), \tag{6}$$

where

$$\Omega_{\mu+1,\nu+1} = \frac{\partial B_{\mu+1}}{\partial x_{\nu+1}} - \frac{\partial B_{\nu+1}}{\partial x_{\mu+1}}, \quad \det(\Omega_{\mu+1,\nu+1}) \neq 0,$$
$$\Omega^{\mu+1,\nu+1}\Omega_{\nu+1,\tau+1} = \delta_{\mu+1,\tau+1}. \tag{7}$$

3. Poisson theory of Birkhoffian systems in event space

Differentiating an arbitrary function $F = F(\tau, x_\alpha)$ with respect to the parameter τ along the dynamical trajectory of the Birkhoffian system (5) in the event space, we have

$$\frac{\bar{\mathrm{d}}F}{\mathrm{d}\tau} = \frac{\partial F}{\partial \tau} + x'_1\frac{\partial F}{\partial x_1} + \Omega^{\mu+1,\nu+1}\Omega_{\nu+1,1}x'_1\frac{\partial F}{\partial x_{\mu+1}}, \tag{8}$$

where

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}\tau} = \frac{\partial}{\partial \tau} + x'_1\frac{\partial}{\partial x_1} + \Omega^{\mu+1,\nu+1}\Omega_{\nu+1,1}x'_1\frac{\partial}{\partial x_{\mu+1}}. \tag{9}$$

From the formula (8), we obtain the following Proposition.

Proposition 1 For a Birkhoffian system (5) in the event space, the necessary and sufficient condition under which $I(\tau, x_a) = C$ is a first integral of the system is

$$\frac{\partial I}{\partial \tau} + x'_1\frac{\partial I}{\partial x_1} + \Omega^{\mu+1,\nu+1}\Omega_{\nu+1,1}x'_1\frac{\partial I}{\partial x_{\mu+1}} = 0. \tag{10}$$

Let $I = B_1$, then the condition (10) becomes

$$0 = \frac{\partial B_1}{\partial \tau} + x'_1\frac{\partial B_1}{\partial x_1} + \Omega^{\mu+1,\nu+1}\Omega_{\nu+1,1}x'_1\frac{\partial B_1}{\partial x_{\mu+1}}$$
$$= \frac{\partial B_1}{\partial \tau} + x'_1\frac{\partial B_1}{\partial x_1} + \Omega^{\mu+1,\nu+1}\frac{\partial B_{\nu+1}}{\partial x_1}x'_1\frac{\partial B_1}{\partial x_{\mu+1}}. \tag{11}$$

Therefore, we have another Proposition below.

Proposition 2 For a Birkhoffian system (5) in the event space, if x_1 is not explicitly contained in B_α $(\alpha = 1, \ldots, 2n+1)$, and τ is not explicitly contained in B_1, then the Birkhoffian $B_1 = B$ is a first integral of the system.

Taking the partial derivative of Eq. (10) with respect to the parameter τ, we obtain

$$\frac{\partial}{\partial \tau}\Big(\frac{\partial I}{\partial \tau}\Big) + \frac{\partial x'_1}{\partial \tau}\frac{\partial I}{\partial x_1} + x'_1\frac{\partial}{\partial \tau}\Big(\frac{\partial I}{\partial x_1}\Big) + \frac{\partial \Omega^{\mu+1,\nu+1}}{\partial \tau}\Omega_{\nu+1,1}x'_1\frac{\partial I}{\partial x_{\mu+1}} + \Omega^{\mu+1,\nu+1}\frac{\partial \Omega_{\nu+1,1}}{\partial \tau}x'_1\frac{\partial I}{\partial x_{\mu+1}}$$
$$+ \Omega^{\mu+1,\nu+1}\Omega_{\nu+1,1}\frac{\partial x'_1}{\partial \tau}\frac{\partial I}{\partial x_{\mu+1}} + \Omega^{\mu+1,\nu+1}\Omega_{\nu+1,1}x'_1\frac{\partial}{\partial \tau}\Big(\frac{\partial I}{\partial x_{\mu+1}}\Big) = 0. \tag{12}$$

If τ is not explicitly contained in B_α $(\alpha = 1, \ldots, 2n+1)$ and x'_1, then the above formula becomes

$$\frac{\partial}{\partial \tau}\Big(\frac{\partial I}{\partial \tau}\Big) + x'_1\frac{\partial}{\partial x_1}\Big(\frac{\partial I}{\partial \tau}\Big) + \Omega^{\mu+1,\nu+1}\Omega_{\nu+1,1}x'_1\frac{\partial}{\partial x_{\mu+1}}\Big(\frac{\partial I}{\partial \tau}\Big) = 0. \tag{13}$$

Therefore, we have the following Proposition 3.

Proposition 3 For a Birkhoffian system (5) in the event space, if $I(\tau, x_\alpha) = C$ is a first integral, and τ is not explicitly contained in B_α $(\alpha = 1, \ldots, 2n+1)$ and x'_1, then $\partial I/\partial \tau$, $\partial^2 I/\partial \tau^2, \ldots$ are also first integrals of the system.

Similarly, we have Proposition 4 below.

Proposition 4 For a Birkhoffian system (5) in the event space, if $I(\tau, x_\alpha) = C$ is a first integral, and x_ρ is not explicitly contained in

B_α ($\alpha = 1, \ldots, 2n+1$) and x_1', then $\partial I/\partial x_\rho$, $\partial^2 I/\partial x_\rho^2$, ... are also first integrals of the system.

Propositions 1–4 constitute the Poisson theory of Birkhoffian systems in the event space.

4. Method of Jacobi last multiplier of Birkhoffian systems in event space

The conception of the last multiplier of a system of differential equation was introduced by Jacobi[29] in 1844. Whittaker gave an application of the last multiplier to a Hamilton system with two degrees of freedom when one integral is known in addition to the integral of energy.[29] Mei extended the method to a generalized Hamilton system.[30] Now we further apply the method of Jacobi last multiplier to the Birkhoffian systems in the event space.

Definition For a Birkhoffian system (5) in the event space, if there exists some function $M = M(\tau, x_\alpha)$ satisfying the following condition

$$\frac{\partial M}{\partial \tau} + \frac{\partial}{\partial x_1}(Mx_1') + \frac{\partial}{\partial x_{\mu+1}}(M\Omega^{\mu+1,\nu+1}\Omega_{\nu+1,1}x_1') = 0, \tag{14}$$

then the function M is called the Jacobi last multiplier of the system. Equation (14) is called the determining equation of Jacobi last multiplier of the Birkhoffian system in the event space.

From the definition above, we have the following Propositions.

Proposition 5 For a Birkhoffian system (5) in the event space, if the following condition is satisfied

$$\frac{\partial}{\partial x_{\mu+1}}(\Omega^{\mu+1,\nu+1}\Omega_{\nu+1,1}x_1') = 0, \tag{15}$$

then $M = 1$ is the Jacobi last multiplier of the system, and all the integrals of the system are its Jacobi last multipliers.

Proposition 6 For a Birkhoffian system (5) in the event space, if $M = M(\tau, x_\alpha)$ and $N = N(\tau, x_\alpha)$ are two Jacobi last multipliers of the system, then its quotient M/N is a first integral of the system.

Proof Using the determining equation (14) and the formula (9), we have

$$\begin{aligned}\frac{\bar{\mathrm{d}}}{\mathrm{d}\tau}\Big(\frac{M}{N}\Big) &= \frac{\partial}{\partial\tau}\Big(\frac{M}{N}\Big) + x_1'\frac{\partial}{\partial x_1}\Big(\frac{M}{N}\Big) + \Omega^{\mu+1,\nu+1}\Omega_{\nu+1,1}x_1'\frac{\partial}{\partial x_{\mu+1}}\Big(\frac{M}{N}\Big)\\ &= \frac{1}{N}\Big(\frac{\partial M}{\partial\tau} + x_1'\frac{\partial M}{\partial x_1} + \Omega^{\mu+1,\nu+1}\Omega_{\nu+1,1}x_1'\frac{\partial M}{\partial x_{\mu+1}}\Big) - \frac{M}{N^2}\Big(\frac{\partial N}{\partial\tau} + x_1'\frac{\partial N}{\partial x_1} + \Omega^{\mu+1,\nu+1}\Omega_{\nu+1,1}x_1'\frac{\partial N}{\partial x_{\mu+1}}\Big)\\ &= \frac{1}{N}\Big[-M\frac{\partial}{\partial x_{\mu+1}}(\Omega^{\mu+1,\nu+1}\Omega_{\nu+1,1}x_1')\Big] - \frac{M}{N^2}\Big[-N\frac{\partial}{\partial x_{\mu+1}}(\Omega^{\mu+1,\nu+1}\Omega_{\nu+1,1}x_1')\Big] = 0.\end{aligned} \tag{16}$$

Therefore, the quotient M/N is a first integral of the system.

Proposition 7 For a Birkhoffian system (5) in the event space, if $2n$ first integrals of the system are known, i.e.

$$I_\sigma(\tau, x_\alpha) = C_\sigma \ (\sigma = 1, \ldots, 2n), \tag{17}$$

then the integral of the system is

$$\int \frac{M^*}{\Delta^*}\Big[\mathrm{d}x_{2n+1} - (\Omega^{2n+1,\nu+1}\Omega_{\nu+1,1}x_1')^*\mathrm{d}\tau\Big] = \text{const.}, \tag{18}$$

where Δ denotes the Jacobian

$$\Delta = \frac{\partial(I_1, I_2, \ldots, I_{2n})}{\partial(x_1, x_2, \ldots, x_{2n})}, \tag{19}$$

here $(\ )^*$ denotes that x_σ ($\sigma = 1, \ldots, 2n$) are expressed as functions of x_{2n+1} and τ.

Proof For the Birkhoffian system (5) in the event space, similar to the deduction given by Whittaker in Ref. [29], we can easily obtain

$$\frac{\partial}{\partial x_{\mu+1}}(\Omega^{\mu+1,\nu+1}\Omega_{\nu+1,1}x_1') = \Delta\Big\{\frac{\partial}{\partial x_{2n+1}}\Big[(\Omega^{2n+1,\nu+1}\Omega_{\nu+1,1}x_1')^*/\Delta^*\Big] + \frac{\partial}{\partial\tau}\Big(\frac{1}{\Delta^*}\Big)\Big\}. \tag{20}$$

From Eq. (20), and using the determining equation (14) of the Jacobi last multiplier, we obtain

$$\frac{1}{\Delta M}\frac{\bar{\mathrm{d}}M}{\mathrm{d}\tau}+\frac{\partial}{\partial x_{2n+1}}\Big[(\Omega^{2n+1,\nu+1}\Omega_{\nu+1,1}x_1')^*/\Delta^*\Big]+\frac{\partial}{\partial\tau}\Big(\frac{1}{\Delta^*}\Big)=0, \tag{21}$$

or

$$\frac{\partial}{\partial x_{2n+1}}\Big[(\Omega^{2n+1,\nu+1}\Omega_{\nu+1,1}x_1')^*M^*/\Delta^*\Big]+\frac{\partial}{\partial\tau}\Big(\frac{M^*}{\Delta^*}\Big)=0, \tag{22}$$

which shows that the expression

$$\frac{M^*}{\Delta^*}\Big[\mathrm{d}x_{2n+1}-(\Omega^{2n+1,\nu+1}\Omega_{\nu+1,1}x_1')^*\mathrm{d}\tau\Big]$$

is the perfect differential of some functions of x_{2n+1} and τ, and this establishes the proposition.

5. Examples

As an example, we study the integration problem of a fourth-order Birkhoffian system in the event space, whose Birkhoff's functions are[23]

$$\begin{aligned}&B_1=-x_3-\frac{1}{2}\Big(x_4^2+x_5^2\Big),\\&B_2=x_4,\quad B_3=x_5,\quad B_4=B_5=0.\end{aligned} \tag{23}$$

The differential equations of motion (5) can be written as

$$\begin{aligned}&x_3'+x_4x_4'+x_5x_5'=0,\\&-x_4'=0,\quad -x_1'-x_5'=0,\\&-x_4x_1'+x_2'=0,\quad -x_5x_1'+x_3'=0.\end{aligned} \tag{24}$$

The determining equation (14) of the Jacobi last multiplier can be written as follows:

$$\begin{aligned}&\frac{\partial M}{\partial\tau}+\frac{\partial}{\partial x_1}(Mx_1')+\frac{\partial}{\partial x_2}(Mx_4x_1')\\&\quad+\frac{\partial}{\partial x_3}(Mx_5x_1')-\frac{\partial}{\partial x_5}(Mx_1')=0.\end{aligned} \tag{25}$$

Equation (25) has a solution

$$M=1. \tag{26}$$

In this problem, the system has the following integrals[23]

$$I_1=x_3+\frac{1}{2}(x_4^2+x_5^2)=C_1, \tag{27}$$

$$I_2=x_4=C_2, \tag{28}$$

$$I_3=x_1x_4+x_1x_5+\frac{1}{2}x_1^2-x_2-x_3=C_3. \tag{29}$$

According to the Proposition 4, we can obtain a first integral by the integral (29) as follows:

$$I_4=x_4+x_5+x_1=C_4. \tag{30}$$

From the formulae (27)–(30), we have

$$\begin{aligned}&x_1=C_4-C_2-x_5,\\&x_2=-C_2x_5+\frac{1}{2}C_4^2-C_1-C_3,\\&x_3=C_1-\frac{1}{2}C_2^2-\frac{1}{2}x_5^2,\quad x_4=C_2.\end{aligned} \tag{31}$$

Making some deriving work, we obtain

$$\Delta=\frac{\partial(I_1,I_2,I_3,I_4)}{\partial(x_1,x_2,x_3,x_4)}=1=\Delta^*. \tag{32}$$

The integral (18) can be written as

$$\int\Big[\mathrm{d}x_5-(-x_1')\mathrm{d}\tau\Big]=C_5. \tag{33}$$

After integral, we obtain

$$x_5=C_5-x_1. \tag{34}$$

Formulae (31) and (34) give the solution of the problem, and it contains four arbitrary constants.

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