Enhancing synchronizability by rewiring networks*

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According to different forms of synchronized region, complex networks are divided into type I (unbounded synchronization region) and type II (bounded synchronization region) networks. This paper presents a rewiring algorithm to enhance the synchronizability of type I and type II networks. By utilizing the algorithm for an unweighted and undirected network, a better synchronizability of network with the same number of nodes and edges can be obtained. Numerical simulations on several different network models are used to support the proposed procedure. The relationship between different topological properties of the networks and the number of rewirings are shown. It finds that the final optimized network is independent of the initial network, and becomes homogeneous. In addition the optimized networks have similar structural properties in the sense of degree, and node and edge betweenness centralities. However, they do not have similar cluster coefficients for type II networks. The research may be useful for designing more synchronizable networks and understanding the synchronization behaviour of networks.

Keywords: synchronizability, rewire networks, complex networks, optimized network, network structural property

PACC: 0250, 0290

1. Introduction

Over the past few years, complex networks have been intensively studied in many fields, such as social, biological, mathematical, and engineering sciences.[1] Generally, a complex network is made up of interconnected nodes in which a node is a basic unit with detailed contents. These interactions between nodes determine many basic properties of a network.[2] To understand well the complex dynamical behaviours of many natural systems, we need to study the topological structures of the underlying networks.[3] In fact, the properties of a complex network are mainly determined by its topological structures, i.e., the connections between nodes.

Understanding the effect of networks on the dynamical processes taking place on them is a central issue. One of the main goals in the study of network science is to understand the relation between the network structure and the dynamical processes occurring on it.[2,3] A typical collective dynamical behaviour of the network system is synchronization, in which all the participants behave alike, or even in exactly the same way.[4,5] Oscillator network models, which are natural representations of real systems consisting of coupling units, have been commonly used to characterize synchronization behaviours. One important issue in studying the synchronization of dynamical networks is their synchronizability. For many applications, it is desired to have networks with high synchronizability.[6] Technological networks with desirable synchronizability are typical examples where properly assigned interaction between dynamical units is of high importance. In particular, designing interaction schemes to optimize the performance of computational tasks based on the synchronization of processes in computer networks[7] and designing networks with optimized synchronizability for sensor networks where synchronization is used as a mechanism for consensus reaching.[8] Neuronal networks are other prototypic examples, where studying optimal synchronizability may advance our understanding of their organizing principles. Other examples could be found in biological and social networks.

The synchronizability theorem provided by Pecora and Carroll[9] indicates that the collective synchronous behaviour of a network is completely deter-

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mined by a master stability function. According to different forms of synchronized region,[10] there are two types of synchronization criteria based on the eigenvalues of the coupling matrix of a network: the second largest eigenvalue criterion[11] and the eigenratio criterion.[9,12] The two synchronization criteria lead to two types of networks: type I networks and type II networks. Scientists have proposed many methods for altering the type II networks synchronizability. Assigning proper connection weights methods are proposed for enhancing synchronizability of type II networks.[13−18] A random rewiring algorithm has been proposed by Donetti *et al.*[19,20] which is able to optimize type II networks for their synchronizability. Their algorithm is powerful in finding a network topology with high synchronizability, but it is very expensive to perform and the completely random rewiring strategy limits its application to relatively small networks. Hagberg *et al.*[21] proposed a rewiring algorithm for enhancing the synchronizability of the type II networks. However, their algorithm is not capable of avoiding local minima. It is natural to ask whether or not some algorithms can enhance the synchronizability of Type I networks. In this work, we consider both type I and type II networks. A rewiring algorithm is designed to enhance synchronizability of type I networks and type II networks. And, the algorithm can fast find a network structure with high synchronizability in a few steps. In addition, the algorithm is capable of avoiding local minima. Numerical simulations on some networks with different topological structures are performed to support the algorithm. Then, we discuss the relationship between different topological properties of the networks and the iteration steps.

2. Synchronizability of dynamical networks

Consider a dynamical network consisting of N linearly and diffusively coupled identical nodes, where each node is an n-dimensional nonlinear oscillator. The state equations of this network can be written as:

$$\dot{x}_i = f(x_i) + c\sum_{j=1}^{N} a_{ij}H(x_j), \quad i = 1, 2, \ldots, N, \quad (1)$$

where $f(\cdot)$ is a nonlinear function describing the dynamics of the isolated nodes. We also have a positive scalar c which can be interpreted as a global coupling strength parameter, and $H(\cdot)$ is the output function. In the case of symmetric and unweighted coupling networks, the elements of A are defined as follows: i) if nodes i and j are connected, then $a_{ij} = a_{ji} = 1$; ii) otherwise, $a_{ij} = a_{ji} = 0$. The diagonal entries are $a_{ii} = -k_i$ where k_i is the degree of node i. In graph theory, $-A$ is usually called the Laplace matrix. In the case of connected networks, A is negative semidefinite, so all the eigenvalues of A are nonpositive real values and the largest eigenvalue is zero with multiplicity 1. All the other eigenvalues of A are strictly negative, and are denoted by

$$0 = \lambda_1 > \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_N.$$

The dynamical network (1) is said to show synchronization if $x_1 \to x_2 \to \cdots \to x_N \to s(t)$, as $t \to \infty$, where because of the diffusive coupling configuration, the synchronous state $s(t)$ is a solution of an individual node, i.e., $\dot{s}(t) = f(s(t))$. The variational equation governing the linear stability of the synchronous state of the network can be diagonalized into N blocks

$$\dot{\xi}_i = [Jf(s(t)) + \alpha JH(s)]\xi_i, \quad (2)$$

where $\alpha = c\lambda_i \in R$, c is coupling strength, $Jf(s(t))$ and $JH(s(t))$ are the Jacobian matrices of functions f and H at $s(t)$, respectively.[9,22] The largest Lyapunov exponent $L_{\max}$ which is a function of α and can be calculated from Eq. (2), is the so-called master stability function. It fully determines the linear stability of the synchronous state. Moreover, the region S of negative real α where $L_{\max}$ is also negative is called the synchronized region. It depends on the dynamical function f, inner coupling matrix H, and synchronization state s. Based on the results of Refs. [9] and [12], the synchronized solution of the dynamical network (1) is locally asymptotically stable if and only if $c\lambda_k \subseteq S$, $k = 2, 3, \ldots, N$. According to different forms of synchronized region S, there are three possible types of networks.[10] Type I networks: the synchronized region S is unbounded, in the form of $(-\infty, \alpha]$, then the second largest eigenvalue λ_2 of A characterizes the network synchronizability. Smaller λ_2 leads to better synchronizability. Type II networks: the synchronized region S is bounded, in the form of $[\alpha_1, \alpha_2]$, then the eigenratio $R = \lambda_N/\lambda_2$ of the network structural matrix A characterizes the synchronizability. Smaller eigenratio $R = \lambda_N/\lambda_2$ leads to better synchronizability. Type III networks: $S = \phi$. Synchronization cannot be achieved for this type of network.

3. Rewiring algorithm for better synchronizability

Considering an undirected and unweighted network with N nodes and average degree $\langle k\rangle$, i.e. the number of edges $\langle k\rangle N/2$. Our objective is to enhance synchronizability of the undirected and unweighted network with the same number of nodes and average degree, and thus the same number of edges.

3.1. Type I networks

For this type of network the second largest eigenvalue λ_2 of A characterizes the network synchronizability, and smaller λ_2 leads to better synchronizability. How to obtain a network with the smaller λ_2? If adding edges in a network, then eigenvalues of A always decrease.[23] This implies that synchronizability of the type I networks is enhanced; on the other hand, if we remove edges in a network, then eigenvalues of A always increase.[23] This implies that synchronizability of type I networks is weakened. Ghosh and Boyd[24] have proposed a convex optimization method for growing well-connected networks. They proposed a heuristic greedy perturbation algorithm for adding proper edges to a base network that has a maximum effect on λ_2, and derived lower and upper bounds on the algebraic connectivity of a graph obtained by adding a single edge to a connected graph. If u_2 is the eigenvector with unit norm corresponding to λ_2, then they proposed that adding a nonexistent edge which maximizes $(u_{2i} - u_{2j})^2$ seems to be a good strategy to decrease λ_2 effectively. Inspired by it, we also strategically choose edges which impact λ_2. Here we examine eigenvector u_2 and remove the existent edge (i, j) which minimizes $(u_{2i} - u_{2j})^2$, add the nonexistent edge (m, n) which maximizes $(u_{2m} - u_{2n})^2$.

We consider three classical network models: Barabási–Albert (BA) scale-free network, Newman–Watts (NW) small-world network, and Erdös–Rényi (ER) random network. The BA scale-free network can be constructed by the following algorithm.[25] Starting with a network of m_0 all-to-all connected nodes, at every time step a new node is introduced, which is connected to m already existent nodes. Such an edge connects to old node i with probability $p_i = k_i/\Sigma_j k_j$ where k_i is the degree of the node i. The NW small-world network can be constructed using the following algorithm.[26] Start with a regular ring in which every node is connected to its first $2k$ neighbours (k on each side). Add a new long-range edge (short-cut) into the network with probability p between randomly chosen pair of nodes, avoiding duplications of edges. The ER model of random networks[27] is defined as a random network having N labelled nodes connected by n edges, which are chosen randomly from all the possible edges. The ER network evolves as follows: start with N nodes, every pair of nodes are connected with the same probability p.

We apply this method to the BA preferential attachment network with $m_0 = m = 5$, NW small-world network with $k = 2$, $p = 0.08$, and ER random network with $p = 0.1$ (if the model produced a disconnected network we used the largest connected component so in that case the number of nodes $n \approx N$). While we varied the number of nodes from 20 to 500 and other network parameters in our explorations, these variations change the results without significant qualitative differences. So here we present one representative case. For each model we used the number of nodes $N = 100$, the number of edges 480, i.e. average degree $\langle k\rangle = 9.6$. In Fig. 1, we show the relationships between the synchronizability and change numbers of edges for BA, NW, and ER network models. When removing edge by eigenvector method successively, synchronizability of type I networks (the second largest eigenvalue λ_2) almost keeps invariable. While, the synchronizability sharply increases with adding edges successively. In addition, we find that the second largest eigenvalue λ_2 is almost linearly related with the number of adding edges, until λ_2 reaches the minimum value $-N$.

Therefore, according to the above-mentioned idea, we present an algorithm of re-connecting edge to enhance synchronizability of the type I networks. Considering a network with N nodes and average degree $\langle k\rangle$, the algorithm consists of the following steps:

(i) The second largest eigenvalue λ_2 of the network is calculated and in the first iteration step $(\lambda_2)_{\min} = \lambda_2$.

(ii) For each pair of nonadjacent nodes i and j, the quantity $(u_{2i} - u_{2j})^2$ is calculated, which is used for choosing a pair of nonadjacent nodes to connect an edge between them. Add an edge (i, j) with the largest $(u_{2i} - u_{2j})^2$.

(iii) For each edge e_{ij} (connecting the i-th and j-th nodes) of the network, the quantity $(u_{2i} - u_{2j})^2$ is calculated, which is used to choose one edge for disconnection, i.e., remove the edge (i, j) with the smallest $(u_{2i} - u_{2j})^2$.

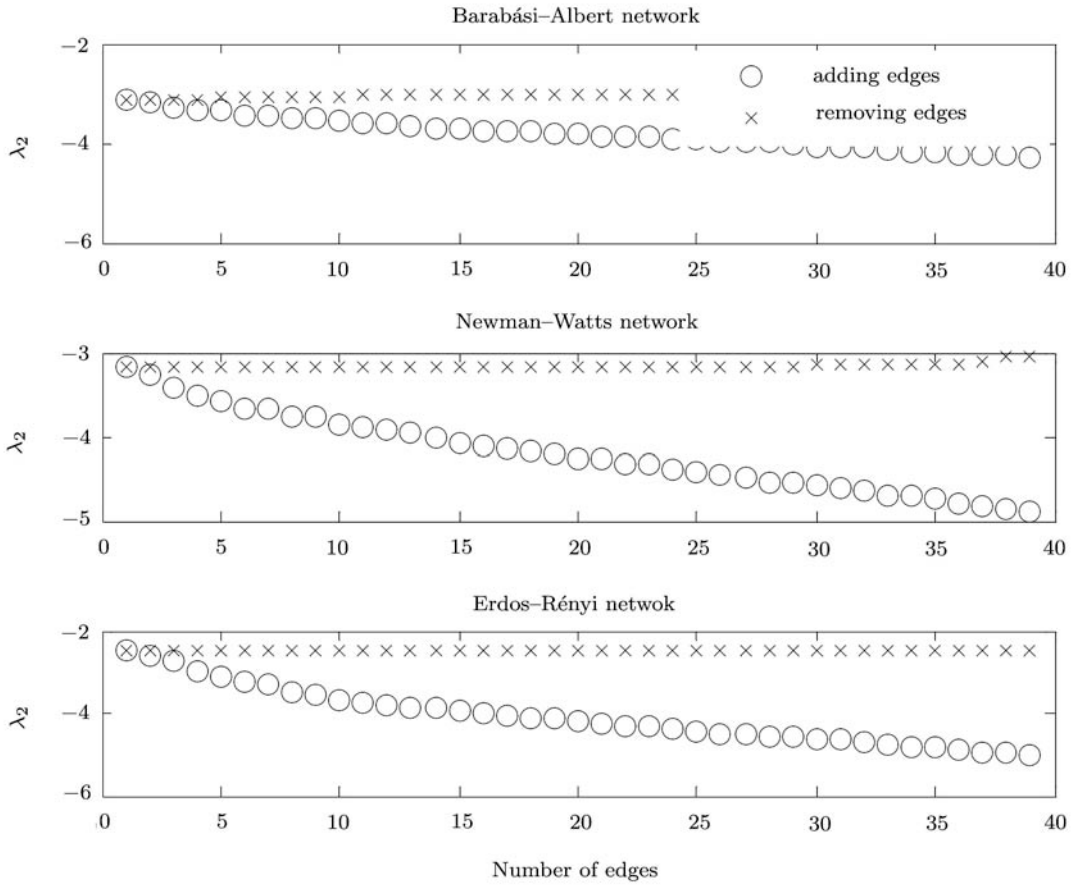


Fig. 1. The second largest eigenvalue λ_2 as a function of number of adding (removing) edges for different networks with $N = 100$ and $\langle k \rangle = 9.6$.

(iv) After rewiring (adding a nonexistent edge and removing an existent one), the cost function λ_2 of the new network is calculated. If eigenvalue $(\lambda_2)_{\rm new}$ is less than the eigenvalue of the old network $(\lambda_2)_{\rm old}$, the rewiring is accepted and $(\lambda_2)_{\rm min} = (\lambda_2)_{\rm new}$. If eigenvalue $(\lambda_2)_{\rm new}$ is more than the eigenvalue of the old network $(\lambda_2)_{\rm old}$, the rewiring is accepted with the probability of ξ (and reject the rewiring with the probability $1 - \xi$), then $(\lambda_2)_{\rm min} = (\lambda_2)_{\rm new}$. If the rewiring is rejected, randomly removing an existent edge and randomly adding a nonexistent edge, i.e., randomly rewiring, then $(\lambda_2)_{\rm min} = (\lambda_2)_{\rm new}$. The ξ is a pre-set parameter which helps this algorithm avoid getting trapped by a local minimum.

(v) The algorithm is stopped after a predetermined number of iterations.

Next we will support this procedure by numerical simulations on some sample networks. The abovementioned BA scale-free network, NW small-world network, ER random network and a class of tunable heterogeneity network model[28] are considered. The tunable heterogeneity network model generates a one-parameter family of networks labeled by $\alpha \in [0, 1]$. This parameter α measures the degree of heterogeneity of the final networks so that $\alpha = 0$ corresponds to the heterogeneous BA network, and $\alpha = 1$ corresponds to the homogeneous ER graphs. For intermediate values of α, one obtains networks which have been grown by combining both preferential attachment and homogeneous randomly linking. Each mechanism in this model is chosen with probabilities $1 - \alpha$ and α, respectively.

The optimization algorithm was applied to these four networks with $N = 100$, $\langle k \rangle = 9.6$, and $\alpha = 0.5$, other network parameters are the same as the abovementioned ones (BA: $m_0 = m = 5$; NW: $k = 2$, $p = 0.08$; and ER: $p = 0.1$). For the algorithm we set $\xi = 0.9$, and the results were averaged over 10 realizations. Figure 2 shows λ_2 as a function of iteration steps. It shows that λ_2 decreases as the algorithm proceeds and a dramatic decrease is obtained by introducing only a few rewirings. This indicates the

effectiveness of the rewiring algorithm for improving the synchronizability of type I networks. As it is seen

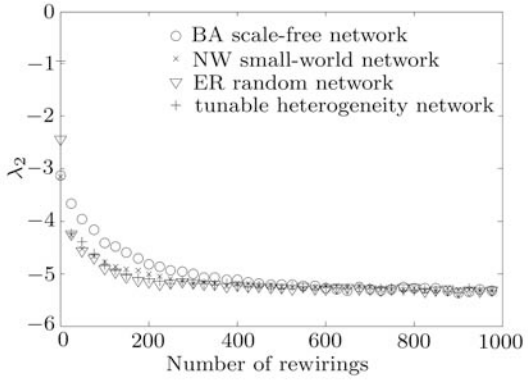


Fig. 2. The changes of the type I networks synchronizability as a function of number of rewirings for different topology structures with $N = 100$, $\langle k \rangle = 9.6$.

in the figure, the four networks have the similar trends as the rewirings proceeding. This implies that the final optimized networks with the similar synchronization properties are independent of the initial networks. In addition, we varied the number of nodes from 50 to 500 and average degree from 3 to 20 in our explorations, these variations change the results without significant qualitative differences.

The synchronizability has been shown to be related with network structural properties, such as node degree, betweenness centrality on nodes, betweenness centrality on edges, cluster. The node betweenness NBC_k is the number of shortest paths between two (other) nodes that pass through node k. Precisely, the betweenness of node k is defined to be $\mathrm{NBC}_k = \Sigma_{(i,j)} g_{ikj}/g_{ij}$.[29,30] where, g_{ij} is the number of geodesic shortest paths between i and j, and g_{ikj} is the number of shortest paths between i and j passing through node k. The summation is to be performed over all pairs of nodes (i, j), such as $i, j \neq k$ and $i \neq j$. Similar to node betweenness centrality, the betweenness centrality of an edge (i, j) is defined as the number of shortest paths between pairs of other nodes that contain i, j.[31] The cluster coefficient C_i of node i is the ratio of the number of edges between a node's neighbours to the total possible number of edges between the node's neighbours.[32] The cluster coefficient for the whole network is given by Watts and Strogatz[32] as the average of the cluster coefficient for each node, $C = 1/N \sum C_i$.

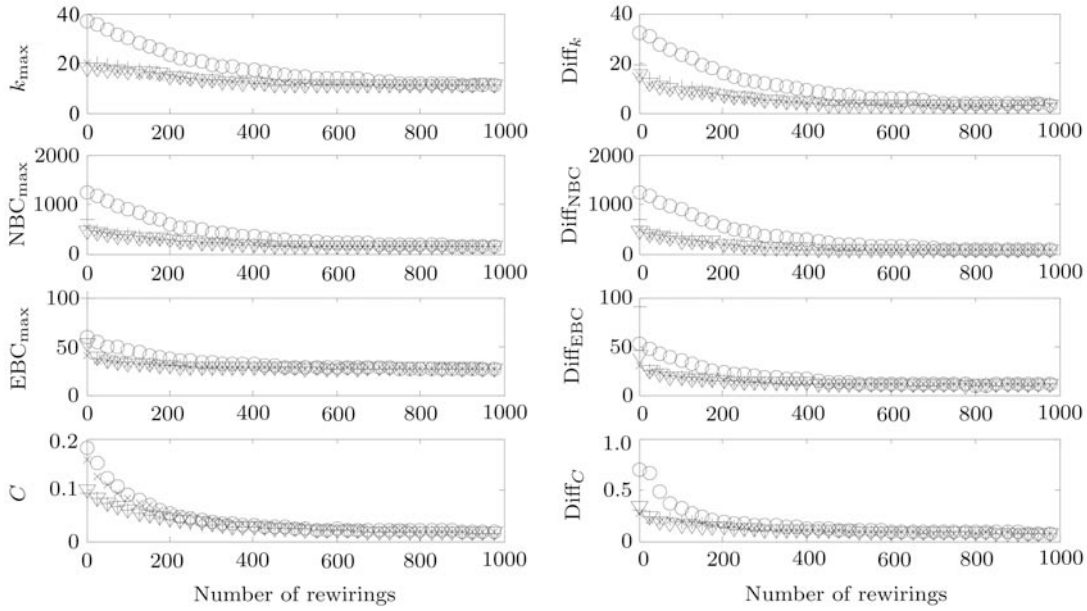


Fig. 3. Type I networks: the curves of maximum node degree ($k_{\max}$), difference between $k_{\max}$ and minimum node degree (Diff_k), maximum node betweenness centrality ($\mathrm{NBC}_{\max}$), difference between $\mathrm{NBC}_{\max}$ and minimum node betweenness centrality ($\mathrm{Diff}_{\mathrm{NBC}}$), maximum edge betweenness centrality ($\mathrm{EBC}_{\max}$), difference between $\mathrm{EBC}_{\max}$ and minimum edge betweenness centrality ($\mathrm{Diff}_{\mathrm{EBC}}$), cluster coefficient ($C$), and difference between maximum node cluster coefficient and minimum node cluster coefficient (Diff_C) as a function of iteration steps for networks with $N = 100$, $\langle k \rangle = 9.6$, and different topology structures, BA scale-free network ('o'), NW small-world network ('×'), ER random network ('∇'), the tunable heterogeneity network ('+').

Further analyses were done on the structural properties of the final optimized networks. Figure 3 shows an evolution of different parameters [maximum node degree ($k_{\max}$), difference between maximum node degree and minimum node degree (Diff_k), maximum node betweenness centrality ($\mathrm{NBC}_{\max}$), difference between maximum node betweenness centrality and minimum node betweenness centrality ($\mathrm{Diff}_{\mathrm{NBC}}$), maximum edge betweenness centrality ($\mathrm{EBC}_{\max}$), difference between maximum edge betweenness centrality and minimum edge betweenness centrality ($\mathrm{Diff}_{\mathrm{EBC}}$), cluster coefficient ($C$), and difference between maximum node cluster coefficient and minimum node cluster coefficient (Diff_C)] of the networks during the optimization process. It can be seen that the optimized networks have same structural properties. And, the optimized networks become a class of homogeneous for degree, cluster, node and edge betweenness centralities. That is to say, networks with optimized synchronizability belong to the class of random homogenous networks with very little difference of degree, cluster, node and edge betweenness centralities.

3.2. Type II networks

For type II networks the eigenratio $R = \lambda_N/\lambda_2$ of A characterizes the network synchronizability, and smaller $R = \lambda_N/\lambda_2$ leads to better synchronizability. First, we consider complement network. The complement of a network places an edge between two nodes precisely when there is not an edge in the original network. The eigenvalues of the complement network is related to the original network by[33] $\lambda_i^C + \lambda_{N-i+2} = -N$, $(\lambda_i < 0,\ 2 \leq i \leq N)$. One easily obtains $\lambda_N = -N - \lambda_2^C$. The $R = \lambda_N/\lambda_2$ decreases with the increase of λ_N or the decrease of λ_2 $(\lambda_i < 0)$. Through the above analysis, it can be seen that adding maximal $(u_{2i} - u_{2j})^2$ edges has dramatic influence on λ_2. And, removing edges corresponds to adding edges in complement graph, then adding the maximal $(u_{2i}^C - u_{2j}^C)^2$ edges has dramatic influence on λ_2^C. In general, the u_2^C associates with u_N. So removing maximal $(u_{Ni} - u_{Nj})^2$ edges makes λ_N have an efficient increase. Therefore, we present an algorithm to enhance the synchronizability of type II networks.

(I) The eigenratio $R = \lambda_N/\lambda_2$ of the network is calculated and in the first iteration step $R_{\min} = \lambda_N/\lambda_2$.

(II) For each pair of nonadjacent nodes i and j, the quantity $(u_{2i} - u_{2j})^2$ is calculated, which is used for a pair of nonadjacent nodes to connect an edge between them. Add an edge (i, j) with the largest $(u_{2i} - u_{2j})^2$.

(III) For each edge e_{ij} of the network, the quantity $(u_{Ni} - u_{Nj})^2$ is calculated, which is used to choose one edge for disconnection, i.e., remove the edge (i, j) with the largest $(u_{Ni} - u_{Nj})^2$.

(IV) After rewiring, the cost function $R = \lambda_N/\lambda_2$ of the new network is calculated. If eigenvalue $(R)_{\mathrm{new}}$ is less than the eigenvalue of the old network $(R)_{\mathrm{old}}$, the rewiring is accepted and $R_{\min} = (R)_{\mathrm{new}}$. If eigenvalue $(R)_{\mathrm{new}}$ is more than eigenvalue of the old network $(R)_{\mathrm{old}}$, the rewiring is accepted with the probability of ξ (and reject the rewiring with the probability $1 - \xi$), then $R_{\min} = (R)_{\mathrm{new}}$. If the rewiring is rejected, randomly by removing an existent edge and randomly by adding a nonexistent edge, i.e., randomly by rewiring, then $R_{\min} = (R)_{\mathrm{new}}$. The ξ is a pre-set parameter which helps this algorithm avoid getting trapped by a local minimum.

(V) The algorithm is stopped after a predetermined number of iterations.

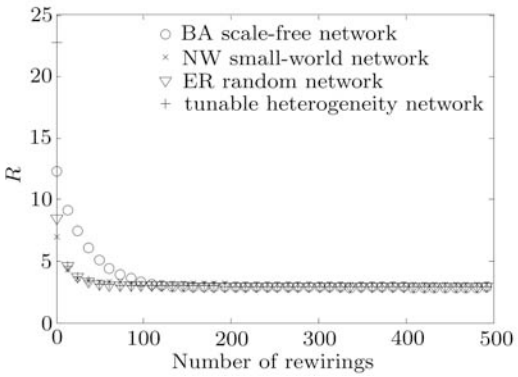


Fig. 4. The changes of the type II networks synchronizability as a function of number of rewirings for different topology structures with $N = 100$, $\langle k \rangle = 9.6$.

Next we will support this algorithm by numerical simulations on the above-mentioned different classes of networks. All network parameters are the same as those listed above. Figure 4 shows R as a function of iteration steps. It shows that eigenratio R is decreases as the algorithm proceeds and a dramatic decrease is obtained by introducing only a few rewirings. This indicates the effectiveness of the rewiring algorithm for improving the ability of type II networks synchronization. From the figure, we can see that N iteration steps are enough to reach optimal value. Comparing with type I networks in Fig. 2, we can faster achieve asymptotic behaviour. The reason is

that adding edges mainly influences λ_2, and removing edges mainly influences λ_N in the combined effect optimization process of type II networks. While, the optimization process of type I networks mainly influences λ_2 only by adding edges. In other words, optimization process of type II networks depends on u_2 and u_N, while optimization process of type I networks only depends on u_2. So the algorithm is faster for type II networks than for type I networks. In addition, the optimized network also is independent of the initial network structure, i.e., no matter what the topology structure of initial network is, the final networks have the same optimized synchronizability.

Figure 5 shows an evolution of different parameters [$k_{\max}$, Diff$_k$, NBC$_{\max}$, Diff$_{\rm NBC}$, EBC$_{\max}$, Diff$_{\rm EBC}$, C, Diff$_C$] of the networks during the optimization process of type II networks. It can be seen that the optimized networks for their synchronizability have similar structural properties in the sense of degree, node and edge betweenness centralities. However, for type II networks, the cluster coefficient seems to be dependent on the initial network. In spite of this, the networks with optimized synchronizability still belong to the class of random homogenous networks with almost zero difference of degree, cluster node and edge betweenness centralities.

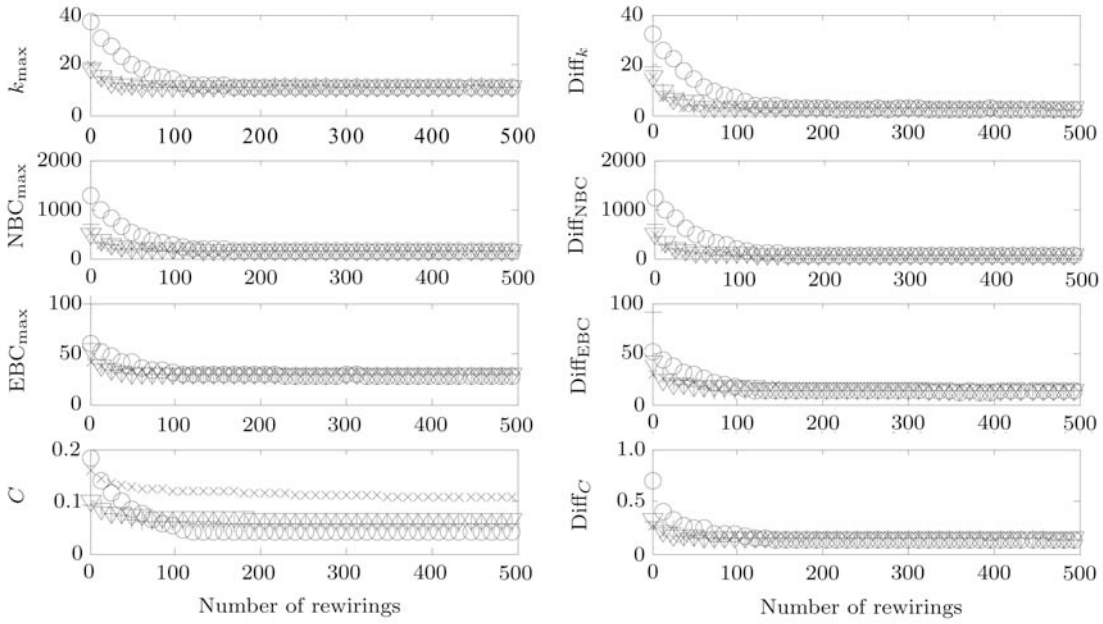


Fig. 5. Type II networks: the curves of $k_{\max}$, Diff$_k$, NBC$_{\max}$, Diff$_{\rm NBC}$, EBC$_{\max}$, Diff$_{\rm EBC}$, C, Diff$_C$ as a function of iteration steps for networks with $N = 100$, $\langle k\rangle = 9.6$, and different topology structure, BA scale-free network ('o'), NW small-world network ('$\times$'), ER random network ('∇'), the tunable heterogeneity network ('+').

4. Conclusions

By choosing $(u_{2i} - u_{2j})^2$ and $(u_{Ni} - u_{Nj})^2$, we presented an algorithm based on a strategically rewired mechanism in order to optimize the synchronizability of dynamical networks. We show that the method is able to enhance the synchronizability of networks of any size and topological properties in a few steps. It is much faster in comparison with other methods with a random rewiring strategy. And, the algorithm is capable of avoiding local minima. We find that the final optimized network is independent of the initial network, and becomes homogeneous. In addition, the optimized network ,for their synchronizability, has similar structural properties in the sense of degree, node and edge betweenness centralities. However, they do not have similar cluster coefficients for type II networks.

For some problems in neuron networks,[34] and ecosystem dynamics,[35] complex networks with good synchronizability are undesired. In those cases our algorithm may be inverted. For type I networks, we add edges with the minimal $(u_{2i} - u_{2j})^2$, and remove edges with the maximal $(u_{2i} - u_{2j})^2$. For type II networks, we add

edges with the maximal $(u_{Ni} - u_{Nj})^2$, and remove edges with the maximal $(u_{2i} - u_{2j})^2$. The method is still effective for the two types of networks.

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