Approximate analytic solutions for a generalized Hirota–Satsuma coupled KdV equation and a coupled mKdV equation*

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(Received 14 December 2009; revised manuscript received 22 January 2010)

This paper applies the variational iteration method to obtain approximate analytic solutions of a generalized Hirota–Satsuma coupled Korteweg-de Vries (KdV) equation and a coupled modified Korteweg-de Vries (mKdV) equation. This method provides a sequence of functions which converges to the exact solution of the problem and is based on the use of the Lagrange multiplier for the identification of optimal values of parameters in a functional. Some examples are given to demonstrate the reliability and convenience of the method and comparisons are made with the exact solutions.

Keywords: approximate analytic solutions, generalized Hirota–Satsuma coupled KdV equation, coupled mKdV equation, variational iteration method

PACC: 0230, 0545

1. Introduction

The investigation of the exact and numerical solution to nonlinear equations plays an important role in the study of nonlinear physical phenomena. In this paper, we consider a generalized Hirota–Satsuma coupled Korteweg-de Vries (KdV) equation and a coupled modified Korteweg-de Vries (mKdV) equation which was introduced by Wu *et al.*[1] They all belong to a new hierarchy of nonlinear evolution equations. The new generalized Hirota–Satsuma coupled KdV equation is

$$\begin{aligned} u_t &= \frac{1}{2}u_{xxx} - 3uu_x + 3(vw)_x, \\ v_t &= -v_{xxx} + 3uv_x, \\ w_t &= -w_{xxx} + 3uw_x. \end{aligned} \tag{1}$$

The coupled mKdV equation is

$$\begin{aligned} u_t &= \frac{1}{2}u_{xxx} - 3u^2u_x + \frac{3}{2}v_{xx} + 3(vu)_x - 3\lambda u_x, \\ v_t &= -v_{xxx} - 3vv_x - 3u_xv_x + 3u^2v_x + 3\lambda v_x. \end{aligned} \tag{2}$$

With $w = v^*$ and $w = v$, equation (1) reduces respectively to a new complex coupled KdV equation and the Hirota–Satsuma equation,[2,3] equation (2) becomes a generalized KdV equation for $u = 0$ and the mKdV equation for $v = 0$, respectively.

During the last few decades, the construction of exact solutions for some nonlinear equations has been done by many researchers. The auxiliary equation method, the Hirota's bilinear method, the Jacobian elliptic function, the powerful inverse scattering transform, the Bäcklund transformations technique, the Painlevé analysis, the Lie group theoretical methods, the direct algebraic method, the tangent hyperbolic method, the variable separation method, the improved mapping approach and the direct reduction method have been used in the construction of exact solutions of some nonlinear equations.[4−21] The discussed generalized Hirota–Satsuma coupled KdV equation has been studied by many researchers via different approaches, for example, the extended tanh-function method,[22]

*Project supported by the National Natural Science Foundation of China (Grant Nos. 10771019 and 10826107).

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the Jacobi elliptic function method,[23] the projective Riccati equations method,[24] the Adomian's decomposition method,[25] the homotopy perturbation method[26] and recently by the homotopy analysis method.[27] In this paper, the variational iteration method (VIM) is introduced and applied to the generalized Hirota–Satsuma coupled KdV equation and the coupled mKdV equation. The VIM was first envisioned by He[28,29] and has been successfully applied to many nonlinear equations.[30−40] Its advantages include easiness of coding and getting semi-analytical solutions. So in this paper, we extend the VIM to the above two equations and some approximate analytic solutions are obtained. Comparisons with exact solutions show that the VIM is also one of the most efficient methods for studying such equations.

2. Variational iteration method

To illustrate the basic concepts of the VIM, we consider the following nonlinear differential equation:

$$Lu + Nu = f(x), \tag{3}$$

where L is a linear operator, N is a nonlinear operator and $f(x)$ is an inhomogeneous term. According to the VIM, we can construct a correction functional as follows:

$$u_{n+1} = u_n + \int_0^x \lambda[Lu_n(\tau) + N\widetilde{u_n}(\tau) - f(x)]\,\mathrm{d}\tau, \quad (n \geq 0), \tag{4}$$

where λ is a general Lagrange multiplier which can be identified optimally via the variational theory, the subscript n denotes the n-th approximation, $\widetilde{u_n}(\tau)$ is considered as a restricted variation, i.e. $\delta\widetilde{u_n}(\tau) = 0$. The iterative formula is

$$u_{n+1} = u_n + \int_0^x \lambda[Lu_n(\tau) + Nu_n(\tau) - f(x)]\,\mathrm{d}\tau, \quad (n \geq 0). \tag{5}$$

This method is very effective and reliable. It can solve a large class of nonlinear problems. For linear problems, its exact solution can be obtained by only one iteration, because the Lagrange multiplier λ can be exactly identified. For nonlinear problems, the exact solution is given by

$$u = \lim_{n\to\infty} u_n.$$

3. The VIM applied to the generalized Hirota–Satsuma coupled KdV equation

In this section, we describe the implementation of the VIM for the generalized Hirota–Satsuma coupled KdV equation in detail. To solve Eq. (1), according to the VIM, the correction functionals in t-direction can be expressed as follows:

$$\begin{aligned}
u_{n+1} &= u_n + \int_0^t \lambda_1 \left(\frac{\partial u_n(x,s)}{\partial s} - \frac{1}{2}\frac{\partial^3 \widetilde{u_n}(x,s)}{\partial x^3} + 3\widetilde{u_n}(x,s)\frac{\partial \widetilde{u_n}(x,s)}{\partial x} - 3\frac{\partial(\widetilde{v_n}(x,s)\widetilde{w_n}(x,s))}{\partial x} \right) \mathrm{d}s, \quad (n \geq 0), \\
v_{n+1} &= v_n + \int_0^t \lambda_2 \left(\frac{\partial u_n(x,s)}{\partial s} + \frac{\partial^3 \widetilde{v_n}(x,s)}{\partial x^3} - 3\widetilde{u_n}(x,s)\frac{\partial \widetilde{v_n}(x,s)}{\partial x} \right) \mathrm{d}s, \quad (n \geq 0), \\
w_{n+1} &= w_n + \int_0^t \lambda_3 \left(\frac{\partial w_n(x,s)}{\partial s} + \frac{\partial^3 \widetilde{w_n}(x,s)}{\partial x^3} - 3\widetilde{u_n}(x,s)\frac{\partial \widetilde{w_n}(x,s)}{\partial x} \right) \mathrm{d}s, \quad (n \geq 0).
\end{aligned} \tag{6}$$

Making the above correction functionals stationary, we obtain the following stationary conditions:

$$\frac{\mathrm{d}\lambda_i}{\mathrm{d}\tau} = 0, \quad 1 + \lambda_i|_{\tau=t} = 0, \quad (i = 1, 2, 3). \tag{7}$$

The Lagrange multipliers, therefore, can be identified as

$$\lambda_i(\tau) = -1, \quad (i = 1, 2, 3). \tag{8}$$

As a result, we obtain the following iteration formulas:

$$u_{n+1} = u_n - \int_0^t \left(\frac{\partial u_n(x,s)}{\partial s} - \frac{1}{2}\frac{\partial^3 u_n(x,s)}{\partial x^3} + 3u_n(x,s)\frac{\partial u_n(x,s)}{\partial x} - 3\frac{\partial(v_n(x,s)w_n(x,s))}{\partial x} \right) \mathrm{d}s, \quad (n \geq 0),$$

$$v_{n+1} = v_n - \int_0^t \left(\frac{\partial u_n(x,s)}{\partial s} + \frac{\partial^3 v_n(x,s)}{\partial x^3} - 3u_n(x,s)\frac{\partial v_n(x,s)}{\partial x} \right) \mathrm{d}s, \quad (n \geq 0),$$
$$w_{n+1} = w_n - \int_0^t \left(\frac{\partial w_n(x,s)}{\partial s} + \frac{\partial^3 w_n(x,s)}{\partial x^3} - 3u_n(x,s)\frac{\partial w_n(x,s)}{\partial x} \right) \mathrm{d}s, \quad (n \geq 0). \tag{9}$$

Considering the initial value conditions, a sequence of approximate analytic solutions can be obtained.

First, we consider the approximate analytic solutions of Eq. (1) with the initial conditions:[22]

$$\begin{aligned} u(x,0) &= \frac{1}{3}(\beta - 2k^2) + 2k^2 \tanh^2(kx), \\ v(x,0) &= -\frac{4k^2 c_0(\beta + k^2)}{3c_1^2} + \frac{4k^2(\beta + k^2)}{3c_1} \tanh(kx), \\ w(x,0) &= c_0 + c_1 \tanh(kx), \end{aligned} \tag{10}$$

where $k, c_0, c_1 \neq 0$, and β are arbitrary constants. Beginning with

$$u_0 = u(x,0), \quad v_0 = v(x,0), \quad w_0 = w(x,0), \tag{11}$$

we illustrate the first approximations as follows:

$$\begin{aligned} u_1(x,t) &= \frac{1}{3}(\beta - 2k^2) + 2k^2 \tanh^2(kx) - t\Big(12k^3 \operatorname{sech}^2(kx)\tanh(kx)\Big(\frac{1}{3}(\beta - 2k^2) \\ &\quad + 2k^2 \tanh^2(kx)\Big) - 1 \cdot k^2(-16k^3 \operatorname{sech}^4(kx)\tanh(kx) + 8k^3 \operatorname{sech}^2(kx)\tanh^3(kx)) \\ &\quad - 3\Big(\frac{4k^3(\beta + k^2)\operatorname{sech}^2(kx)(c_0 + c_1\tanh(kx))}{3c_1} \\ &\quad + c_1 k \operatorname{sech}^2(kx)\Big(- \frac{4c_0 k^2(\beta + k^2)}{3c_1^2} + \frac{4k^2(\beta + k^2)\tanh(kx)}{3c_1}\Big)\Big)\Big), \\ v_1(x,t) &= -\frac{4k^2 c_0(\beta + k^2)}{3c_1^2} + \frac{4k^2(\beta + k^2)}{3c_1}\tanh(kx) \\ &\quad - t\Big(- \frac{4k^2(\beta + k^2)\operatorname{sech}^2(kx)\Big(\frac{1}{3}(\beta - 2k^2) + 2k^2\tanh^2(kx)\Big)}{c_1} \\ &\quad + \frac{4k^2(\beta + k^2)(-2k^3\sec^4(kx) + 4k^3\operatorname{sech}^2(kx)\tanh^2(kx))}{3c_1}\Big), \\ w_1(x,t) &= c_0 + c_1\tanh(kx) - t\Big(- 3c_1 k \operatorname{sech}^2(kx)\Big(\frac{1}{3}(\beta - 2k^2) + 2k^2\tanh(kx)\Big) \\ &\quad + c_1(-2k^3\operatorname{sech}^4(kx) + 4k^3\operatorname{sech}^2(kx))\Big). \end{aligned} \tag{12}$$

To examine the accuracy and reliability of the VIM for the generalized Hirota–Satsuma coupled KdV equation, we can also consider the different initial values:[22]

$$\begin{aligned} u(x,0) &= \frac{1}{3}(\beta - 8k^2) + 4k^2\tanh(kx), \\ v(x,0) &= -\frac{4k^2(3k^2c_0 - 2\beta c_2 + 4k^2 c_2)}{3c_2^2} + \frac{4k^2}{c_2}\tanh^2(kx), \\ w(x,0) &= c_0 + c_2\tanh^2(kx), \end{aligned} \tag{13}$$

where $k, c_0, c_2 \neq 0$, and β are arbitrary constants. In the same manner, we can obtain the first approximations as follows:

$$u_1(x,t) = \frac{1}{3}(\beta - 8k^2) + 4k^2\tanh(kx) - t\Big(24k^3\operatorname{sech}^2(kx)\tanh(kx)\Big(\frac{1}{3}(\beta - 8k^2)$$

$$
\begin{aligned}
&+ 4k^2\tanh^2(kx)\Big) - 2\cdot k^2(-16k^3\operatorname{sech}^4(kx)\tanh(kx) + 8k^3\operatorname{sech}^2(kx)\tanh^3(kx))\\
&- 3\Big(\frac{8k^3\operatorname{sech}^2(kx)\tanh(kx)(c_0 + c_2\tanh^2(kx))}{c_2}\\
&+ 2c_2k\operatorname{sech}^2(kx)\tanh(kx)\Big(-\frac{4k^2(-2\beta c_2 + 3c_0k^2 + 4c_2k^2)}{3c_2^2} + \frac{4k^2\tanh^2(kx)}{c_2}\Big)\Big)\Big),\\
v_1(x,t) &= -\frac{4k^2(3k^2c_0 - 2\beta c_2 + 4k^2c_2)}{3c_2^2} + \frac{4k^2}{c_2}\tanh^2(kx)\\
&- t\Big(-\frac{24k^3\operatorname{sech}^2(kx)\tanh(kx)\Big(\frac{1}{3}(\beta - 8k^2) + 4k^2\tanh^2(kx)\Big)}{c_2}\\
&+ \frac{4k^2(-16k^3\sec^4(kx)\tanh(kx) + 8k^3\operatorname{sech}^2(kx)\tanh^3(kx))}{c_2}\Big),\\
w_1(x,t) &= c_0 + c_2\tanh^2(kx) - t(-6c_2k\operatorname{sech}^2(kx)\Big(\frac{1}{3}(\beta - 8k^2) + 4k^2\tanh^2(kx)\Big)\\
&+ c_2(-16k^3\operatorname{sech}^4(kx)\tanh(kx) + 8k^3\operatorname{sech}^2(kx)\tanh^3(kx))).
\end{aligned}
\tag{14}
$$

4. The VIM applied to the coupled mKdV equation

In this section, we describe the implementation of the VIM for the coupled mKdV equation in detail. To solve Eq. (2), according to the VIM, the correction functionals in t-direction can be expressed as follows:

$$
\begin{aligned}
u_{n+1} &= u_n + \int_0^t \lambda_1\Big(\frac{\partial u_n(x,s)}{\partial s} - \frac{1}{2}\frac{\partial^3\widetilde{u_n}(x,s)}{\partial x^3} + 3\widetilde{u_n^2}(x,s)\frac{\partial\widetilde{u_n}(x,s)}{\partial x}\\
&- \frac{3}{2}\frac{\partial^2\widetilde{u_n}(x,s)}{\partial x^2} - 3\frac{\partial(\widetilde{v_n}(x,s)\widetilde{u_n}(x,s))}{\partial x} + 3\lambda\frac{\partial\widetilde{u_n}(x,s)}{\partial x}\Big)\mathrm{d}s,\quad (n\geq 0),\\
v_{n+1} &= v_n + \int_0^t \lambda_2\Big(\frac{\partial v_n(x,s)}{\partial s} + \frac{\partial^3\widetilde{v_n}(x,s)}{\partial x^3} - 3\widetilde{u_n^2}(x,s)\frac{\partial\widetilde{v_n}(x,s)}{\partial x}\\
&+ 3\frac{\partial\widetilde{u_n}(x,s)}{\partial x}\frac{\partial\widetilde{v_n}(x,s)}{\partial x} + 3\widetilde{v_n}(x,s)\frac{\partial\widetilde{v_n}(x,s)}{\partial x} - 3\lambda\frac{\partial\widetilde{v_n}(x,s)}{\partial x}\Big)\mathrm{d}s,\quad (n\geq 0).
\end{aligned}
\tag{15}
$$

Making the above correction functionals stationary, we obtain the following stationary conditions:

$$
\frac{\mathrm{d}\lambda_i}{\mathrm{d}\tau} = 0,\quad 1 + \lambda_i|_{\tau=t} = 0,\quad (i = 1, 2). \tag{16}
$$

The Lagrange multipliers, therefore, can be identified as

$$
\lambda_i(\tau) = -1,\quad (i = 1, 2). \tag{17}
$$

As a result, we obtain the following iteration formulas:

$$
\begin{aligned}
u_{n+1} &= u_n - \int_0^t \Big(\frac{\partial u_n(x,s)}{\partial s} - \frac{1}{2}\frac{\partial^3 u_n(x,s)}{\partial x^3} + 3u_n^2(x,s)\frac{\partial u_n(x,s)}{\partial x}\\
&- \frac{3}{2}\frac{\partial^2 u_n(x,s)}{\partial x^2} - 3\frac{\partial(v_n(x,s)u_n(x,s))}{\partial x} + 3\lambda\frac{\partial u_n(x,s)}{\partial x}\Big)\mathrm{d}s,\quad (n\geq 0),\\
v_{n+1} &= v_n - \int_0^t \Big(\frac{\partial v_n(x,s)}{\partial s} + \frac{\partial^3 v_n(x,s)}{\partial x^3} - 3u_n^2(x,s)\frac{\partial v_n(x,s)}{\partial x}\\
&+ 3\frac{\partial u_n(x,s)}{\partial x}\frac{\partial v_n(x,s)}{\partial x} + 3v_n(x,s)\frac{\partial v_n(x,s)}{\partial x} - 3\lambda\frac{\partial v_n(x,s)}{\partial x}\Big)\mathrm{d}s,\quad (n\geq 0).
\end{aligned}
\tag{18}
$$

Giving the initial value conditions, we can obtain a sequence of approximate analytic solutions.

First, we consider the approximate analytic solutions of Eq. (2) with the initial conditions:[22]

$$\begin{aligned} u(x,0) &= \frac{b_1}{2k} + k\tanh(kx), \\ v(x,0) &= \frac{\lambda}{2}\left(1+\frac{k}{b_1}\right) + b_1\tanh(kx), \end{aligned} \tag{19}$$

where $k, b_1 \neq 0$, and λ are arbitrary constants. Beginning with

$$u_0 = u(x,0), \quad v_0 = v(x,0), \tag{20}$$

we illustrate the first approximations as follows:

$$\begin{aligned} u_1(x,t) &= \frac{b_1}{2k} + k\tanh(kx) - t\Bigg(3k^2\lambda\,\mathrm{sech}^2(kx) + 3\cdot b_1k^2\,\mathrm{sech}^2(k,x)\tanh(kx) \\ &\quad + 3k^2\,\mathrm{sech}^2(kx)\left(\frac{b_1}{2k} + k\tanh(kx)\right)^2 - 0.5k(-sk^3\,\mathrm{sech}^4(kx) + 4k^3\,\mathrm{sech}^2(kx)\tanh^2(kx)) \\ &\quad - 3\left(k^2\,\mathrm{sech}^2(kx)\left(0.5\left(1+\frac{k}{b_1}\right)\lambda + b_1\tanh(kx)\right) + b_1k\,\mathrm{sech}^2(kx)\left(\frac{b_1}{2k} + k\tanh(kx)\right)\right)\Bigg), \\ v_1(x,t) &= \frac{\lambda}{2}\left(1+\frac{k}{b_1}\right) + b_1\tanh(kx) - t\Bigg(-3b_1k\lambda\,\mathrm{sech}^2(kx) + 3b_1k^3\,\mathrm{sech}^4(k,x) \\ &\quad + 3b_1k\,\mathrm{sech}^2(kx)\left(0.5\left(1+\frac{k}{b_1}\right) + b_1\tanh(kx)\right) - 3b_1k\,\mathrm{sech}^2(kx)\left(\frac{b_1}{2k} + k\tanh(kx)\right)^2 \\ &\quad + b_1(-2k^3\,\mathrm{sech}^4(kx) + 4k^3\,\mathrm{sech}^2(x,t)\tanh^2(kx))\Bigg). \end{aligned} \tag{21}$$

We can also consider the different initial values:[22]

$$u(x,0) = k\tanh(kx), \quad v(x,0) = \frac{1}{2}(4k^2+\lambda) - 2k^2\tanh^2(kx), \tag{22}$$

where $k \neq 0$, and λ are arbitrary constants. In the same manner, we can get the first approximations as follows:

$$\begin{aligned} u_1(x,t) &= k\tanh(kx) - t(3k^2\lambda\,\mathrm{sech}^2(kx) + 3k^4\,\mathrm{sech}^4(kx)\tanh^2(kx) + 3k^2(2k^2\,\mathrm{sech}^4(kx) \\ &\quad - 4k^2\,\mathrm{sech}^2(kx)\tanh^2(kx)) - 0.5k(-2k^3\,\mathrm{sech}^4(kx) + 4k^2\,\mathrm{sech}^2(kx)\tanh^2(kx)) \\ &\quad - 3(-4k^4\,\mathrm{sech}^2(kx)\tanh^2(kx) + k^2\,\mathrm{sech}^2(kx)(0.5(4k^2+\lambda) - 2k^2\tanh^2(kx)))) \\ &\quad + 4k^3\,\mathrm{sech}^2(kx)\tanh^2(kx)) - 3\Bigg(k^2\,\mathrm{sech}^2(kx)\left(0.5\left(1+\frac{k}{b_1}\right)\lambda \right. \\ &\quad \left. + b_1\tanh(kx)\right) + b_1k\,\mathrm{sech}^2(kx)\left(\frac{b_1}{2k} + k\tanh(kx)\right)\Bigg)\Bigg), \\ v_1(x,t) &= \frac{1}{2}(4k^2+\lambda) - 2k^2\tanh^2(kx) - t(12k^3\lambda\,\mathrm{sech}^2(kx)\tanh(kx) \\ &\quad - 12k^5\,\mathrm{sech}^4(k,x)\tanh(kx) + 12k^5\,\mathrm{sech}^2(kx)\tanh^3(kx) \\ &\quad - 12k^3\,\mathrm{sech}^2(kx)\tanh(kx)(0.5(4k^2+\lambda) - 2k^2\tanh^2(kx)) \\ &\quad - 2k^2(-16k^3\,\mathrm{sech}^4(kx)\tanh(kx) + 8k^3\,\mathrm{sech}^2(x,t)\tanh^3(kx))). \end{aligned} \tag{23}$$

5. Comparison of results with exact solutions

To demonstrate the convergence of the VIM, the comparisons between numerical results and exact solutions are presented below. The exact solutions of Eq. (1) with the initial conditions (10) are[22]

$$\begin{aligned} u(x,t) &= \frac{1}{3}(\beta - 2k^2) + 2k^2\tanh^2(k(x+\beta t)), \\ v(x,t) &= -\frac{4k^2c_0(\beta+k^2)}{3c_1^2} + \frac{4k^2(\beta+k^2)}{3c_1}\tanh(k(x+\beta t)), \end{aligned}$$

$$w(x,t) = c_0 + c_1 \tanh(k(x+\beta t)). \tag{24}$$

The exact solutions of Eq. (1) with the initial conditions (13) are

$$\begin{aligned} u(x,t) &= \frac{1}{3}(\beta - 8k^2) + 4k^2 \tanh(k(x+\beta t)), \\ v(x,t) &= -\frac{4k^2(3k^2c_0 - 2\beta c_2 + 4k^2 c_2)}{3c_2^2} + \frac{4k^2}{c_2}\tanh^2(k(x+\beta t)), \\ w(x,t) &= c_0 + c_2 \tanh^2(k(x+\beta t)). \end{aligned} \tag{25}$$

The exact solutions of Eq. (2) with the initial conditions (19) are

$$\begin{aligned} u(x,t) &= \frac{b_1}{2k} + k\tanh\left(k\left(x + \frac{1}{4}\left(-4k^2 - 6\lambda + \frac{6k\lambda}{b_1} + \frac{3b_1^2}{k^2}\right)t\right)\right), \\ v(x,t) &= \frac{\lambda}{2}\left(1 + \frac{k}{b_1}\right) + b_1 \tanh\left(k\left(x + \frac{1}{4}\left(-4k^2 - 6\lambda + \frac{6k\lambda}{b_1} + \frac{3b_1^2}{k^2}\right)t\right)\right). \end{aligned} \tag{26}$$

The exact solutions of Eq. (2) with the initial conditions (22) are

$$\begin{aligned} u(x,t) &= k\tanh\left(k\left(x + \frac{1}{2}(-2k^2 - 3\lambda)t\right)\right), \\ v(x,t) &= \frac{1}{2}(4k^2 + \lambda) - 2k^2 \tanh^2\left(k\left(x + \frac{1}{2}(-2k^2 - 3\lambda)t\right)\right). \end{aligned} \tag{27}$$

The absolute errors of the first approximations with some choices of t and x are listed in Tables 1–4. Both the exact results and the approximate solutions obtained by the second approximations are plotted in Figs. 1–10. It is evident that the approximate solutions are in excellent agreement with the exact solutions.

Table 1. The first approximations by VIM for $u(x,t), v(x,t)$ and $w(x,t)$ in comparison with the exact solutions when $c_0 = 1.5, c_1 = 0.1, \beta = 1.5$ and $k = 0.1$, for the solutions of Eq. (1) with the initial conditions (10).

(x,t)	$\lvert u_{\text{exact}} - u_{\text{VIM}}\rvert$	$\lvert v_{\text{exact}} - v_{\text{VIM}}\rvert$	$\lvert w_{\text{exact}} - w_{\text{VIM}}\rvert$
(0.1,0.1)	4.49573×10^{-6}	6.79260×10^{-7}	3.37381×10^{-7}
(0.1,0.2)	1.79676×10^{-5}	3.62130×10^{-6}	1.79866×10^{-6}
(0.1,0.3)	4.03807×10^{-5}	1.01791×10^{-5}	5.05583×10^{-6}
(0.2,0.1)	4.48854×10^{-6}	1.13150×10^{-6}	5.62003×10^{-7}
(0.2,0.2)	1.79317×10^{-5}	5.42835×10^{-6}	2.69620×10^{-6}
(0.2,0.3)	4.02839×10^{-5}	1.42395×10^{-5}	7.07258×10^{-6}
(0.3,0.1)	4.47777×10^{-6}	1.58283×10^{-6}	7.86175×10^{-7}
(0.3,0.2)	1.78815×10^{-5}	7.23107×10^{-6}	3.59159×10^{-6}
(0.3,0.3)	4.01550×10^{-5}	1.82885×10^{-5}	9.08368×10^{-6}

Table 2. The first approximations by VIM for $u(x,t), v(x,t)$ and $w(x,t)$ in comparison with the exact solutions when $c_0 = 1.5, c_2 = 1.5, \beta = 1.5$ and $k = 0.1$, for the solutions of Eq. (1) with the initial conditions (13).

(x,t)	$\lvert u_{\text{exact}} - u_{\text{VIM}}\rvert$	$\lvert v_{\text{exact}} - v_{\text{VIM}}\rvert$	$\lvert w_{\text{exact}} - w_{\text{VIM}}\rvert$
(0.1,0.1)	1.47701×10^{-5}	5.99430×10^{-6}	3.37180×10^{-4}
(0.1,0.2)	1.15879×10^{-5}	2.39568×10^{-5}	1.34757×10^{-3}
(0.1,0.3)	9.47672×10^{-6}	5.38410×10^{-5}	3.02856×10^{-3}
(0.2,0.1)	3.85556×10^{-5}	5.98471×10^{-6}	3.36640×10^{-4}
(0.2,0.2)	5.92019×10^{-5}	2.39090×10^{-5}	1.34480×10^{-3}
(0.2,0.3)	6.20302×10^{-5}	5.37118×10^{-5}	3.02129×10^{-3}
(0.3,0.1)	6.23671×10^{-5}	5.97035×10^{-6}	3.35832×10^{-4}
(0.3,0.2)	1.06882×10^{-4}	2.38420×10^{-5}	1.34111×10^{-3}
(0.3,0.3)	1.33658×10^{-4}	5.34540×10^{-5}	3.01163×10^{-3}

Table 3. The first approximations by VIM for $u(x,t)$ in comparison with the exact solutions when $b_1 = 0.1, k = 1$ and $\lambda = 1$, for Eq. (2) with the initial conditions (19).

(x,t)	$\lvert u_{\text{exact}} - u_{\text{VIM}}\rvert$	$\lvert v_{\text{exact}} - v_{\text{VIM}}\rvert$
(0.1,0.1)	5.66467×10^{-5}	5.66467×10^{-5}
(0.1,0.2)	2.06305×10^{-5}	2.06305×10^{-5}
(0.1,0.3)	4.18230×10^{-4}	4.18230×10^{-4}
(0.2,0.1)	1.14088×10^{-4}	1.14088×10^{-4}
(0.2,0.2)	4.38137×10^{-4}	4.38137×10^{-4}
(0.2,0.3)	9.44095×10^{-4}	9.44095×10^{-4}
(0.3,0.1)	1.63006×10^{-4}	1.63006×10^{-4}
(0.3,0.2)	6.37102×10^{-4}	6.37102×10^{-4}
(0.3,0.3)	1.39887×10^{-3}	1.39887×10^{-3}

Table 4. The first approximations by VIM for $v(x,t)$ in comparison with the exact solutions when $k = 0.1$ and $\lambda = 1.5$, for Eq. (2) with the initial conditions (22).

(x,t)	$\lvert u_{\text{exact}} - u_{\text{VIM}}\rvert$	$\lvert v_{\text{exact}} - v_{\text{VIM}}\rvert$
(0.1,0.1)	1.25978×10^{-7}	7.78381×10^{-6}
(0.1,0.2)	1.03445×10^{-6}	4.84287×10^{-6}
(0.1,0.3)	5.78347×10^{-6}	3.77920×10^{-5}
(0.2,0.1)	6.36549×10^{-7}	2.57731×10^{-5}
(0.2,0.2)	1.00760×10^{-6}	3.11233×10^{-5}
(0.2,0.3)	1.19253×10^{-6}	1.61019×10^{-5}
(0.3,0.1)	1.14661×10^{-6}	4.37418×10^{-5}
(0.3,0.2)	3.04883×10^{-6}	6.70647×10^{-5}
(0.3,0.3)	3.39935×10^{-6}	6.99829×10^{-5}

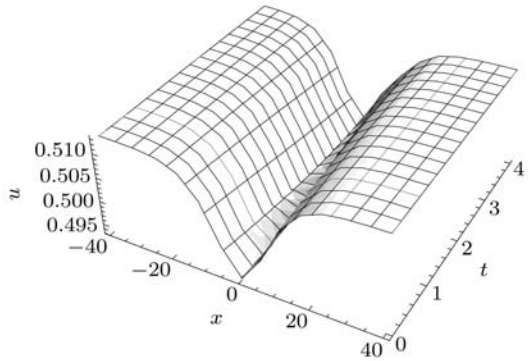


Fig. 1. The second approximation for $u(x,t)$ obtained by using the VIM when $c_0 = 1.5, c_1 = 0.1, \beta = 1.5$ and $k = 0.1$ for Eq. (1) with initial conditions (10).

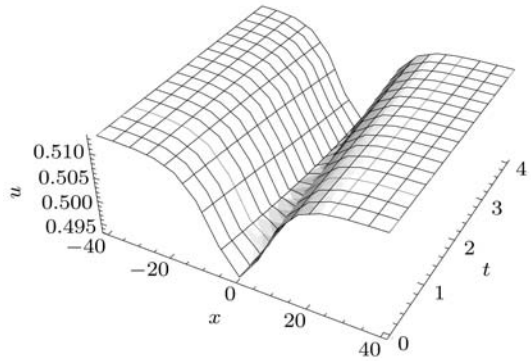


Fig. 2. The exact solution for $u(x,t)$ when $c_0 = 1.5, c_1 = 0.1, \beta = 1.5$ and $k = 0.1$ for Eq. (1) with initial conditions (10).

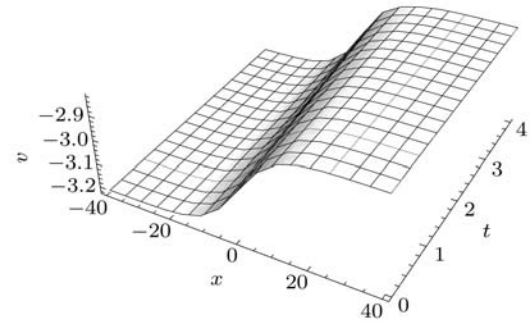


Fig. 3. The second approximation for $v(x,t)$ obtained by using the VIM when $c_0 = 1.5, c_1 = 0.1, \beta = 1.5$ and $k = 0.1$ for Eq. (1) with initial conditions (10).

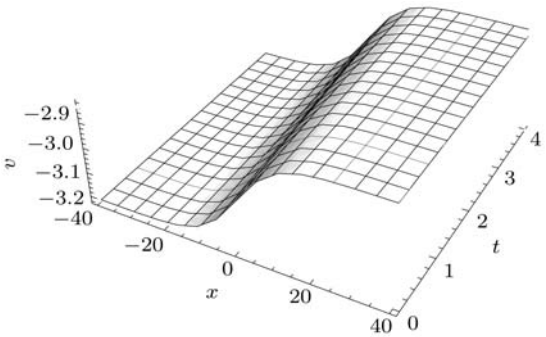


Fig. 4. The exact solution for $v(x,t)$ when $c_0 = 1.5, c_1 = 0.1, \beta = 1.5$ and $k = 0.1$ for Eq. (1) with initial conditions (10).

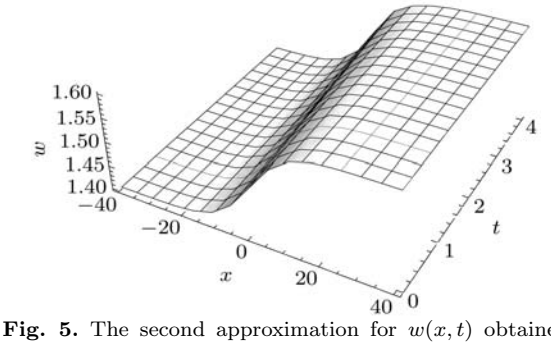


Fig. 5. The second approximation for $w(x,t)$ obtained by using the VIM when $c_0 = 1.5, c_1 = 0.1, \beta = 1.5$ and $k = 0.1$ for Eq. (1) with initial conditions (10).

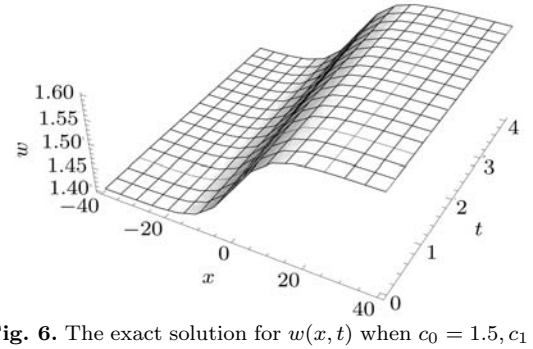


Fig. 6. The exact solution for $w(x,t)$ when $c_0 = 1.5, c_1 = 0.1, \beta = 1.5$ and $k = 0.1$ for Eq. (1) with initial conditions (10).

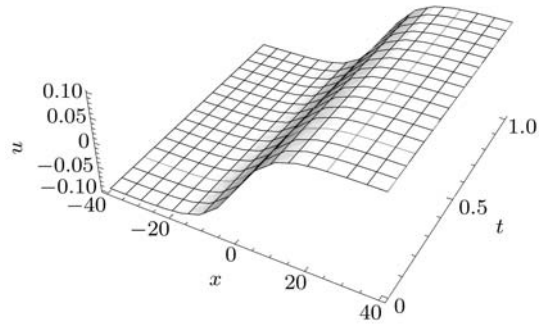


Fig. 7. The second approximation for $u(x,t)$ obtained by using the VIM when $\lambda = 1.5$ and $k = 0.1$ for Eq. (2) with initial conditions (22).

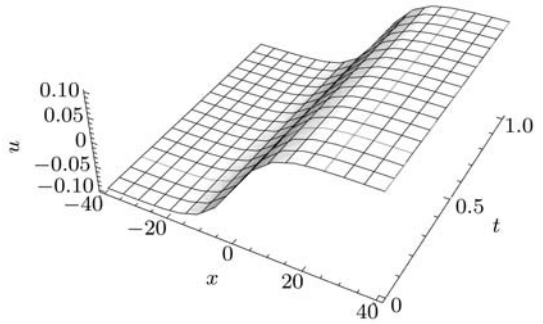


Fig. 8. The exact solution for $u(x,t)$ when $\lambda = 1.5$ and $k = 0.1$ for Eq. (2) with initial conditions (22).

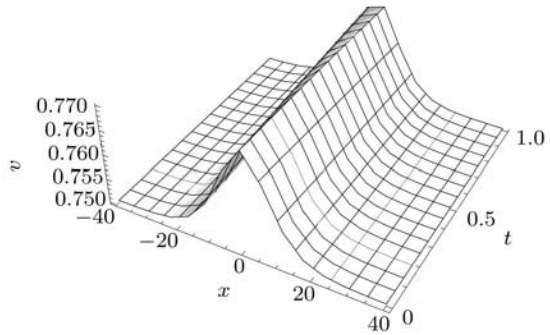


Fig. 9. The second approximation for $v(x,t)$ obtained by using the VIM when $\lambda = 1.5$ and $k = 0.1$ for Eq. (2) with initial conditions (22).

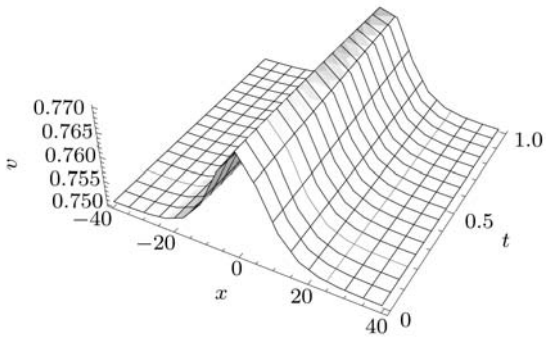


Fig. 10. The exact solution for $v(x,t)$ when $\lambda = 1.5$ and $k = 0.1$ for Eq. (2) with initial conditions (22).

6. Conclusions

In this paper, we have achieved the approximations for solutions of a generalized Hirota–Satsuma coupled KdV equation and a coupled mKdV equation by applying the VIM. To our best knowledge, this is the first attempt to extend the use of the VIM to such equations. The numerical results show that the VIM is one of the most reliable and efficient methods in finding the approximate analytic solutions of such equations. The solutions obtained by the VIM can do the analytical manipulation without any difficulties. It is more important that the initial approximation of the VIM can be chosen arbitrarily. So we can find other approximate analytic solutions in the same manner.

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