Approximate homotopy similarity reduction for the generalized Kawahara equation via Lie symmetry method and direct method*

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This paper studies the generalized Kawahara equation in terms of the approximate homotopy symmetry method and the approximate homotopy direct method. Using both methods it obtains the similarity reduction solutions and similarity reduction equations of different orders, showing that the approximate homotopy direct method yields more general approximate similarity reductions than the approximate homotopy symmetry method. The homotopy series solutions to the generalized Kawahara equation are consequently derived.

Keywords: approximate homotopy symmetry method, approximate homotopy direct method, generalized Kawahara equation, homotopy series solutions

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1. Introduction

It is well known that many phenomena in scientific fields can be described by nonlinear partial differential equations (PDEs) and, in general, such nonlinear equations are often very difficult to solve explicitly. One of the most successful methods for solving nonlinear equations is to reduce them to ordinary differential equations which may be easier to solve. There are several methods of computing similarity reductions of PDEs, among which, the Lie group method also known as the classical method is the fundamental one.[1−5] The classical Lie group method based on the invariance of PDEs under point transformations (point symmetries) can be further extended to include nonclassical symmetry groups, known as the nonclassical method due to Bluman and Cole.[6] As a different approach, the direct method which is proposed by Clarkson and Kruskal[7] and used in many problems (see, e.g., Refs. [8]–[10]) does not use techniques of group analysis.

Because of the complexity of nonlinear problems, perturbation, asymptotic and numerical methods are often used, with much success, to obtain approximate solutions of these equations. Another powerful analytical method for nonlinear problems, called the homotopy analysis method (HAM), was first envisioned by Liao[11] and successfully utilized in solving many problems for strong perturbations. The essence of the HAM is to apply a kind of linear property of homotopy to transform a nonlinear problem into an infinite number of linear sub-problems. Because this kind of transformation is independent of small parameters, the HAM can overcome the foregoing limitations of perturbation techniques. The applications of HAM in nonlinear problems have also been presented by many researchers.[12−14] Recently, the HAM was further improved in Refs. [15] and [16].

Integrating HAM with the Lie group theory and the direct method gives rise to the approximate homotopy symmetry method and the approximate homotopy direct method respectively, which were proposed and used in Refs. [17] and [18].

The well-known Kawahara equation

$$u_t + uu_x + u_{xxx} - u_{xxxxx} = 0 \tag{1}$$

with the indices t and x denoting differentiation with respect to time and a spatial coordinate, respectively, describes long nonlinear waves in weakly dispersive media (see, for example, Refs. [19]–[22]). Equation (1) is also called the KdV equation of fifth order or the singularly perturbed KdV equation.[23,24] There are many generalizations of the Kawahara equation according to the models of similar wave processes in inhomogeneous media. We just consider the general-

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ized Kawahara equation in the form

$$u_t + \tau\, uu_x + u^2u_x + u_{xxx} - u_{xxxxx} = 0 \qquad (2)$$

with τ an arbitrary constant.

In this paper, we are interested in applying the approximate homotopy symmetry method and the approximate homotopy direct method to obtain the similarity reduction solutions and similarity reduction equations for approximate equations of different orders, and eventually to obtain the homotopy series solutions of the generalized Kawahara equation.

For simplicity later, we construct the following homotopy for Eq. (2):

$$(1-q)(u_t + \tau\, uu_x + u^2u_x + u_{xxx}) - q(u_t + \tau\, uu_x + u^2u_x + u_{xxx} - u_{xxxxx}) = 0, \qquad (3)$$

where $q \in [0,\ 1]$ is an embedding homotopy parameter. Obviously, when $q = 1$, equation (3) is the generalized Kawahara equation.

We first make the ansatz that the homotopy model (3) has the perturbation expansion

$$u = \sum_{i=0}^{\infty} u_i q^i. \qquad (4)$$

Substituting this expansion into Eq. (3) and letting all coefficients of different powers of q equal zero, one has

$$O(q^0): \quad \tau\, u_0u_{0,x} + u_{0,t} + u_0^2u_{0,x} + u_{0,xxx} = 0, \qquad (5)$$

$$O(q^1): \quad \tau\, u_0u_{1,x} - u_{0,xxxxx} + 2u_0u_1u_{0,x} + u_0^2u_{1,x} + u_{1,t} + u_{1,xxx} + \tau\, u_1u_{0,x} = 0, \qquad (6)$$

$$O(q^2): \quad \tau\, u_0u_{2,x} + 2u_0u_1u_{1,x} + 2u_0u_2u_{0,x} + u_{2,xxx} - u_{1,xxxxx} + u_1^2u_{0,x} + u_{2,t} + u_0^2u_{2,x} + \tau\, u_1u_{1,x} + \tau\, u_2u_{0,x} = 0, \qquad (7)$$

$$O(q^3): \quad -u_{2,xxxxx} + 2u_0u_2u_{1,x} + 2u_1u_2u_{0,x} + 2u_0u_1u_{2,x} + u_{3,xxx} + \tau\, u_0u_{3,x} + \tau\, u_2u_{1,x} + 2u_0u_3u_{0,x} + u_1^2u_{1,x} + u_{3,t} + u_0^2u_{3,x} + \tau\, u_3u_{0,x} + \tau\, u_1u_{2,x} = 0, \qquad (8)$$

$$\cdots,$$

$$O(q^k): \quad u_{k,t} + \tau \sum_{i=0}^{k} u_{k-i}u_{i,x} + \sum_{j=0}^{k}\sum_{i=0}^{j} u_{i,x}u_{j-i}u_{k-j} + u_{k,xxx} - u_{k-1,xxxxx} = 0, \qquad (9)$$

$$\cdots$$

with $u_{-1} = 0$.

This paper is organized as follows. Sections 2 and 3 are devoted to applying the symmetry reduction method and direct method to Eq. (9), respectively. Section 4 concerns conclusion and discussion.

2. Approximate homotopy symmetry method to Eq. (2)

A symmetry is characterized by an infinitesimal transformation which leaves the given differential equation invariant under the transformation of all independent and dependent variables. To apply the classical method to Eq. (9), we search for the Lie point symmetry transformations in the vector form

$$V = X\frac{\partial}{\partial x} + T\frac{\partial}{\partial t} + \sum_{k=0}^{\infty} U_k \frac{\partial}{\partial u_k},$$

where X, T, and U_k are functions with respect to x, t, and u_k, $k = 0,\ 1,\ \ldots$. The symmetry in the vector form V can also be written in the function form

$$\sigma_k = Xu_{k,x} + Tu_{k,t} - U_k,\ (k = 0,\ 1,\ \ldots), \qquad (10)$$

where σ_k are functions of x and t.

We can obtain the linearized equations for Eq. (9) directly from the definition of symmetry,

$$\sigma_{k,t} + \tau \sum_{i=0}^{k}(u_{i,x}\sigma_{k-i} + \sigma_{i,x}u_{k-i}) + \sum_{j=0}^{k}\sum_{i=0}^{j}(\sigma_{i,x}u_{j-i}u_{k-j} + u_{i,x}\sigma_{j-i}u_{k-j} + u_{i,x}u_{j-i}\sigma_{k-j}) + \sigma_{k,xxx} - \sigma_{k-1,xxxxx} = 0, \quad (k = 0,\ 1,\ \ldots) \qquad (11)$$

with $\sigma_{-1} = 0$.

To figure out X, T, and U_k ($k = 0,\ 1,\ \ldots$), we should substitute Eq. (10) into the linearized Eq. (11) and eliminate $u_{k,t}$ in terms of Eq. (9) for $k = 0,\ 1,\ 2,\ \ldots$. After repeating computation several times, we easily to find that

$$X = 2C_1x - C_1\tau^2t + x_0, \quad T = 6C_1t + t_0,$$

$$U_0 = -C_1(2u_0+\tau),\ \ U_1 = -6C_1u_1,$$
$$U_2 = -10C_1u_2,\ \ U_3 = -14C_1u_3,\ \ \ldots,$$
$$U_k = -2C_1(2k+1)u_k - \delta_{k,0}C_1\tau,$$
$$(k=0,1,\ldots), \tag{12}$$

where C_1, x_0 and t_0 are arbitrary constants. The notation $\delta_{k,0}$ satisfying $\delta_{0,0}=1$ and $\delta_{k,0}=0$ $(k\neq 0)$ is adopted in the following text.

Having determined the infinitesimals, the similarity solutions to Eq. (9) are found by solving the invariant surface condition $\sigma_k = 0$ in Eq. (10), or, equivalently, by solving the characteristic equations

$$\frac{\mathrm{d}x}{X} = \frac{\mathrm{d}t}{T},\ \frac{\mathrm{d}u_0}{U_0} = \frac{\mathrm{d}t}{T},$$
$$\frac{\mathrm{d}u_1}{U_1} = \frac{\mathrm{d}t}{T},\ \cdots,\ \frac{\mathrm{d}u_k}{U_k} = \frac{\mathrm{d}t}{T},\ \cdots, \tag{13}$$

which will be discussed in detail for the following two subcases.

Case 1 $C_1 \neq 0$, in this case, without loss of generality, we make the transformations $x_0 \mapsto x_0C_1$, $t_0 \mapsto t_0C_1$, so equation (12) can be rewritten as

$$X = (2x - \tau^2 t + x_0)C_1,\ T = (6t+t_0)C_1,$$
$$U_k = -2C_1(2k+1)u_k - \delta_{k,0}C_1\tau,$$
$$(k=0,1,\ldots). \tag{14}$$

From Eqs. (14) and (13), we obtain the invariant $I(x,t)$, and invariant functions I_k, $k=0,\ 1,\ 2,\ \ldots$

$$I(x,t)$$
$$= \xi = \left(x + \frac{1}{2}x_0 + \frac{1}{8}t_0\tau^2 + \frac{1}{4}\tau^2 t\right)(6t+t_0)^{-1/3}, \tag{15}$$
$$I_0(x,t,u_0) = V_0 = \left(u_0 + \frac{\tau}{2}\right)(t_0+6t)^{1/3}, \tag{16}$$
$$I_1(x,t,u_0) = V_1 = (t_0+6t)u_1, \tag{17}$$
$$I_2(x,t,u_0) = V_2 = (t_0+6t)^{5/3}u_2, \tag{18}$$
$$I_3(x,t,u_0) = V_3 = (t_0+6t)^{7/3}u_3, \tag{19}$$
$$\cdots.$$

From the above equations, considering V_k $(k=0,\ 1,\ \ldots)$ as functions of the similarity variable

$$\xi = \left(x + \frac{x_0}{2} - \frac{t_0\tau^2}{8} - \frac{\tau^2 t}{4}\right)(6t+t_0)^{-1/3},$$

we obtain the symmetry reductions to Eq. (9)

$$u_0 = -\frac{\tau}{2} + (6t+t_0)^{-1/3}V_0(\xi), \tag{20}$$
$$u_1 = V_1(\xi)(6t+t_0)^{-1}, \tag{21}$$
$$u_2 = (6t+t_0)^{-5/3}V_2(\xi), \tag{22}$$
$$u_3 = (6t+t_0)^{-7/3}V_3(\xi), \tag{23}$$
$$\cdots$$
$$u_k = (6t+t_0)^{-(2k+1)/3}V_k(\xi) - \frac{1}{2}\delta_{k,0}\tau, \tag{24}$$
$$\cdots$$

The corresponding similarity reduction equations which are satisfied by V_k $(k=0,\ 1,\ \ldots)$ can be obtained by substituting u_k from Eq. (24) into Eq. (9) as

$$V_{0,\xi\xi\xi} + V_{0,\xi}V_0^2 - 2V_0 - 2V_{0,\xi}\xi = 0, \tag{25}$$
$$-2V_{1,\xi}\xi - 6V_1 - V_{0,\xi\xi\xi\xi\xi} + 2V_1V_{0,\xi}V_0 + V_0^2V_{1,\xi} + V_{1,\xi\xi\xi} = 0, \tag{26}$$
$$2V_0V_2V_{0,\xi} + V_0^2V_{2,\xi} + V_{0,\xi}V_1^2 - V_{1,\xi\xi\xi\xi\xi} + 2V_0V_{1,\xi}V_1 - 2\xi V_{2,\xi} - 10V_2 + V_{2,\xi\xi\xi} = 0, \tag{27}$$
$$V_{3,\xi}V_0^2 + 2V_0V_{2,\xi}V_1 + 2V_0V_3V_{0,\xi} + 2V_0V_2V_{1,\xi} + V_{1,\xi}V_1^2 + 2V_2V_{0,\xi}V_1 - 2\xi V_{3,\xi}$$
$$+\ V_{3,\xi\xi\xi} - V_{2,\xi\xi\xi\xi\xi} - 14V_3 = 0, \tag{28}$$
$$\cdots$$
$$V_{k,\xi\xi\xi} + \sum_{j=0}^{k}\sum_{i=0}^{j} V_{i,\xi}V_{j-i}V_{k-j} - 2\xi V_{k,\xi} - (4k+2)V_k - V_{k-1,\xi\xi\xi\xi\xi} = 0, \tag{29}$$
$$\cdots$$

with $V_{-1}=0$.

Hence the series solution to Eq. (3) is obtained from Eqs. (4) and (24)

$$u = \sum_{k=0}^{\infty}(6t+t_0)^{-(2k+1)/3}V_k(\xi)q^k - \frac{\tau}{2}, \tag{30}$$

then the homotopy series solution to the generalized Kawahara equation can be obtained by further setting $q=1$

$$u = \sum_{k=0}^{\infty}(6t+t_0)^{-(2k+1)/3}V_k(\xi) - \frac{\tau}{2}. \tag{31}$$

It is easily shown that when $k = 0$ of Eq. (29), equation (25) is of Painlevé II type.

Noting that the k-th similarity reduction equation in Eq. (29) contains V_0, V_1, ..., V_{k-1}, V_k, we can theoretically compute V_0, V_1, ..., V_{k-1}, V_k one after another. The k-th similarity reduction equation is in fact a third order linear ordinary differential equation of V_k when the previous V_0, V_1, ..., V_{k-1} are known. To see this point we rewrite Eq. (29) as

$$\begin{aligned} &V_{k,\xi\xi\xi} - 2\xi V_{k,\xi} - (4k+2)V_k \\ &\quad + (2-2\delta_{k,0})V_{0,\xi}V_kV_0 + V_{k,\xi}V_0^2 = f_k(\xi), \\ &\quad (k = 1,\ 2,\ \ldots), \end{aligned} \tag{32}$$

where $f_k(\xi)$ is an only function of $\{V_0,\ V_1,\ \ldots,\ V_{k-1}\}$ as follows:

$$\begin{aligned} f_k(\xi) = {} & V_{k-1,\xi\xi\xi\xi\xi} - \sum_{j=1}^{k-1}\sum_{i=0}^{j} V_{i,\xi}V_{j-i}V_{k-j} \\ & - \sum_{i=1}^{k-1} V_{i,\xi}V_{k-i}V_0. \end{aligned}$$

Case 2 When $C_1 = 0$ and $t_0 \neq 0$, from Eqs. (13) and (12), we obtain the following similarity solutions:

$$u_0 = V_0(\xi), \tag{33}$$

$$u_1 = V_1(\xi), \tag{34}$$

$$u_2 = V_2(\xi), \tag{35}$$

$$\ldots$$

$$u_k = V_k(\xi), \tag{36}$$

$$\ldots$$

with the similarity variable $\xi = t - (t_0/x_0)x$ which can be redefined as the traveling wave form $\xi = x + ct$, where c is an arbitrary velocity constant.

Substituting Eq. (36) into Eq. (9), we obtain the corresponding similarity reduction equations as

$$\tau V_{0,\xi}V_0 + V_{0,\xi\xi\xi} + V_{0,\xi}V_0^2 + V_{0,\xi}c = 0, \tag{37}$$

$$V_{1,\xi\xi\xi} + \tau V_1(\xi)V_{0,\xi} + \tau V_0(\xi)V_{1,\xi} + 2V_1V_{0,\xi}V_0 + V_{1,\xi}V_0^2 + cV_{1,\xi} - V_{0,\xi\xi\xi\xi\xi} = 0, \tag{38}$$

$$\begin{aligned} &V_{2,\xi}c - V_{1,\xi\xi\xi\xi\xi} + \tau V_1V_{1,\xi} + 2V_0V_2V_{0,\xi} + V_{2,\xi\xi\xi} + 2V_0V_{1,\xi}V_1 + V_{0,\xi}V_1^2 + V_{2,\xi}V_0^2 \\ &\quad + \tau V_0V_{2,\xi} + \tau V_{0,\xi}V_2 = 0, \end{aligned} \tag{39}$$

$$\begin{aligned} &\tau V_1V_{2,\xi} + \tau V_2V_{1,\xi} + 2V_0V_3V_{0,\xi} + V_{3,\xi}c + V_{1,\xi}V_1^2 + V_{3,\xi}V_0^2 + 2V_0V_{2,\xi}V_1 + \tau V_0V_{3,\xi} \\ &\quad + 2V_2V_{0,\xi}V_1 - V_{2,\xi\xi\xi\xi\xi} + \tau V_{0,\xi}V_3 + V_{3,\xi\xi\xi} + 2V_0V_2V_{1,\xi} = 0, \end{aligned} \tag{40}$$

$$\ldots$$

$$V_{k,\xi\xi\xi} + \tau\sum_{i=0}^{k} V_{k-i}V_{i,\xi} + \sum_{j=0}^{k}\sum_{i=0}^{j} V_{i,\xi}V_{j-i}V_{k-j} + cV_{k,\xi} - V_{k-1,\xi\xi\xi\xi\xi} = 0, \tag{41}$$

$$\ldots$$

with $V_{-1} = 0$.

From Eq. (36), it is easy for us to derive the series solution to Eq. (3)

$$u = \sum_{k=0}^{\infty} V_k(\xi)q^k, \tag{42}$$

after setting $q = 1$ to Eq. (42) we obtain the homotopy series solution

$$u = \sum_{k=0}^{\infty} V_k(\xi). \tag{43}$$

After integration, equation (41) can also be rewritten as

$$\begin{aligned} &cV_k + V_{k,\xi\xi} + \left(1 - \frac{2}{3}\delta_{k,0}\right)V_0^2V_k \\ &+ \left(\tau - \frac{\tau}{2}\delta_{k,0}\right)V_0V_k = g_k(\xi),\ (k = 0,\ 1,\ \ldots), \end{aligned} \tag{44}$$

where $g_k(\xi)$ is a function of $\{V_0,\ V_1,\ \ldots,\ V_{k-1}\}$,

$$\begin{aligned} g_k(\xi) = {} & -\frac{1}{3}\sum_{j=1}^{k-1}\sum_{i=0}^{j} V_iV_{j-i}V_{k-j} \\ & - \frac{1}{3}\sum_{i=1}^{k-1} V_iV_{k-i}V_0 + V_{k-1,\xi\xi\xi\xi} \\ & - \frac{1}{2}\tau\sum_{i=1}^{k-1} V_iV_{k-i} - A_k \end{aligned}$$

with A_k arbitrary integral constants. Equation (44) is a second order linear ordinary differential equation of V_k.

When $k = 0$, equation (44) can be solved by the

hyperbolic sech function

$$V_0 = -\frac{\tau}{2} - \frac{1}{2}\sqrt{6\tau^2 - 24c}\ \mathrm{sech}\left(\mathrm{c}_0 + \frac{\sqrt{-4\mathrm{c} + \tau^2}}{2}\xi\right), \tag{45}$$

where c_0 is an arbitrary constant.

When $k > 0$, equation (44) can be integrated out step by step, we obtain

$$V_k = V_{0,\xi}\left[C_k + \int V_{0,\xi}^{-2}(B_k + V_0 g_k)\mathrm{d}\xi\right] \tag{46}$$

with arbitrary integral constants B_k and C_k.

3. Approximate homotopy direct method to Eq. (2)

Now we apply the direct method proposed by Clarkson and Kruskal to find the similarity reduction of Eq. (9). According to the method, the similarity reduction of Eq. (9) can be assumed in the general form

$$u_k = U_k(x, t, P_k(z(x,t))),\ (k = 0, 1, \ldots), \tag{47}$$

where U_k, P_k and z are functions with respect to the indicated variables, $P_k(z)$ satisfies a system of ordinary differential equations, which can be obtained by substituting Eq. (47) into Eq. (9).

For Eq. (9), it turns out to be sufficient to take Eq. (47) in the special form:

$$u_k = \alpha_k(x,t) + \beta_k(x,t)P_k(z(x,t)),\ (k = 0, 1, \ldots). \tag{48}$$

To see this, we substitute Eq. (47) into only one term $u_{k,xxx}$ in Eq. (9), since only this term generates $P_{k,zzz}$ and $P_{k,zz}P_{k,z}$ after the substitution. It is easily seen that the coefficients for $P_{k,zzz}$ and $P_{k,zz}P_{k,z}$ are $U_{k,P_k}z_x^3$ and $3U_{k,P_kP_k}z_x^3$ respectively. We reserve uppercase Greek letters for undetermined functions of z hereafter. Because it must be an ordinary differential equation in $P_k(z)$ after substituting Eq. (47) into Eq. (9), the ratios of the coefficients are functions of z, namely,

$$3U_{k,P_kP_k}z_x^3 = U_{k,P_k}z_x^3\Gamma_k(z),\ (k = 0, 1, \ldots) \tag{49}$$

with the solution $U_k = F_k(x,t) + G_k(x,t)\mathrm{e}^{\Gamma_k(z)P_k/3}$. So, it is sufficient to seek similarity reductions of Eqs. (9) in the form (48).

Remark Without loss of generality, there are three freedoms in the determination of $\alpha_k(x,t)$, $\beta_k(x,t)$ and $z(x,t)$.

(i) If $\alpha_k(x,t)$ has the form $\alpha_k(x,t) = \alpha_k'(x,t) + \beta_k\Omega(z)$, then one can take $\Omega(z) = 0$;

(ii) If $\beta_k(x,t)$ has the form $\beta_k(x,t) = \beta_k'(x,t)\Omega(z)$, then one can take $\Omega(z) =$ constant;

(iii) If $z(x,t)$ is determined by $\Omega(z) = z_0(x,t)$, where $\Omega(z)$ is any invertible function, then one can take $\Omega(z) = z$.

We substitute Eq. (48) into Eq. (9) in the case $k = 0$ and single out the coefficients for P_{0zzz}, $P_{0z}P_0^2$, $P_{0z}P_0$ and P_{0zz}, which are $\beta_0 z_x^3$, $\beta_0^3 z_x$, $2\alpha_0\beta_0^2 z_x$ and $3\beta_{0x}z_x^2 + 3\beta_0 z_x z_{xx}$, respectively. The assumption that P_k is only a function of z requires that

$$\beta_0^3 z_x = \beta_0 z_x^3 \Psi_0(z), \tag{50}$$

$$2\alpha_0\beta_0^2 z_x = \beta_0 z_x^3 \Phi_0(z), \tag{51}$$

$$3\beta_{0x}z_x^2 + 3\beta_0 z_x z_{xx} = \beta_0 z_x^3 \Omega_0(z), \tag{52}$$

and thus, from Eq. (50) and Remark (ii), we have

$$\beta_0 = z_x. \tag{53}$$

From Eqs. (51), (53) and Remark (i), we have

$$\alpha_0 = -\frac{\tau}{2}. \tag{54}$$

From Eqs. (52), (53) and Remark (iii), after redefining z we have

$$z = \theta(t)x + \sigma(t), \tag{55}$$

where $\theta(t)$ and $\sigma(t)$ are undetermined functions of integration. Then in the case $k = 0$ equation (9) is simplified to

$$\theta^4 P_0^2 P_{0z} + \theta_t P_0 + \left(\theta_t z - \theta_t\sigma + \theta\sigma_t - \frac{1}{4}\tau^2\theta^2\right)P_{0z} + \theta^4 P_{0zzz} = 0. \tag{56}$$

For this to be an ordinary differential equation in $P_0(z)$, the coefficients of P_{0zzz}, zP_{0z} and P_{0z} must satisfy the following relations:

$$\theta_t = A_1\theta^4, \quad -\theta_t\sigma + \theta\sigma_t - \frac{1}{4}\tau^2\theta^2 = A_2\theta^4, \tag{57}$$

where A_1 and A_2 are arbitrary constants.

We discuss the results in the following two subcases for Eq. (57).

Case 1 When $A_1 \neq 0$, equation (57) has the solution

$$\begin{aligned}\theta &= -(3A_1t - t_1)^{-1/3},\\ \sigma &= -(3A_1t - t_1)^{-1/3}\left(\frac{\tau^2 t}{4} - \frac{\tau^2 t_1}{12A_1} - t_2\right)\\ &\quad - \frac{A_2}{A_1},\end{aligned} \tag{58}$$

where t_1 and t_2 are arbitrary constants.

In order to determine α_k, β_k, we substitute Eq. (48) into Eq. (9) and single out the coefficients of $P_{k,zzz}$, $P_{k-1,zzzzz}$ and $P_{0z}P_0$, which are $\beta_k z_x^3$, $\beta_{k-1} z_x^5$ and $2\alpha_k \beta_0^2 z_x$, respectively. We require that

$$\beta_{k-1} z_x^5 = \beta_k z_x^3 \Psi_k(z), \ (k = 1, 2, \ldots), \tag{59}$$

$$2\alpha_k \beta_0^2 z_x = \beta_k z_x^3 \Phi_k(z), \ (k = 1, 2, \ldots). \tag{60}$$

From Eqs. (53), (59) and Remark (ii), we obtain

$$\beta_k = z_x^{2k+1}, \ (k = 0, 1, \ldots). \tag{61}$$

From Eqs. (53), (60), (61) and Remark (i), we obtain

$$\alpha_k = -\frac{\tau}{2}\delta_{k,0}, \ (k = 0, 1, \ldots). \tag{62}$$

In terms of Eqs. (4), (48), (55), (58), (61) and (62), we obtain the similarity solutions to Eq. (9)

$$\begin{aligned} u_k \ = \ & (-1)^{2k+1}(3A_1 t - t_1)^{-(2k+1)/3} P_k(z) \\ & - \frac{\tau}{2}\delta_{k,0}, \end{aligned} \tag{63}$$

where the similarity variable

$$\begin{aligned} z \ = \ & -x(3A_1 t - t_1)^{-1/3} \\ & - (3A_1 t - t_1)^{-1/3} \left(\frac{\tau^2 t}{4} - \frac{\tau^2 t_1}{12 A_1} - t_2 \right) - \frac{A_2}{A_1}. \end{aligned}$$

So the homotopy series solution to Eq. (2) can be derived from Eqs. (4) and (63) by setting $q = 1$

$$u = \sum_{j=0}^{\infty} (-1)^{2j+1}(3A_1 t - t_1)^{-(2j+1)/3} P_j(z) - \frac{\tau}{2}. \tag{64}$$

The related similarity reduction equations are

$$\begin{aligned} & P_{k,zzz} + (A_1 z + A_2) P_{k,z} + \sum_{j=0}^{k} \sum_{i=0}^{j} P_{i,z} P_{j-i} P_{k-j} \\ & \quad + \ (2k+1) A_1 P_k - P_{k-1,zzzzz} = 0, \\ & \quad (k = 0, 1, \ldots) \end{aligned} \tag{65}$$

with $P_{-1} = 0$.

We now compare Eqs. (63) and (65) with Eqs. (24) and (29). Making the transformation $t_2 \to -x_0/2$, $A_2 \to 0$, $A_1 \to -2$ and $t_1 \to t_0$ simultaneously, we see that the similarity variable becomes

$$z = \left(x + \frac{x_0}{2} - \frac{t_0 \tau^2}{8} - \frac{\tau^2 t}{4} \right) (6t + t_0)^{-1/3}.$$

Equations (63) and (65) are changed into

$$\begin{aligned} u_k \ = \ & (6t + t_0)^{-(2k+1)/3} P_k(\xi) - \frac{1}{2}\delta_{k,0}\tau, \\ & (k = 0, \ 1, \ \ldots), \end{aligned} \tag{66}$$

and

$$\begin{aligned} & P_{k,zzz} + \sum_{j=0}^{k} \sum_{i=0}^{j} P_{i,z} P_{j-i} P_{k-j} - 2z P_{k,z} \\ & \quad - \ (4k+2) P_k - P_{k-1,zzzzz} = 0, \\ & \quad (k = 0, 1, \ldots) \end{aligned} \tag{67}$$

with $P_{-1} = 0$. Obviously, equations (66) and (67) are exactly the same as Eqs. (24) and (29).

When $k = 0$, equation (65) degenerates to Painlevé II type equation. Equation (65) is in fact a third order linear variable coefficients ordinary differential equation of P_k with the assumption that P_j for $j < k$ are determined in advance, since we can rewrite Eq. (65) as

$$\begin{aligned} & P_{k,zzz} + (A_1 z + A_2) P_k + (2k+1) A_1 P_k \\ & \quad + (2 - 2\delta_{k,0}) P_{0z} P_0 P_k + P_0^2 P_{k,z} = f_k, \\ & \quad (k = 0, 1, \ldots), \end{aligned} \tag{68}$$

where

$$\begin{aligned} f_k \ = \ & -\sum_{j=1}^{k-1} \sum_{i=0}^{j} P_{i,z} P_{j-i} P_{k-j} - P_0 \sum_{i=1}^{k-1} P_{i,z} P_{k-i} \\ & + P_{k-1,zzzzz}. \end{aligned}$$

Case 2 When $A_1 = 0$, it is straightforward from Eq. (57) to find that

$$\theta = \theta_0, \ \sigma = \left(\frac{\theta_0 \tau^2}{4} + \theta_0^3 A_2 \right) t + \sigma_0, \tag{69}$$

where θ_0 and σ_0 are arbitrary constants. From Eqs. (55) and (69), we can redefine the similarity variable as the travelling wave form $z = x + c't$ with c' being an arbitrary velocity constant.

So, from Eqs. (4), (48), (55), (61) and (62) we obtain the homotopy series solution to Eq. (2) by setting $q = 1$

$$u = \sum_{k=0}^{\infty} P_k(z) - \frac{\tau}{2}, \tag{70}$$

where all $P_k(z)$ satisfy

$$\begin{aligned} & c' P_{k,z} + \sum_{j=0}^{k} \sum_{i=0}^{j} P_{i,z} P_{j-i} P_{k-j} + P_{k,zzz} \\ & \quad - P_{k-1,zzzzz} - \frac{\tau^2}{4} P_{k,z} = 0, \\ & \quad (k = 0, 1, \ldots) \end{aligned} \tag{71}$$

with $P_{-1} = 0$. Taking $k = 0$ in Eq. (71), we find that the solution can be given by

$$P_0 = -\frac{1}{2}\sqrt{6\tau^2 - 24c'} \ \mathrm{sech}\left(c_1 + \frac{\sqrt{-4c' + \tau^2}}{2} z \right), \tag{72}$$

where c_1 is an arbitrary constant. Equation (71) can also be rewritten as

$$cP_{k,z} + \left(1 - \frac{2}{3}\delta_{k,0}\right) P_0^2 P_k + P_{j,zz} - \frac{\tau^2}{4} P_k = g_k, \ (k = 0, 1, \ldots), \tag{73}$$

where

$$g_k = P_{k-1,zzzz} - \frac{1}{3}\sum_{j=1}^{k-1}\sum_{i=0}^{j} P_i P_{j-i} P_{k-j} - \frac{1}{3} P_0 \sum_{i=1}^{k-1} P_i P_{k-i} - a_k.$$

From Eq. (73), we can obtain P_k ($j = 1,\ 2,\ \ldots$) through the following recursion equation:

$$P_k = P_{0,z}\left[d_k + \int P_{0,z}^{-2}\left(e_k + \int g_j P_{0,z} \mathrm{d}z\right)\mathrm{d}z\right], \tag{74}$$

where d_k and e_k are arbitrary integral constants.

4. Conclusion

In this paper, we have presented the application of the approximate homotopy symmetry method and the approximate homotopy direct method to the nonlinear generalized Kawahara equation. For both methods, the similarity reduction solutions and similarity reduction equations for approximate equations of different orders are constructed. The homotopy series solutions to the generalized Kawahara equation were thus generated. Zero-order similarity solutions are equivalent to Painlevé II and hyperbolic sech function solutions. The k-order similarity reduction equations are linear ordinary differential equations of P_k which can be solved step by step through the given equations. From the comparison of the results for the two methods, we discover that the approximate homotopy direct method produces more general approximate similarity reduction than the approximate homotopy symmetry perturbation method.

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