Applications of the first integral method to nonlinear evolution equations

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In this paper, we establish travelling wave solutions for some nonlinear evolution equations. The first integral method is used to construct the travelling wave solutions of the modified Benjamin–Bona–Mahony and the coupled Klein–Gordon equations. The obtained results include periodic and solitary wave solutions. The first integral method presents a wider applicability to handling nonlinear wave equations.

Keywords: travelling wave solutions, first integral method, modified Benjamin–Bona–Mahony equation, coupled Klein–Gordon equation

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1. Introduction

Over the last few decades, directly searching for exact solutions of nonlinear partial differential equations (NPDEs) has become a more attractive topic in physical science and nonlinear science. The investigation of the travelling wave solutions for nonlinear partial differential equations plays an important role in the study of nonlinear physical phenomena. Nonlinear wave phenomena appear in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fibres, biology, solid state physics, chemical kinematics, chemical physics and geochemistry. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations. In the past several decades, new exact solutions may help to find new phenomena. A variety of powerful methods, such as inverse scattering method,[1,2] bilinear transformation,[3] the tanh-sech method,[4−6] extended tanh method,[7−9] sine–cosine method,[10,11] homogeneous balance method,[12] exp-function method[13−15] and Riccati equation method[16] were used to address nonlinear dispersive and dissipative problems.

Also, He *et al.*[17] gave physical meaning to approximate solutions and exact solutions of nonlinear equations with initial-boundary conditions. Dai *et al.* used a three-wave method[18] and double exp-function method[19] for finding exact multi-solitary solutions of nonlinear evolution equations.

In the pioneering work, Feng[20] introduced the first integral method for a reliable treatment of the NPDEs. The useful first integral method is widely used in many papers such as in Refs. [21]–[23] and the reference therein. Chow derived an exact and periodic solution of the nonlinear model.[24]

Our first interest in the present work is in implementing the first integral method to stress its power in handling nonlinear equations, so that one can apply it to models of various types of nonlinearity. In Section 2, we describe this method for finding exact travelling wave solutions of nonlinear evolution equations. In Sections 3 and 4, we illustrate this method in detail with the celebrated modified Benjamin–Bona–Mahony and coupled Klein–Gordon equations. In Section 5, some conclusions are given.

2. The first integral method

Raslan summarized the procedure for using the first integral method.[25]

Step 1 Consider a general NPDE in the form

$$P(u, u_t, u_x, u_{xx}, u_{tt}, u_{xt}, u_{xxx}, ...) = 0. \quad (1)$$

Using a wave variable $\xi = x - ct$, we can rewrite Eq. (1) in the following nonlinear ordinary differential equation (ODE)

$$Q(U, U', U'', U''', \ldots) = 0, \quad (2)$$

where the prime denotes the derivation with respect to ξ. Equation (2) is then integrated as long as in all terms containing derivatives the integration constants are considered to be zeros.

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Step 2 Suppose that the solution of ODE (2) can be written as

$$u(x,t) = f(\xi). \tag{3}$$

Step 3 We introduce a new independent variable

$$X(\xi) = f(\xi), \qquad Y = f_\xi(\xi), \tag{4}$$

which leads to a system of

$$\begin{aligned} X_\xi(\xi) &= Y(\xi), \\ Y_\xi(\xi) &= F(X(\xi), Y(\xi)). \end{aligned} \tag{5}$$

Step 4 By the qualitative theory of ordinary differential equations,[26] if we can find the integrals of Eq. (5) under the same conditions, then the general solutions to Eq. (5) can be solved directly. However, in general, it is really difficult for us to realize this even for one first integral, because for a given plane autonomous system, there is no systematic theory that can tell us how to find its first integrals, nor is there a logical way for showing us what these first integrals are. We will use the division theorem to obtain one first integral of Eq. (5) which reduces Eq. (2) to a first order integrable ODE. An exact solution to Eq. (1) is then obtained by solving this equation. Now, we come to recall the division theorem.

Division theorem Suppose that $P(w,z)$ and $Q(w,z)$ are polynomials in $C(w,z)$, and that $P(w,z)$ is irreducible in $C(w,z)$. If $Q(w,z)$ vanishes at all zero points of $P(w,z)$, then there exists a polynomial $G(w,z)$ in $C(w,z)$ such that

$$Q[w,z] = P[w,z]G[w,z]. \tag{6}$$

3. Modified Benjamin–Bona–Mahony equation

Consider the modified Benjamin–Bona–Mahony equation[27]

$$u_t + u_x + au^2u_x + bu_{xxt} = 0, \tag{7}$$

where a and b are known constants.

Using wave variable $\xi = x - ct$, equation (7) is carried out to the following ODE:

$$(1-c)u' + au^2u' - bcu''' = 0, \tag{8}$$

where the prime denotes the derivation with respect to ξ. Integrating Eq. (8) with respect to ξ and considering the constants for integration to be zeros, we obtain

$$(1-c)u + a\frac{u^3}{3} - bcu'' = 0. \tag{9}$$

Using Eq. (4) we obtain

$$\dot{X}(\xi) = Y(\xi), \tag{10}$$

$$\dot{Y}(\xi) = \frac{(1-c)}{cb}X(\xi) + \frac{a}{3bc}X^3(\xi). \tag{11}$$

According to the first integral method, we suppose that $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of Eqs. (10) and (11), and $q(X,Y) = \sum_{i=0}^{m} a_i(X)Y^i$ is an irreducible polynomial in complex domain $C[X,Y]$ such that

$$q[X(\xi), Y(\xi)] = \sum_{i=0}^{m} a_i(X)Y^i = 0, \tag{12}$$

where $a_i(X)$, $(i = 0, 1, \ldots, m)$ are polynomials of X and $a_m(X) \neq 0$. Equation (12) is called first integral Eqs. (10) and (11). According to the division theorem, there exists a polynomial $g(X) + h(X)Y$ in complex domain $C[X,Y]$ such that

$$\begin{aligned} \frac{\mathrm{d}q}{\mathrm{d}\xi} &= \frac{\partial q}{\partial X}\frac{\partial X}{\partial \xi} + \frac{\partial q}{\partial Y}\frac{\partial Y}{\partial \xi} \\ &= [\,g(X) + h(X)Y\,]\sum_{i=0}^{m} a_i(X)Y^i. \end{aligned} \tag{13}$$

In this example, we assume that $m = 1$ in Eq. (12). Suppose that $m = 1$, by equating the coefficients of Y^i $(i = 0, 1, 2)$ on both sides of Eq. (13), we have

$$\dot{a}_1(X) = h(X)a_1(X), \tag{14}$$

$$\dot{a}_0(X) = g(X) + h(X)a_0(X), \tag{15}$$

$$\begin{aligned} a_1(X)\dot{Y} &= g(X)a_0(X) \\ &= a_1(X)\left(\frac{(1-c)}{cb}X + \frac{a}{3bc}X^3\right). \end{aligned} \tag{16}$$

Since $a_i(X)$ $(i = 0, 1)$ are polynomials, then from Eq. (14) we deduce that $a_1(X)$ is constant and $h(X) = 0$. For simplicity, take $a_1(X) = 1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that $\deg(g(X)) = 1$ only. Suppose that $g(X) = A_1X + B_0$ and $A_1 \neq 0$, then we find that $a_0(X)$ is expressed as

$$a_0(X) = \frac{A_1}{2}X^2 + B_0X + A_0. \tag{17}$$

Substituting $a_0(X), a_1(X)$ and $g(X)$ into Eq. (16) and setting all the coefficients of powers X to be zeros, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$A_0 = \frac{1-c}{cbA_1}, \quad B_0 = 0, \quad c = \frac{a}{bA_1^2}, \tag{18}$$

where A_1 is the free parameter. Inserting Eq. (18) into Eq. (12), we obtain

$$Y(\xi) = -\frac{A_1}{2}X^2(\xi) + \frac{a - bA_1^2}{abA_1}. \tag{19}$$

Combining Eq. (19) with Eq. (10), we obtain the exact solution to Eq. (11) and then the exact solution to the modified Zakharov–Kuznetsov equation (7) can be written as

$$X(\xi) = -\frac{K}{A_1}\tan\left[\frac{K}{2}(\xi + c_0)\right], \tag{20}$$

where c_0 is the integration constant and K is

$$K = \sqrt{\frac{2bA_1^2 - 2a}{ab}}. \tag{21}$$

Thus the travelling wave solution to the modified Benjamin–Bona–Mahony equation (7) can be written as:

$$u(x,t) = -\frac{K}{A_1}\tan\left[\frac{K}{2}\left(x - \frac{a}{bA_1^2}t + c_0\right)\right]. \tag{22}$$

Our results can be compared with Bekir's results.[27] If proper K values are chosen, then it can be seen that the results are same.

4. Coupled Klein–Gordon equation

Now, we consider the Klein–Gordon equations coupled with scaler field v, in the form[28]

$$\begin{aligned} &u_{xx} - u_{tt} - u + 2u^3 + 2uv = 0,\\ &v_x - v_t - 4uu_t = 0. \end{aligned} \tag{23}$$

Using the wave variable $\xi = x - ct$ and proceeding as before we find

$$\begin{aligned} &(1-c^2)u'' - u + 2u^3 + 2uv = 0,\\ &(1+c)v' + 4cuu' = 0, \end{aligned} \tag{24}$$

where by integrating the second equation we find

$$v = \frac{-2cu^2}{1+c}. \tag{25}$$

Substituting Eq. (25) into the first equation of Eq. (24) and integrating the resulting equation, we obtain

$$(1-c^2)u'' - u + \frac{2(1-c)}{1+c}u^3 = 0. \tag{26}$$

Using Eq. (4) we have

$$\dot{X}(\xi) = Y(\xi), \tag{27}$$

$$\dot{Y}(\xi) = \frac{1}{1-c^2}X(\xi) - \frac{2}{(1+c)^2}X^3(\xi), \quad c \neq \mp 1. \tag{28}$$

According to the first integral method, we suppose that $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of Eqs. (27) and (28), and $q(X,Y) = \sum_{i=0}^{m} a_i(X)Y^i$ is an irreducible polynomial in complex domain $C[X,Y]$ such that

$$q[X(\xi), Y(\xi)] = \sum_{i=0}^{m} a_i(X)Y^i = 0, \tag{29}$$

where $a_i(X)$, $(i = 0, 1, \ldots, m)$ are polynomials of X and $a_m(X) \neq 0$. Equation (29) is called first integral of Eqs. (27) and (28), due to the division theorem, there exists a polynomial $g(X) + h(X)Y$ in the complex domain $C[X,Y]$ such that

$$\begin{aligned} \frac{\mathrm{d}q}{\mathrm{d}\xi} &= \frac{\partial q}{\partial X}\frac{\partial X}{\partial \xi} + \frac{\partial q}{\partial Y}\frac{\partial Y}{\partial \xi}\\ &= [\,g(X) + h(X)Y]\sum_{i=0}^{m} a_i(X)Y^i. \end{aligned} \tag{30}$$

In this example, we assume that $m = 1$ in Eq. (29). Suppose that $m = 1$, by equating the coefficients of Y^i $(i = 0, 1, 2)$ on both sides of Eq. (30), we have

$$\dot{a}_1(X) = h(X)a_1(X), \tag{31}$$

$$\dot{a}_0(X) = g(X) + h(X)a_0(X), \tag{32}$$

$$\begin{aligned} a_1(X)\dot{Y} &= g(X)a_0(X)\\ &= a_1(X)\left(\frac{1}{1-c^2}X - \frac{2}{(1+c)^2}X^3\right). \end{aligned} \tag{33}$$

Since $a_i(X)$ $(i = 0, 1)$ are polynomials, then from Eq. (31) we deduce that $a_1(X)$ is constant and $h(X) = 0$. For simplicity, take $a_1(X) = 1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that $\deg(g(X)) = 1$ only. Suppose that $g(X) = A_1X + B_0$ and $A_1 \neq 0$, then we find $a_0(X)$ to be given as

$$a_0(X) = \frac{A_1}{2}X^2 + B_0X + A_0. \tag{34}$$

Substituting $a_0(X), a_1(X)$ and $g(X)$ into Eq. (33) and setting all the coefficients of powers X to be zeros, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$A_1 = \pm\frac{2A_0}{\mathrm{i}\sqrt{2}A_0 \pm 1}, \quad B_0 = 0, \quad c = 1 \pm \frac{1}{\mathrm{i}\sqrt{2}A_0}. \tag{35}$$

where A_0 is free parameter.

Similarly, as for the case of Eq. (35), the exact solutions are

$u_1(x,t)$

$$= -K_1 \tan\left[\frac{A_0}{K_1}\left(x - \frac{K_1^2}{K_1^2-1}t + c_0\right)\right], \quad (36)$$

$$u_2(x,t)$$

$$= -K_2 \tanh\left[\frac{A_0}{K_2}\left(x - \frac{K_2^2}{K_1^2+1}t + c_0\right)\right], \quad (37)$$

$$v_1(x,t)$$

$$= \frac{-2K_1^4}{2K_1-1}\tan^2\left[\frac{A_0}{K_1}\left(x - \frac{K_1^2}{K_1^2-1}t + c_0\right)\right], \quad (38)$$

$$v_2(x,t)$$

$$= \frac{-2K_2^4}{2K_1+1}\tanh^2\left[\frac{A_0}{K_2}\left(x - \frac{K_2^2}{K_2^2+1}t + c_0\right)\right], \quad (39)$$

where c_0 is the integration constant and K_1 and K_2 are

$$K_1 = \sqrt{\mathrm{i}A_0\sqrt{2}+1}, \quad (40)$$

$$K_2 = \sqrt{\mathrm{i}A_0\sqrt{2}-1}. \quad (41)$$

A comparison between our results and Bekir's results,[29] shows that some of the obtained results are new, and the rest are the same. We foresee that our results can be found to have potentially useful applications in mathematical physics and applied mathematics including numerical simulation.

5. Conclusion

The first integral method is successfully used to establish travelling wave solutions. Many well-known nonlinear wave equations are handled by this method. The performance of this method is found to be reliable and effective, and it gives more solutions. This method has the advantages of being direct and concise. The availability of computer systems like Maple facilitates the tedious algebraic calculations. The method which we proposed in this paper is also a standard, direct and computerizable method, which allows us to solve complicated and tedious algebraic calculations.

References

[1] Ablowitz M J and Clarkson P A 1990 *Solitons, Nonlinear Evolution Equations and Inverse Scattering Transform* (Cambridge: Cambridge University Press)

[2] Vakhnenko V O, Parkes E J and Morrison A J 2003 *Chaos, Solitons and Fractals* **17** 683

[3] Hirota R *Direct Method of Finding Exact Solutions of Nonlinear Evolution Equations* in: Bullough R and Caudrey P (eds.) 1980 *Backlund Transformations* (Berlin: Springer) p. 1157

[4] Malfliet W and Hereman W 1996 *Physica Scripta* **54** 563

[5] Wazwaz A M 2004 *Appl. Math. Comput.* **154** 713

[6] Yusufoğlu E and Bekir A 2008 *Chaos, Solitons and Fractals* **37** 842

[7] El-Wakil S A and Abdou M A 2007 *Chaos, Solitons and Fractals* **31** 840

[8] Fan E 2000 *Phys. Lett.* A **277** 212

[9] Wazwaz A M 2007 *Appl. Math. Comput.* **187** 1131

[10] Wazwaz A M 2004 *Math. Comput. Modelling* **40** 499

[11] Bekir A 2008 *Physica Scripta* **77** 501

[12] Fan E and Zhang H 1998 *Phys. Lett.* A **246** 403

[13] Boz A and Bekir A 2008 *Comput. and Math. Appl.* **56** 1451

[14] He J H and Wu X H 2006 *Chaos, Solitons and Fractals* **30** 700

[15] Zhang S 2007 *Phys. Lett.* A **365** 448

[16] Bekir A 2009 *Chaos, Solitons and Fractals* **40** 1467

[17] He J H, Wu G C and Austin F *Nonlinear Science Letters* A **1** 1

[18] Dai Z D, Wang C J, Lin S Q, Li D L and Mu G 2010 *Nonlinear Science Letters* A **1** 77

[19] Fu H M and Dai Z D 2009 *J. Nonlinear Sci. Numer. Simul.* **10** 927

[20] Feng Z S 2002 *J. Phys.* A: *Math. Gen.* **35** 343

[21] Feng Z S and Wang X H 2003 *Phys. Lett.* A **308** 173

[22] Ahmed Ali A H and Raslan K R 2007 *Int. J. Nonlinear Sci.* **4** 109

[23] Deng X J, Yan Z Z and Han L B 2009 *Chin. Phys.* B **18** 3169

[24] Chow K W 2010 *Nonlinear Science Letters* A **1** 83

[25] Raslan K R2008 2008 *Nonlinear Dynamics* **53** 281

[26] Ding T R and Li C Z 1996 *Ordinary Differential Equations* (Beijing: Peking University Press)

[27] Yusufoğlu E and Bekir A 2008 *Chaos, Solitons and Fractals* **38** 1126

[28] Alagesan T, Chung Y and Nakkeeran K 2004 *Chaos, Solitions and Fractals* **21** 879

[29] Yusufoğlu E and Bekir A 2008 *Math. Comput. Modelling* **48** 1694