Relation between space-charge-limiting current and electric field enhancement factor at curved surface cathode

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A two-dimensional solution of space-charge-limiting current for a high current vacuum diode with a spherical cathode is presented. The relation between space-charge-limiting current and electric field enhancement factor at the cathode surface for the diode with a curved surface cathode is also discussed. It is shown that compared with the current given by the conventional Child–Langmuir law, which describes the one-dimensional space-charege-limiting current, the two-dimensional space-charge-limiting current in such a diode is enhanced due to the electric-field enhancement along the cathode surface. Among practical parameter ranges, enhancement factor $\eta_{\rm b}$ approximately satisfies $\eta_{\rm b} \simeq A\beta^n$, where β is the electric field enhancement factor at the cathode surface, and n is a constant between 1 and 2, which is confirmed to be universal for the diodes with curved surface cathodes.

Keywords: diode, space-charge-limiting current, intense electron beam

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1. Introduction

Many vacuum diodes contain cathodes with curved shapes, such as spherical, cylindrical, or even pin-shape cathodes. These kinds of diodes are widely used to generate high current electron beams which serve as a part of the pulsed power systems. The space-charge-limiting currents of curved surface-cathode diode are necessary and important for the design of this kind of diode. However, there are formidable difficulties and it is almost impossible to obtain a completely analytic solution for this two-dimensional problem. Recently, the space-charge-limiting current emitted over a strip patch of width W and a circular patch of radius R on a planar cathode surface were studied by a particle-in-cell (PIC) simulation and semi-analytic method, and some useful results were obtained,[1−7] showing that such a cathode could produce a current density significantly exceeding that predicted by classical one-dimensional (1D) Child–Langmuir law. A simple theory for the two-dimensional (2D) Child–Langmuir law was presented in Ref. [1], the result is explicative and readily usable. On the other hand, none of those studies have taken into account the influence of electric field enhancement along the cathode surface on the space-charge-limiting current. In the present paper we first develop the space-charge-limiting current for a spherical cathode diode. Then we have a discussion on the general relation in diodes with curved surface cathodes between the space-charge-limiting current and the field enhancement factor on the cathode surface by an analytic approach. The results show that the space-charge-limiting current in such diodes is directly proportional to the product of the space-charge-limiting current of an infinite plate diode (the classical 1D Child–Langmuir value) and the nth power of field enhancement factor on the cathode surface approximately.

2. Space-charge-limiting current for spherical cathode diode

First, we try to deduce the space-charge-limiting current for a spherical cathode diode. However, it is formidably difficult and almost impossible to obtain a complete analytic solution for this two-dimensional problem. The method similar to that in Ref. [1] can be taken to obtain the space-charge-limiting current for this kind of diode. For the space-charge-limiting current emission, the electric field vanishes at the sur-

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face of the emitting cathode due to the space charge of the beam, so the space-charge-limiting current in a vacuum diode is directly related to the electric field strength at the cathode surface, and the stronger E-field on the cathode surface leads to the emission of more electrons and therefore a greater current can be obtained. We assume that there are sufficient space charges in the cathode plasma covering the whole emission surface to drive the normal surface electric field to zero, in other words, there is no limitation by the cathode emission capability as the field-enhanced explosive emission cathode. As shown in Fig. 1, the electron beam emitted from the spherical cathode expands as long as they leave the cathode surface, distributing in a variable range of angle θ_0. When a high enough external magnetic field is applied, $\theta_0 \simeq 0$.

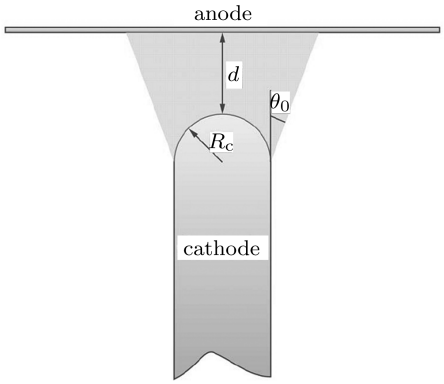


Fig. 1. Schematic diagram of spherical cathode diode.

We first discuss the spherical cathode diode immersed in a magnetic field. For the convenience of deduction, we assume beam electrons to be uniformly distributed along r and θ directions, i.e. $\rho(z,r,\theta) = \rho_0(z)$. Point charge $\Delta\rho(z)$ at positon (z,r) will produce electric field ΔE in the centre of the cathode (0, 0), the z-component of the electric field is

$$\Delta E_z(0,0) = \frac{\Delta\rho z}{4\pi\varepsilon_0 (r^2+z^2)^{3/2}}. \quad (1)$$

Then the total electric field at (0, 0) is obtained by the integration over all space charges,

$$E_z(0,0) = \int_0^d \int_0^{R_c} \int_0^{2\pi} \frac{z\rho_0(z)}{4\pi\varepsilon_0 (r^2+z^2)^{3/2}} r\,\mathrm{d}\theta\,\mathrm{d}r\,\mathrm{d}z. \quad (2)$$

Integrating Eq. (2) along r and θ directions gives

$$E_z(0,0) = \frac{1}{2\varepsilon_0}\int_0^d z\rho_0(z)\,\mathrm{d}z\left[\frac{1}{z} - \frac{1}{(R_c^2+z^2)^{1/2}}\right]. \quad (3)$$

Conservation of charge implies that the beam current density $J = \rho(z)v(z) = \text{const.}$, where $v(z)$ is the electron velocity. The total electric field due to the space charge can be written as

$$E_z(0,0) = \frac{G_0 J_{\mathrm{sph}}(2)}{2\varepsilon_0}\int_0^d \frac{z\,\mathrm{d}z}{v(z)}\left[\frac{1}{z} - \frac{1}{(R_c^2+z^2)^{1/2}}\right], \quad (4)$$

where G_0 is the multiplication factor to account for the contribution due to the image charges. The 2D limiting current density for the spherical cathode diode is reached when this total self-electric field equals the vacuum electric field in the centre of cathode. It gives

$$G_0\frac{J_{\mathrm{sph}}(2)}{2\varepsilon_0}\int_0^d \frac{z\,\mathrm{d}z}{v(z)}\left[\frac{1}{z} - \frac{1}{(R_c^2+z^2)^{1/2}}\right] = \beta\frac{V}{d}, \quad (5)$$

on the assumption that field enhancement factor β is equal to the ratio of the maximum field at the cathode surface, $E_{\max}$, to V/d, in which V is the voltage and d is the distance between the two electrodes, that is, $\beta = E_{\max}d/V$. And the current enhancement factor, η_b of space-charge-limiting current is defined as ratio $\eta_b = J_{\mathrm{sph}}(2)/J_{\mathrm{cl}}(1)$, where $J_{\mathrm{sph}}(2)$ is the space-charge-limiting current of the curved surface cathode diode and $J_{\mathrm{cl}}(1)$ is the space-charge-limiting current of an equivalent planar diode with the same gap between cathode and anode, which is given by classical (1D) Child–Langmuir law. The 1D Child–Langmuir law is obtained by setting R_c to be infinity. Thus from Eq. (5) we have

$$G_0\frac{J_{\mathrm{cl}}(1)}{2\varepsilon_0}\int_0^d \frac{\mathrm{d}z}{v(z)} = \frac{V}{d}. \quad (6)$$

The value of η_b can be given by the ratio between Eqs. (5) and (6) as

$$\eta_b = \frac{J_{\mathrm{sph}}(2)}{J_{\mathrm{cl}}(1)} = \beta\frac{\displaystyle\int_0^d \frac{\mathrm{d}z}{v(z)}}{\displaystyle\int_0^d \frac{z\,\mathrm{d}z}{v(z)}\left[\frac{1}{z} - \frac{1}{(R_c^2+z^2)^{1/2}}\right]}. \quad (7)$$

From the well-known density profile, $\rho(z) \propto z^{-2/3}$, in the 1D Child–Langmuir solution, then $v(z) \propto z^{2/3}$, thus we have

$$\eta_b = \beta\frac{\displaystyle\int_0^d \frac{\mathrm{d}z}{z^{2/3}}}{\displaystyle\int_0^d z^{1/3}\mathrm{d}z\left[\frac{1}{z} - \frac{1}{(R_c^2+z^2)^{1/2}}\right]}. \quad (8)$$

In practical cases, the cathode of the diode of this kind usually is made up of a half sphere connected to the end of a cylinder, serving as the cathode shank (shown in Fig. 1). The electric field enhancement effect arising from the cylinder cannot be ignored. Numerical simulations on this effect are carried out by the code POISSON SUPERFISH,[8] and for practical data range $0.5 < d/R_{\rm c} < 50$, we obtain the field enhancement factor

$$\beta = 0.87 + 0.9\,(d/R_{\rm c})^{0.92}\,. \tag{9}$$

Substituting Eq. (9) in Eq. (8), and integrating numerically in data range of $d/R_{\rm c}$ from 0.5 to 50, $\eta_{\rm b}$ can be approximately expressed as

$$\eta_{\rm b} = 1.05\beta^{1.4}. \tag{10}$$

Then the space-charge-limiting current of the spherical cathode diode, $I_{\rm sph}$, can be written as

$$I_{\rm sph}(2) = \eta_{\rm b}\pi R_{\rm c}^2 J_{\rm cl}(1) = 1.05\beta^{1.4}\pi R_{\rm c}^2 J_{\rm cl}(1), \tag{11}$$

in which $J_{\rm cl}(1)$ is given by the classical (1D) Child–Langmuir law as follows:[1,4,5,9]

$$J_{\rm cl}(1) = \frac{4\sqrt{2}}{9}\frac{\varepsilon_0\,(e/m_0)^{1/2}\,V^{3/2}}{d^2}. \tag{12}$$

The assumption of uniform space-charge density in the proceeding calculation holds true if the shielding effect of electrons is small enough. However, this condition is violated in many applications of intense relativistic beam. Now we extend our model to the case of nonuniform distribution. Since any distribution function can be expanded to linear superposition of several power functions accurately, in the following we will deduce the space charge limiting current of spherical cathode diode immersed in magnetic field with a power function density distribution profile.

Firstly, we assume the beam electrons to distribute in the form $\rho(z,r,\theta) = \rho_0(z)(1 + c\cdot r/R_{\rm c})$, a similar deduction process gives the current enhancement factor

$$\eta_{\rm b} = \frac{I_{\rm sph}(2)}{I_{\rm cl}\,(1)} = \frac{\beta\,(1+2c/3)}{F_1 - cF_2}, \tag{13}$$

where we introduce a dimensionless variable $x = z/d$, then

$$F_1 = \frac{1}{3}\int_0^1 x^{1/3}\left(\frac{1}{x} - \frac{1}{\sqrt{(R_{\rm c}/d)^2 + x^2}}\right)\mathrm{d}x, \tag{14}$$

$$F_2 = \frac{1}{3}\int_0^1 x^{1/3}\left[\frac{1}{\sqrt{1+(d/R_{\rm c})^2\,x^2}} - \log\left(\frac{R_{\rm c}}{d}\frac{1}{x} + \sqrt{1+\left(\frac{R_{\rm c}}{d}\right)^2\frac{1}{x^2}}\right)\right]\mathrm{d}x. \tag{15}$$

Integration by numerical method gives that the current enhancement factor $\eta_{\rm b}$ can be expressed as

$$\eta_{\rm b} \simeq A\beta^{1.4}, \tag{16}$$

where A is a constant related to space charge distribution, so it is a function of c. For $c = 1$, 2 and 10, $A = 1.74$, 2.42 and 7.53, respectively.

Secondly, we assume that the space charge distributes following $\rho\,(z,r) = \rho_0\,(z)\left(1 + cr^2/R_{\rm c}^2\right)$, space charge current enhancement factor can be obtained by a similar deduction process and expressed as

$$\eta_{\rm b} = \frac{I_{\rm sph}(2)}{I_{\rm cl}\,(1)} = \frac{\beta\,(1+c/2)}{F_1 - cF_3}, \tag{17}$$

$$F_3 = \frac{1}{3}\int_0^1 x^{1/3}\left(2x\left(\frac{d}{R_{\rm c}}\right)^2 - \frac{1+2\,(d/R_{\rm c})^2\,x^2}{\sqrt{(R_{\rm c}/d)^2 + x^2}}\right)\mathrm{d}x. \tag{18}$$

Results calculated by numerical method illustrates that the current enhancement factor $\eta_{\rm b}$ of space-charge-limiting current can also be expressed as the expression of Eq. (16) accurately, and for $c = 1$, 2 and 10, $A = 1.38$, 1.64 and 2.60, respectively.

Higher power function distributions give the same results. Therefore generally space charges with any distributions will follow the same laws, since any distribution function can be approximated to linear superposition of several power functions. Only coefficient A varies with distribution.

The space-charge-limiting current of the spherical cathode diode immersed in a high enough external magnetic field has been obtained. For the case where no external magnetic field is applied, the electrons emitted by the spherical cathode expand as long as they leave the cathode surface, and distribute in a variable range of angle θ_0. Angle θ_0 varies with distance d between the two electrodes and radius $R_{\rm c}$ of spherical cathode. A similar method can be taken to deduce the current enhancement factor with different angles of divergence. It is given that

$$\eta_{\rm b} = \frac{I_{\rm sph}(2)}{I_{\rm cl}(1)} = \beta \frac{\displaystyle\int_0^d \frac{{\rm d}z}{\upsilon(z)}}{\displaystyle\int_0^d \frac{R_{\rm c}^2 z\,{\rm d}z}{\upsilon(z)(R_{\rm c}+\alpha z)^2}\left[\frac{1}{z} - \frac{1}{\left((R_{\rm c}+\alpha z)^2+z^2\right)^{1/2}}\right]}, \tag{19}$$

in which $\alpha = \tan\theta_0$. The numerical integration in a practical data range of $0.5 < d/R_{\rm c} < 50$ gives

$$\eta_{\rm b} = A\beta^{1.4}. \tag{20}$$

For $\theta_0 = 5°, 10°, 15°, 20°, 30°, 45°$ and $60°$, $A = 1.08$, 1.10, 1.13, 1.16, 1.22, 1.32, 1.49, respectively.

The above result obtained indicates that the space-charge-limiting current in the spherical cathode diode is enhanced by a factor of $\eta_{\rm b}$, which can be determined by $\eta_{\rm b} = A\beta^n$, where β is the electric field enhancement factor at the cathode surface, and $n = 1.4$ for spherical cathode diode, A is a constant related to space charge distribution.

3. Discussion on relation between space-charge limiting current for curved cathodes diode and electric field enhancement factor at the cathode surface

In Section 2, we show that in a spherical cathode diode, there exists a relation between space-charge-limiting current and electric field enhancement factor at the cathode surface. The similar relation also exists in other curved surface cathode diodes. Now we discuss concentric cylindrical and spherical electrodes as examples.

A concentric cylindrical diode or coaxial diode consists of two cylindrical electrodes as shown in Fig. 2, serving as the cathode and anode respectively. The cathode radius is $R_{\rm c}$, and the anode radius is $R_{\rm a}$, $R_{\rm a} > R_{\rm c}$, i.e. electrons are emitted outwards from the inner cylinder surface.

From Refs. [10] and [11], we know that in the non-relativistic case, the space-charge-limiting beam current density of the coaxial diode is given by

$$I_{\rm cd} = \frac{4\varepsilon_0\sqrt{2e/m}V^{3/2}}{9R_{\rm c}R_{\rm a}B^2}. \tag{21}$$

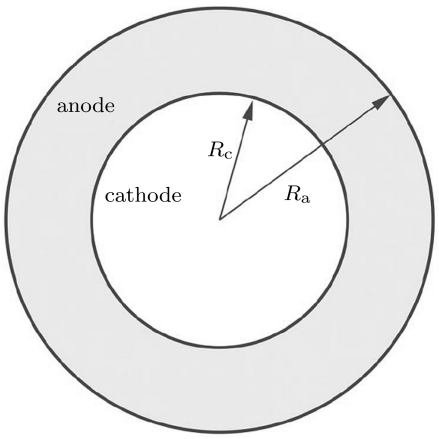


Fig. 2. Schematic diagram of coaxial diode.

Coefficient B satisfies

$$B(x) = x - 0.4x^2 + 0.09167x^3 - 0.01424x^4 + 0.00168x^5 + \dots, \tag{22}$$

where $x = \ln(R_{\rm a}/R_{\rm c})$.

From Eqs. (21) and (12), the current enhancement factor $\eta_{\rm cd}$, is obtained to be

$$\eta_{\rm cd} = \frac{d^2}{R_{\rm c}R_{\rm a}B^2} = \frac{(1-R_{\rm c}/R_{\rm a})(R_{\rm a}/R_{\rm c}-1)}{B^2}, \tag{23}$$

where d is the distance between the anode and the cathode, i.e. $d = R_{\rm a} - R_{\rm c}$.

The electric field enhancement of the cathode is readily obtained to be

$$\beta_{\rm cd} = \frac{d/R_{\rm c}}{\ln(R_{\rm a}/R_{\rm c})} = \frac{(R_{\rm a}/R_{\rm c}-1)}{\ln(R_{\rm a}/R_{\rm c})}. \tag{24}$$

From Eqs. (23) and (24), we know that current enhancement factor $\eta_{\rm cd}$ and field enhancement factor $\beta_{\rm cd}$ are both functions of $R_{\rm a}/R_{\rm c}$. Combining Eqs. (23) and (24) together, the analytic solution of $\eta_{\rm cd}$ can be obtained. In most practical cases, we usually have $1 \le R_{\rm a}/R_{\rm c} \le 20$, thus current enhancement factor $\eta_{\rm cd}$ can be expressed as

$$\eta_{\rm cd} \simeq \beta^{1.5}. \tag{25}$$

For a diode composed of two concentric spheres with cathode radius $R_{\rm c}$ and anode radius $R_{\rm a}(R_{\rm a} >$

R_c), similarly, we have current enhancement factor

$$\eta_{csp} = \frac{d^2}{R_c^2 G^2} = \frac{(R_a/R_c - 1)^2}{G^2}, \tag{26}$$

where G is the parameter determined by R_a/R_c. Introducing $x = \ln(R_a/R_c)$, the coefficient G is approximately expressed as

$$G(x) = x - 0.3x^2 + 0.075x^3 - 0.0143x^4 + 0.00216x^5 + \dots . \tag{27}$$

The electric field enhancement factor is readily obtained, and we have

$$\beta_{csp} = R_a/R_c. \tag{28}$$

From Eqs. (26) and (28), the relationship between space-charge-limiting current enhancement factor η_{csp} and electric field enhancement factor β_{csp} is

$$\eta_{csp} = \frac{\beta_{csp}^2}{G^2 \ln \beta_{csp}}. \tag{29}$$

Similarly, taking $1 \le R_a/R_c \le 20$, space-charge-limiting current enhancement factor η_{csp} can be expressed by a power function of electric field enhancement factor β_{csp}, or

$$\eta_{csp} \simeq \beta^{1.65}. \tag{30}$$

In summary, when curved cathode is introduced in a diode, current emitted by the cathode increases, owing to electric field enhancement. In practical cases where $1 \le R_a/R_c \le 20$, current enhancement factor η_b is a function of electric field enhancement factor β, and written as $\eta_b \simeq A\beta^n$, where n is a real number between 1 and 2.

4. Conclusion

From Sections 2 and 3, we know that there exists a relation between the space-charge-limiting current and electric field enhancement factor at the cathode surface for a curved surface cathode diode. This is not hard to understand, because it is well known that the space-charge-limiting current in a vacuum diode is directly related to the electric field strength at the cathode surface. In general, a stronger E-field leads to the emission of more electrons and therefore a greater current can be obtained on the assumption that there are sufficient space charges in the cathode plasma covering the whole emission surface to drive the surface normal electric field to zero, i.e. without limitation by the cathode emission. It is reasonable that the space-charge-limiting current enhancement factor η_b is a function of field enhancement factor β at the cathode surface. The result indicates that in vacuum diode, when a curved cathode is used, the space-charge-limiting current is enhanced by a factor of η_b, which is determined by $\eta_b \simeq A\beta^n$ in most practical cases, where n is a real number between 1 and 2, A is a constant related to space charge distribution.

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