Inversion formula and Parseval theorem for complex continuous wavelet transforms studied by entangled state representation*

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In a preceding letter (2007 *Opt. Lett.* **32** 554) we propose complex continuous wavelet transforms and found Laguerre–Gaussian mother wavelets family. In this work we present the inversion formula and Parseval theorem for complex continuous wavelet transform by virtue of the entangled state representation, which makes the complex continuous wavelet transform theory complete. A new orthogonal property of mother wavelet in parameter space is revealed.

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1. Introduction

Wavelet transforms (WTs) are very useful in signal analysis and detection[1−3] since it can overcome the shortcomings of nonlocality behaviour of classical Fourier analysis and thus enriches the theory of Fourier optics.[4] The continuous WT of a signal function $f(x) \in L^2(R)$ by a mother wavelet $\psi(x)$(restricted by the admissibility condition $\int_{-\infty}^{\infty}\psi(x)\,\mathrm{d}x=0$) is defined as

$$W_\psi f(\mu,s)=\frac{1}{\sqrt{\mu}}\int_{-\infty}^{\infty}f(x)\,\psi^*\left(\frac{x-s}{\mu}\right)\mathrm{d}x, \quad (1)$$

where $\mu\ (>0)$ is a scaling parameter and $s\,(\in R)$ is a translation parameter. The inversion of Eq.(1) is

$$f(x)=\frac{1}{C_\psi}\int_0^{\infty}\frac{\mathrm{d}\mu}{\mu^2}\int_{-\infty}^{\infty}W_\psi f(\mu,s)\times\psi\left(\frac{x-s}{\mu}\right)\frac{\mathrm{d}s}{\sqrt{\mu}}, \quad (2)$$

where $C_\psi=\int_0^{\infty}(|\psi(p)|^2/p)\mathrm{d}p<\infty$ and $\psi(p)$ is the Fourier transform of $\psi(x)$, for proving Eq.(2) we have employed the Dirac's representation theory,[5] which has the merit of rigor and simplicity.

In Refs. [6] and [7], Fan and Lu linked the one-dimensional (1D) WT with the unitary transform (squeezing and displacement) in quantum mechanics, i.e. they expressed the WT as a matrix element of the single-mode squeezing–displacing operator between the mother wavelet state vector $\langle\psi|$ and the state vector to be transformed, such that the admissibility condition for mother wavelets is examined in the context of quantum mechanics, in doing so a family of the Hermite–Gaussian mother wavelets are found. Further, by introducing the bipartite entangled state representation $|\eta\rangle$[8]

$$|\eta\rangle=\exp\left(-\frac{1}{2}|\eta|^2+\eta a_1^\dagger-\eta^* a_2^\dagger+a_1^\dagger a_2^\dagger\right)|00\rangle, \quad (3)$$

Fan and Lu then proposed the continuous complex wavelet transforms (CCWT) for $g(\eta)\equiv\langle\eta|\,g\rangle$,

$$W_\psi g(\mu,\kappa)=\frac{1}{\mu}\int\frac{\mathrm{d}^2\eta}{\pi}g(\eta)\,\psi^*\left(\frac{\eta-\kappa}{\mu}\right), \quad (4)$$

where $\kappa\in C$. Correspondingly, the admissibility condition for mother wavelets, $\int\frac{\mathrm{d}^2\eta}{2\pi}\psi(\eta)=0$, is examined in the entangled state representations and a family of new mother wavelets (named the Laguerre–Gaussian wavelets) are found to match the CCWT,[9] i.e. the qualified mother wavelets $\psi(\eta)$ satisfying the admissibility condition can be expressed as a function

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of $|\eta|$,

$$\psi(\eta) = \mathrm{e}^{-|\eta|^2/2}\sum_{n=0}^{\infty} K_{n,n}(-1)^n H^*_{n,n}(\eta,\eta^*) = \mathrm{e}^{-|\eta|^2/2}\sum_{n=0}^{\infty} n!K_{n,n}L_n\left(|\eta|^2\right), \quad (5)$$

where $L_n(x)$ is the Laguerre function and $H_{m,n}(x,y)$ is the two-variable Hermite polynomial,[10] whose generating function is

$$H_{m,n}(x,y) = \frac{\partial^{m+n}}{\partial t'^n \partial t^m}\exp\left[-tt' + tx + t'y\right]_{t=t'=0}. \quad (6)$$

We emphasize that the CCWT differs from the direct product of two 1D WTs since the squeezing transform involved in Eq. (4) is of two-mode

$$\frac{1}{\mu}\psi^*\left(\frac{\eta-\kappa}{\mu}\right) = \frac{1}{\mu}\left\langle\psi\left|\frac{\eta-\kappa}{\mu}\right\rangle = \langle\psi|S_2|\eta-\kappa\rangle\right., \quad (7)$$

where S_2 is the two-mode squeezing operator, and $S_2 = \exp[(a_1^\dagger a_2^\dagger - a_1 a_2)\ln\mu]$,[11,12] which is in sharp contrast to the direct-product of two single-mode squeezing (dilation) operators, and the two-mode squeezed state is simultaneously an entangled state.

In order to complete the CCWT theory, we must ask whether the corresponding Parseval theorem exists. This is important because the inversion formula of CCWT may appear as a lemma of this theorem. We shall solve this issue by virtue of the merits of entangled state in quantum mechanics, to be more specific, we shall use the property that the two-mode squeezing operator has its natural representation in the entangled state basis (see Eq. (10) below). Noting that the CCWT involves two-mode squeezing transform, the corresponding Parseval theorem differs from that of the direct-product of two 1D wavelet transforms, too.

2. Quantum mechanical version of CCWT

We begin by putting the CCWT into the context of quantum mechanics. Following the idea of quantum entanglement initiated by Einstein–Podolsky–Rosen (EPR),[13] Fan and Klauder[8] constructed the entangled state representation in two-mode Fock space, $|\eta\rangle$ in Eq. (8) is the common eigenvector of relative position $X_1 - X_2$ between two particles and their momentum $P_1 + P_2$, expressed as

$$(X_1 - X_2)|\eta\rangle = \sqrt{2}\eta_1|\eta\rangle,$$

and

$$(P_1 + P_2)|\eta\rangle = \sqrt{2}\eta_2|\eta\rangle, \quad (8)$$

where $X_j = (a_j + a_j^\dagger)/\sqrt{2}$, $P_j = (a_j - a_j^\dagger)/(\sqrt{2}\,\mathrm{i})$, $(j = 1,2)$. $|\eta\rangle$ is complete, i.e. $\int \frac{\mathrm{d}^2\eta}{\pi}|\eta\rangle\langle\eta| = 1$ ($\mathrm{d}^2\eta \equiv \mathrm{d}\eta_1\mathrm{d}\eta_2$, $\eta = \eta_1 + \mathrm{i}\,\eta_2$), and orthonormal, i.e. $\langle\eta|\,\eta'\rangle = \pi\delta(\eta-\eta')\,\delta(\eta^*-\eta'^*) \equiv \pi\delta^{(2)}(\eta-\eta')$.

Using $\langle\eta|$ and $\psi(\eta) = \langle\eta|\,\psi\rangle$ we can recast the CCWT in Eq. (4) into

$$W_\psi g(\mu,\kappa) = \langle\psi|U_2(\mu,\kappa)|g\rangle, \quad (9)$$

and $U_2(\mu,\kappa)$ is a two-mode squeezing–translating operator, which has its natural expression in EPR entangled state representation,

$$U_2(\mu,\kappa) \equiv \frac{1}{\mu}\int\frac{\mathrm{d}^2\eta}{\pi}\left|\frac{\eta-\kappa}{\mu}\right\rangle\langle\eta|, \quad (10)$$

when $\kappa = 0$, and $U_2(\mu,0) = S_2$.

3. Parseval theorem in the CCWT

Now we prove the Parseval theorem for CCWT,

$$\int_0^\infty\frac{\mathrm{d}\mu}{\mu^3}\int\frac{\mathrm{d}^2\kappa}{\pi}W_\psi g_1(\mu,\kappa)\,W^*_\psi g_2(\mu,\kappa) = C'_\psi\int\frac{\mathrm{d}^2\eta}{\pi}g_2^*(\eta)\,g_1(\eta), \quad (11)$$

where $\kappa = \kappa_1 + \mathrm{i}\,\kappa_2$,

$$C'_\psi = 4\int_0^\infty\frac{\mathrm{d}|\xi|}{|\xi|}|\psi(\xi)|^2. \quad (12)$$

$\psi(\xi)$ is the Fourier transform of $\psi(\eta)$, a mother wavelet. According to Eqs. (10) and (9) the quantum mechanical version of Parseval theorem should be

$$\int_0^\infty\frac{\mathrm{d}\mu}{\mu^3}\int\frac{\mathrm{d}^2\kappa}{\pi}\langle\psi|U_2(\mu,\kappa)|g_1\rangle\langle g_2|U_2^\dagger(\mu,\kappa)|\psi\rangle = C'_\psi\langle g_2|g_1\rangle, \quad (13)$$

where $\psi(\eta) = \langle\eta|\,\psi\rangle$, so $\psi(\xi) = \langle\xi|\,\psi\rangle$, $|\xi\rangle$ is the conjugate state to $|\eta\rangle$, and given as

$$|\xi\rangle = \exp\left\{-\frac{1}{2}|\xi|^2 + \xi a_1^\dagger + \xi^* a_2^\dagger - a_1^\dagger a_2^\dagger\right\}|00\rangle = (-1)^{a_2^\dagger a_2}|\eta\rangle_{\eta=\xi}, \quad \xi = \xi_1 + \mathrm{i}\,\xi_2, \quad (14)$$

which is the common eigenstate of centre-of-mass coordinate and the relative momentum operators, i.e.

$$(X_1 + X_2)|\xi\rangle = \sqrt{2}\xi_1|\xi\rangle, \quad (P_1 - P_2)|\xi\rangle = \sqrt{2}\xi_2|\xi\rangle, \quad (15)$$

and it is complete, i.e.

$$\int \frac{\mathrm{d}^2\xi}{\pi} |\xi\rangle \langle\xi| = 1. \tag{16}$$

The overlap between $\langle\xi|$ and $|\eta\rangle$ is[14]

$$\begin{aligned} \langle\xi\,|\eta\rangle &= \frac{1}{2}\exp[\frac{1}{2}(\xi^*\eta - \xi\eta^*)] \\ &= \frac{1}{2}\exp\left[\mathrm{i}\left(\xi_1\eta_2 - \xi_2\eta_1\right)\right], \end{aligned} \tag{17}$$

so

$$\begin{aligned} \psi(\xi) &= \langle\xi|\,\psi\rangle = \int \frac{\mathrm{d}^2\eta}{\pi} \langle\xi|\,\eta\rangle\langle\eta|\,\psi\rangle \\ &= \int \frac{\mathrm{d}^2\eta}{2\pi} \exp\left[\left(\xi^*\eta - \xi\eta^*\right)/2\right]\psi(\eta). \end{aligned} \tag{18}$$

Equation (11) indicates that once the state vector $\langle\psi|$ corresponding to mother wavelet is known, for any two states $|g_1\rangle$ and $|g_2\rangle$, their overlap (multiplied by factor C_ψ determined by Eq. (12)) is just their corresponding overlap of CCWTs in the (μ,κ) parametric space.

Proof of Eq. (11) or (13)

We start with calculating $U_2^\dagger(\mu,\kappa)|\xi\rangle$. Using Eqs. (10) and (17), we have

$$\begin{aligned} U_2^\dagger(\mu,\kappa)|\xi\rangle &= \frac{1}{\mu}\int \frac{\mathrm{d}^2\eta}{\pi} |\eta\rangle \left\langle \frac{\eta-\kappa}{\mu}\right| \xi\rangle \\ &= \frac{1}{\mu}\int \frac{\mathrm{d}^2\eta}{2\pi} |\eta\rangle\, \mathrm{e}^{(\mathrm{i}/\mu)(\xi_2\eta_1 - \xi_1\eta_2 + \xi_1\kappa_2 - \xi_2\kappa_1)} \\ &= \frac{1}{\mu}\left|\frac{\xi}{\mu}\right\rangle \mathrm{e}^{(\mathrm{i}/\mu)(\xi_1\kappa_2 - \xi_2\kappa_1)}, \end{aligned} \tag{19}$$

and it follows

$$\begin{aligned} &\int \frac{\mathrm{d}^2\kappa}{\pi} U_2^\dagger(\mu,\kappa)|\xi'\rangle\langle\xi|\,U_2(\mu,\kappa) \\ &= \frac{1}{\mu^2}\int \frac{\mathrm{d}^2\kappa}{\pi} \mathrm{e}^{(\mathrm{i}/\mu)[(\xi_1'-\xi_1)\kappa_2 + (\xi_2-\xi_2')\kappa_1]} \left|\frac{\xi'}{\mu}\right\rangle\left\langle\frac{\xi}{\mu}\right| \\ &= 4\pi \left|\frac{\xi'}{\mu}\right\rangle\left\langle\frac{\xi}{\mu}\right| \delta(\xi_1'-\xi_1)\,\delta(\xi_2-\xi_2'). \end{aligned} \tag{20}$$

Using Eqs. (16) and (20), the term on the left-hand side (LHS) of Eq. (13) can be reformed into

$$\begin{aligned} &\text{LHS of Eq. (18)} \\ &= \int_0^\infty \frac{\mathrm{d}\mu}{\mu^3}\int \frac{\mathrm{d}^2\kappa\mathrm{d}^2\xi\mathrm{d}^2\xi'}{\pi^3} \langle\psi|\,\xi\rangle \\ &\quad\times\langle\xi|\,U_2(\mu,\kappa)\,|g_1\rangle\langle g_2|\,U_2^\dagger(\mu,\kappa)\,|\xi'\rangle\langle\xi'|\,\psi\rangle \\ &= 4\int_0^\infty \frac{\mathrm{d}\mu}{\mu^3}\int \frac{\mathrm{d}^2\xi\mathrm{d}^2\xi'}{\pi} \langle g_2\left|\frac{\xi'}{\mu}\right\rangle\left\langle\frac{\xi}{\mu}\right|g_1\rangle \\ &\quad\times\psi^*(\xi)\,\psi(\xi')\,\delta(\xi_1'-\xi_1)\,\delta(\xi_2-\xi_2') \\ &= 4\int_0^\infty \frac{\mathrm{d}\mu}{\mu^3}\int \frac{\mathrm{d}^2\xi}{\pi} |\psi(\xi)|^2 \langle g_2\left|\frac{\xi}{\mu}\right\rangle\left\langle\frac{\xi}{\mu}\right|g_1\rangle \\ &= \int \frac{\mathrm{d}^2\xi}{\pi}\left\{4\int_0^\infty \frac{\mathrm{d}\mu}{\mu}|\psi(\mu\xi)|^2\right\}\langle g_2\,|\xi\rangle\langle\xi\,|g_1\rangle, \end{aligned} \tag{21}$$

where the integration value in $\{\ldots\}$ is actually $\xi-$independent. Noting that the mother wavelet $\psi(\eta)$ in Eq. (5) is just a function of $|\eta|$, $\psi(\xi)$ is also a function of $|\xi|$. In fact, using Eqs. (5), (6) and (18), we have

$$\psi(\xi) = \mathrm{e}^{-1/2|\xi|^2}\sum_{n=0}^\infty K_{n,n}H_{n,n}(|\xi|,|\xi|), \tag{22}$$

where we have used the integral formula

$$\int \frac{\mathrm{d}^2 z}{\pi}\mathrm{e}^{\zeta|z|^2+\xi z+\eta z^*} = -\frac{1}{\zeta}\mathrm{e}^{-\frac{\xi\eta}{\zeta}},\ \mathrm{Re}(\zeta)<0. \tag{23}$$

So we can rewrite Eq.(21) as

$$\begin{aligned} \text{LHS of Eq. (13)} &= C_\psi'\int \frac{\mathrm{d}^2\xi}{\pi}\langle g_2\,|\xi\rangle\langle\xi\,|g_1\rangle \\ &= C_\psi'\langle g_2\,|g_1\rangle, \end{aligned} \tag{24}$$

where

$$C_\psi' = 4\int_0^\infty \frac{\mathrm{d}\mu}{\mu}|\psi(\mu\xi)|^2 = 4\int_0^\infty \frac{\mathrm{d}\,|\xi|}{|\xi|}|\psi(\xi)|^2. \tag{25}$$

Then we have completed the proof of the Parseval theorem for CCWT in Eq. (13). Here, we should emphasize that Eq. (13) is different not only from the product of two 1D WTs, but also from the usual wavelet transform in 2D.

When $|g_2\rangle = |\eta\rangle$, by using Eq. (10) we see

$$\langle\eta|\,U_2^\dagger(\mu,\kappa)\,|\psi\rangle = \frac{1}{\mu}\psi\left(\frac{\eta-\kappa}{\mu}\right), \tag{26}$$

then substituting it into Eq. (13) yields

$$g_1(\eta) = \frac{1}{C_\psi'}\int_0^\infty \frac{\mathrm{d}\mu}{\mu^3}\int \frac{\mathrm{d}^2\kappa}{\pi\mu} W_\psi g_1(\mu,\kappa)\,\psi\left(\frac{\eta-\kappa}{\mu}\right), \tag{27}$$

which is just the inverse transform of the CCWT.

Especially, when $|g_1\rangle = |g_2\rangle$, Eq. (13) reduces to

$$\begin{aligned} &\int_0^\infty \frac{\mathrm{d}\mu}{\mu^3}\int \frac{\mathrm{d}^2\kappa}{\pi}|W_\psi g_1(\mu,\kappa)|^2 \\ &= C_\psi'\int \frac{\mathrm{d}^2\eta}{\pi}|g_1(\eta)|^2, \\ \text{or}\quad &\int_0^\infty \frac{\mathrm{d}\mu}{\mu^3}\int \frac{\mathrm{d}^2\kappa}{\pi}|\langle\psi|\,U_2(\mu,\kappa)\,|g_1\rangle|^2 \\ &= C_\psi'\langle g_1\,|g_1\rangle, \end{aligned} \tag{28}$$

which is named isometry of energy.

4. New orthogonal property of mother wavelet in parameter space

On the other hand, when $|g_1\rangle = |\eta\rangle$, and $|g_2\rangle = |\eta'\rangle$, Eq.(13) becomes

$$\frac{1}{C'_\psi}\int_0^\infty \frac{\mathrm{d}\mu}{\mu^5}\int \frac{\mathrm{d}^2\kappa}{\pi}\psi\left(\frac{\eta'-\kappa}{\mu}\right)\psi^*\left(\frac{\eta-\kappa}{\mu}\right) = \pi\delta^{(2)}(\eta-\eta'), \tag{29}$$

which is a new orthogonal property of mother wavelet in parameter space spanned by (μ,κ). In a similar way, we take $|g_1\rangle = |g_2\rangle = |m,n\rangle$, a two-mode number state, since $\langle m,n\,|m,n\rangle = 1$, we have

$$\int_0^\infty \frac{\mathrm{d}\mu}{\mu^3}\int \frac{\mathrm{d}^2\kappa}{\pi}\left|\langle\psi|\,U_2(\mu,\kappa)\,|m,n\rangle\right|^2 = C'_\psi, \tag{30}$$

or take $|g_1\rangle = |g_2\rangle = |z_1,z_2\rangle$, $|z\rangle = \exp\left(-|z|^2/2 + za^\dagger\right)|0\rangle$ is the coherent state, then

$$\int_0^\infty \frac{\mathrm{d}\mu}{\mu^3}\int \frac{\mathrm{d}^2\kappa}{\pi}\left|\langle\psi|\,U_2(\mu,\kappa)\,|z_1,z_2\rangle\right|^2 = C'_\psi. \tag{31}$$

This indicates that C'_ψ is $|g_1\rangle$-independent, which coincides with expression (12).

Next we examine a particular example. When mother wavelet $\psi(\eta)$ is taken as the following form:

$$\psi_M(\eta) = \langle\eta\,|\psi\rangle = \mathrm{e}^{-1/2|\eta|^2}(1-\frac{1}{2}|\eta|^2), \tag{32}$$

which is different from $\mathrm{e}^{-(x^2+y^2)/2}(1-x^2)\left(1-y^2\right)$, the direct-product of two 1D Mexican hat wavelets (we represent entangled Mexican hat wavelets with EMHWs), using Eq. (18) we have

$$\psi(\xi) = \frac{1}{2}|\xi|^2\,\mathrm{e}^{-\frac{1}{2}|\xi|^2}, \tag{33}$$

which leads to

$$C'_\psi = \int_0^\infty |\xi|^3\,\mathrm{e}^{-|\xi|^2}\mathrm{d}\,|\xi| = \frac{1}{2}. \tag{34}$$

Thus for EMHWs in Eq. (32), we see

$$2\int_0^\infty \frac{\mathrm{d}\mu}{\mu^5}\int \frac{\mathrm{d}^2\kappa}{\pi}\psi_M\left(\frac{\eta'-\kappa}{\mu}\right)\psi_M^*\left(\frac{\eta-\kappa}{\mu}\right) = \pi\delta^{(2)}(\eta-\eta'). \tag{35}$$

Equation (35) can be checked as follows. Using Eq. (32) and the integral formula

$$\int_0^\infty u\left(1-\frac{ux^2}{2}\right)\left(1-\frac{uy^2}{2}\right)\mathrm{e}^{-u\frac{x^2+y^2}{2}}\mathrm{d}u = -\frac{4(x^4-4x^2y^2+y^4)}{(x^2+y^2)^4},\ \mathrm{Re}\left(x^2+y^2\right)>0, \tag{36}$$

we can put the left-hand side (LHS) of (35) into

$$\begin{aligned}&\text{LHS of Eq. (35)}\\ &= -\int_0^\infty \frac{\mathrm{d}\frac{1}{\mu^2}}{\mu^2}\int\frac{\mathrm{d}^2\kappa}{\pi}\mathrm{e}^{-\frac{x^2+y^2}{2\mu^2}}\left(1-\frac{x^2}{2\mu^2}\right)\left(1-\frac{y^2}{2\mu^2}\right)\\ &= \int_0^\infty u\mathrm{d}u\int\frac{\mathrm{d}^2\kappa}{\pi}\left(1-\frac{ux^2}{2}\right)\left(1-\frac{uy^2}{2}\right)\mathrm{e}^{-u\frac{x^2+y^2}{2}}\\ &= -\int\frac{\mathrm{d}^2\kappa}{\pi}\frac{4(x^4-4x^2y^2+y^4)}{(x^2+y^2)^4},\end{aligned} \tag{37}$$

where $x^2 = |\eta'-\kappa|^2$, $y^2 = |\eta-\kappa|^2$.

When $\eta' = \eta$ and $x^2 = y^2$, we have

$$\begin{aligned}\text{LHS of Eq. (35)} &= \int\frac{\mathrm{d}^2\kappa}{2\pi\,|\kappa-\eta|^4}\\ &= \int_0^\infty\int_0^{2\pi}\frac{\mathrm{d}r\mathrm{d}\theta}{2\pi r^3}\to\infty.\end{aligned} \tag{38}$$

On the other hand, when $\eta\neq\eta'$ and noticing that

$$\begin{aligned}x^2 &= \left(\eta'_1-\kappa_1\right)^2+\left(\eta'_2-\kappa_2\right)^2,\\ y^2 &= \left(\eta_1-\kappa_1\right)^2+\left(\eta_2-\kappa_2\right)^2,\end{aligned} \tag{39}$$

we have

$$\mathrm{d}x^2\mathrm{d}y^2 = 4\,|J|\,\mathrm{d}\kappa_1\mathrm{d}\kappa_2, \tag{40}$$

where $\boldsymbol{J}(x,y) = \begin{vmatrix}\kappa_1-\eta'_1 & \kappa_2-\eta'_2\\ \kappa_1-\eta_1 & \kappa_2-\eta_2\end{vmatrix}$. As a result of Eq. (39), Eq. (37) reduces to

$$\begin{aligned}\text{LHS of Eq. (35)} &= -4\int_{-\infty}^\infty\frac{\mathrm{d}x\mathrm{d}y}{\pi}\frac{xy(x^4-4x^2y^2+y^4)}{|J|\,(x^2+y^2)^4}\\ &= 0,\end{aligned} \tag{41}$$

where we have noticed that $J(x,y)$ is a function of $\left(x^2,y^2\right)$. Thus we have

$$\begin{aligned}\text{LHS of Eq. (35)} &= \begin{cases}\infty,\ \eta=\eta',\\ 0,\ \eta\neq\eta'.\end{cases}\\ &= \text{RHS of Eq. (35)},\end{aligned} \tag{42}$$

where RHS stands for right hand side.

In summary, we propose the Parseval theorem corresponding to the CCWT in the context of quantum mechanics. Our calculations are simplified greatly by using the quantum state representations of two-mode squeezing operators. Finally, we should emphasize that since the CCWT corresponds to two-mode squeezing transform which differs from the direct product of two single-mode squeezing operators, the Parseval theorem of CCWT defined in this paper differs from that of the direct product of two 1-D wavelet transforms. For more discussion about optical transformations in the context of quantum optics, readers may refer to Refs. [15] and [16].

References

[1] Jaffard S, Meyer Y and Ryan R D 2001 *Wavelets, Tools for Science & Technology* (Philadelphia: Society for Industrial and Applied Mathematics)

[2] Daubechies I 1992 *Ten Lectures on Wavelets, CBMS-NSF Series in Applied Mathematics* (SIAM) (Philadelphia: Baker & Taylor Books)

[3] Pinsky M A 2002 *Introduction to Fourier Analysis and Wavelets* (New York: Books/Cole)

[4] Goodman J W 1968 *Introduction to Fourier Optics* (New York: McGraw-Hill)

[5] Hu L Y and Fan H Y 2008 *J. Mod. Opt.* **55** 1835

[6] Fan H Y and Lu H L 2006 *Opt. Lett.* **31** 407

[7] Fan H Y and Lu J F 2004 *Commun. Theor. Phys.* **41** 681

[8] Fan H Y and Klauder J R 1994 *Phys. Rev.* A **49** 704

[9] Fan H Y and Lu H L 2007 *Opt. Lett.* **32** 554

[10] Wunsche A 2000 *J. Phys.* A *Math. and Gen.* **33** 1603

[11] Loudon R, and Knight P L 1987 *J. Mod. Opt.* **34** 709

[12] Scully M O and Zubairy M S 1997 *Quantum Optics* (Berlin: Cambridge University)

[13] Einstein A, Podolsky B and Rosen N 1935 *Phys. Rev.* **47** 777

[14] Fan H Y and Lu H L 2005 *Int. J. Mod. Phys.* B **19** 799

[15] Fan H Y and Hu L Y 2009 *Chin. Phys.* B **18** 0611

[16] Fan H Y and Hu L Y 2008 *Chin. Phys.* B **17** 1640