Synchronization between two different chaotic systems with noise perturbation*

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This paper investigates the chaotic synchronization between the noise-perturbed Lorenz system and one of the noise-perturbed Chen and Lü systems. Based on the active control method and the Lyapunov theory in stochastic differential equations, sufficient conditions for the stability of the error dynamics are derived. Numerical simulations are also shown to demonstrate the effectiveness of these theoretic results.

Keywords: chaotic synchronization, noise
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1. Introduction

Since the pioneering work by Pecora and Carroll,[1] there has been particular interest in chaotic synchronization,[2−19] mainly due to its potential applications in secure communications, ecological systems, system identification, etc. During past decades, many methods and experimental techniques have been presented to realize the synchronization of two identical chaotic systems, such as adaptive control,[6] nonlinear control,[7] nonlinear feedback control.[8] However, the method of the synchronization of two different chaotic systems is far from being straightforward. In fact, in systems such as laser array, biological systems to cognitive processes, it is hardly the case that every component can be assumed to be identical. Consequently, in these years, more and more applications of chaotic synchronization in secure communications make it much more important to synchronize two different chaotic systems. Recently active control has been applied to synchronize two different chaotic systems.[10−13]

Besides, noise is ubiquitous in both nature and manmade systems, and in the processes of applications, synchronizing effect is influenced by noise unavoidably. Therefore, the synchronization of systems with noise perturbation has become a field of great interest recently.[14−20] However, to the best of our knowledge, there are no previous results on the chaotic synchronization in two different noise-perturbed systems. Therefore, it is necessary and important to propose the synchronization conditions for two different noise-perturbed chaotic systems.

In this paper, we apply active control theory to synchronizing two different chaotic systems. This method is applied to achieve chaotic synchronization for the noise-perturbed Lorenz system and one of the noise-perturbed Chen and Lü systems. Based on the Lyapunov theory in stochastic differential equations, we derive sufficient conditions for the synchronization. We support the analytical results with numerical simulations.

The rest of this paper is organized as follows. In Section 2, brief descriptions of the Lorenz, Chen and Lü systems are introduced. The synchronization between noise-perturbed Lorenz system and Chen system is investigated in Section 3. The synchronization between noise-perturbed Lorenz system and Lü system is also investigated in Section 4. The simulation results are presented in Section 5. Finally, some conclusions are given in Section 6.

2. Systems description

The Lorenz system is known to be a simplified model of several physical systems. At the origin, it was derived from a model of the earth's atmospheric convection flow heated from below and cooled from above.[21] The Lorenz system is described by

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$$\begin{cases} \dot{x}_1 = a(x_2 - x_1), \\ \dot{x}_2 = cx_1 - x_1x_3 - x_2, \\ \dot{x}_3 = x_1x_2 - bx_3, \end{cases} \tag{1}$$

which has a chaotic attractor when $a = 10$, $b = 8/3$, $c = 28$.

Chen system is a typical chaos anti-control model, which has a more complicated topological structure than Lorenz attractor.[22] The nonlinear differential equations that describe the Chen system are

$$\begin{cases} \dot{y}_1 = \alpha(y_2 - y_1), \\ \dot{y}_2 = (\gamma - \alpha)y_1 - y_1y_3 - \gamma y_2, \\ \dot{y}_3 = y_1y_2 - \beta y_3, \end{cases} \tag{2}$$

which has a chaotic attractor when $\alpha = 35$, $\beta = 3$, $\gamma = 28$.

The Lü system is a typical transition system, which connects the Lorenz and Chen attractors and represents the transition from one to the other.[23] The Lü system is described by

$$\begin{cases} \dot{z}_1 = \rho(z_2 - z_1), \\ \dot{z}_2 = -z_1z_3 - \nu z_2, \\ \dot{z}_3 = z_1z_2 - \mu z_3, \end{cases} \tag{3}$$

which has a chaotic attractor when $\rho = 36$, $\nu = 20$, $\mu = 3$.

In the following sections, we will study chaotic synchronization between two different noise-perturbed chaotic systems by using active control. This method is applied to achieve chaotic synchronization between the noise-perturbed Lorenz system and each of the noise-perturbed Chen and Lü systems.

3. Synchronization between noise-perturbed Lorenz system and Chen system

In the following sections, we will study the synchronization of two different chaotic systems. For simplifying the problem, we assume that the drive system is the noise-perturbed Lorenz system throughout this paper. In this section, the synchronizing behaviours between the noise-perturbed Lorenz system and Chen system will be studied. Our aim is to design a controller and make the response system trace the drive system and become ultimately the same. Therefore, we define the drive and response systems as follows:

$$\begin{cases} \dot{x}_1 = a(x_2 - x_1) + g_1(\boldsymbol{x}, t)\dot{\boldsymbol{W}}, \\ \dot{x}_2 = cx_1 - x_1x_3 - x_2 + g_2(\boldsymbol{x}, t)\dot{\boldsymbol{W}}, \\ \dot{x}_3 = x_1x_2 - bx_3 + g_3(\boldsymbol{x}, t)\dot{\boldsymbol{W}}, \end{cases} \tag{4}$$

and

$$\begin{cases} \dot{y}_1 = \alpha(y_2 - y_1) + u_1(t) + g_1(\boldsymbol{y}, t)\dot{\boldsymbol{W}}, \\ \dot{y}_2 = (\gamma - \alpha)y_1 - y_1y_3 - \gamma y_2 + u_2(t) + g_2(\boldsymbol{y}, t)\dot{\boldsymbol{W}}, \\ \dot{y}_3 = y_1y_2 - \beta y_3 + u_3(t) + g_3(\boldsymbol{y}, t)\dot{\boldsymbol{W}}, \end{cases} \tag{5}$$

where $\boldsymbol{x} = (x_1, x_2, x_3)^{\mathrm{T}}$, $\boldsymbol{y} = (y_1, y_2, y_3)^{\mathrm{T}}$, and $\boldsymbol{U} = (u_1, u_2, u_3)^{\mathrm{T}}$ is the active control function to be designed, $g_i : \boldsymbol{R}^3 \times \boldsymbol{R}^+ \to \boldsymbol{R}^{1\times m}(i = 1, 2, 3)$, and $\boldsymbol{W} = (w_1, \ldots, w_m)^{\mathrm{T}}$ is an m-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with a natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Accordingly, $\dot{\boldsymbol{W}}$ is an m-dimensional white noise. This kind of noise term in Eqs. (4) and (5) is utilized to describe that both the drive system and the response system are influenced by environmental fluctuation, such as inaccurate design of some other important parameters. If every element of $g_i(\boldsymbol{x}, t)$ is the linear function of $\boldsymbol{x}$, then let $\sigma_i(\boldsymbol{e}, t) = g_i(\boldsymbol{y}, t) - g_i(\boldsymbol{x}, t)(i = 1, 2, 3)$ where $\boldsymbol{e} = \boldsymbol{y} - \boldsymbol{x}$ is the error vector. Through out this paper, it is assumed that $\sigma_i(\boldsymbol{e}, t)$ is local Lipschitz continuous and satisfies the linear growth condition. Moreover, assume also that $\sigma_i(\boldsymbol{0}, t) \equiv \boldsymbol{0}$ for the purpose of the complete synchronization. Let $\boldsymbol{\sigma}^{\mathrm{T}} = (\boldsymbol{\sigma}_1^{\mathrm{T}}, \boldsymbol{\sigma}_2^{\mathrm{T}}, \boldsymbol{\sigma}_3^{\mathrm{T}})$, and assume that the matrix $\boldsymbol{\sigma}$ could be estimated by

$$\frac{1}{2}\mathrm{trace}(\boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\sigma}) \leq q_1e_1^2 + q_2e_2^2 + q_3e_3^2, \tag{6}$$

where q_i $(i = 1, 2, 3)$ are nonnegative numbers. There are many functions $\boldsymbol{\sigma}$ satisfy above assumptions, for example, when $\dot{\boldsymbol{W}}$ is a one-dimensional white noise one can set $g_i(\boldsymbol{x}, t) = k_ix_i, g_i(\boldsymbol{y}, t) = k_iy_i$, then $\boldsymbol{\sigma}_i = k_ie_i$ and $\mathrm{trace}(\boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\sigma}) = k_1^2e_1^2 + k_2^2e_2^2 + k_3^2e_3^2$.

Subtracting Eq. (4) from Eq. (5) yields the error dynamical system,

$$\begin{cases} \dot{e_1} = \alpha(e_2 - e_1) + (\alpha - a)(x_2 - x_1) + u_1(t) + \sigma_1(\boldsymbol{e}, t)\dot{\boldsymbol{W}}, \\ \dot{e_2} = (\gamma - \alpha)e_1 - \gamma e_2 + (\gamma - \alpha - c)x_1 - (\gamma - 1)x_2 - y_3e_1 - x_1e_3 + u_2(t) + \sigma_2(\boldsymbol{e}, t)\dot{\boldsymbol{W}}, \\ \dot{e_3} = -\beta e_3 + y_2e_1 + x_1e_2 - (\beta - b)x_3 + u_3(t) + \sigma_3(\boldsymbol{e}, t)\dot{\boldsymbol{W}}. \end{cases} \tag{7}$$

The active control function U is defined as follows:

$$\begin{cases} u_1 = (a-\alpha)(x_2 - x_1), \\ u_2 = (\alpha + c - \gamma)x_1 + (\gamma - 1)x_2 + y_3 e_1, \\ u_3 = -y_2 e_1 + (\beta - b)x_3. \end{cases} \tag{8}$$

Hence the error system Eq. (7) becomes

$$\begin{cases} \dot{e_1} = -\alpha e_1 + \alpha e_2 + \sigma_1(\boldsymbol{e}, t)\dot{\boldsymbol{W}}, \\ \dot{e_2} = (\gamma - \alpha)e_1 - \gamma e_2 - x_1 e_3 + \sigma_2(\boldsymbol{e}, t)\dot{\boldsymbol{W}}, \\ \dot{e_3} = -\beta e_3 + x_1 e_2 + \sigma_3(\boldsymbol{e}, t)\dot{\boldsymbol{W}}. \end{cases} \tag{9}$$

And the error system can be rewritten in the following matrix form:

$$\dot{\boldsymbol{e}} = \begin{pmatrix} -\alpha & \alpha & 0 \\ \gamma - \alpha & -\gamma & 0 \\ 0 & 0 & -\beta \end{pmatrix} \boldsymbol{e} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -x_1 \\ 0 & x_1 & 0 \end{pmatrix} \boldsymbol{e} + \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} \dot{\boldsymbol{W}}. \tag{10}$$

Therefore, from the theory of stochastic differential equation[24] the error dynamics Eq. (10) possesses a unique global solution at $t \geq 0$, denoted by $\boldsymbol{e}(\boldsymbol{e}_0, t)$ for any initial data $\boldsymbol{e}_0$. And $\boldsymbol{e}(\boldsymbol{0}, t) \equiv \boldsymbol{0}$ is a trivial solution of the error dynamics Eq. (10). Obviously, if this trivial solution is globally asymptotically stable in physical sense, the complete synchronization between noise-perturbed Lorenz system and Chen system could be realized.

To this end, construct a function $V = \frac{1}{2}\boldsymbol{e}^{\mathrm{T}}\boldsymbol{e}$, then, according to Eq. (A2) in the Appendix, the operator $\mathcal{L}$ onto the function V along with the error system Eq. (10) gives

$$\mathcal{L}V = \boldsymbol{e}^{\mathrm{T}} \begin{pmatrix} -\alpha & \gamma/2 & 0 \\ \gamma/2 & -\gamma & 0 \\ 0 & 0 & -\beta \end{pmatrix} \boldsymbol{e} + \frac{1}{2}\mathrm{trace}(\sigma^{\mathrm{T}}\sigma)\,. \tag{11}$$

Therefore, with Eq. (6), one obtains an estimation,

$$\mathcal{L}V \leq \boldsymbol{e}^{\mathrm{T}}\boldsymbol{\Xi}\boldsymbol{e}, \tag{12}$$

where

$$\boldsymbol{\Xi} = \begin{pmatrix} q_1 - \alpha & \gamma/2 & 0 \\ \gamma/2 & q_2 - \gamma & 0 \\ 0 & 0 & q_3 - \beta \end{pmatrix}.$$

If we select q_i $(i = 1, 2, 3)$ such that the matrix $\boldsymbol{\Xi}$ is negatively definite, then

$$\mathcal{L}V \leq \lambda \boldsymbol{e}^{\mathrm{T}}\boldsymbol{e}, \tag{13}$$

where $\lambda < 0$ is the largest eigenvalue of the matrix $\boldsymbol{\Xi}$. Then from Lemma 1 in Appendix, we can obtain

$$\lim_{t\to\infty} \boldsymbol{e}(t) = \boldsymbol{0} \ \text{ almost surely.}$$

We conclude the above argument as Theorem 1:

Theorem 1 Assume that the noise intensity matrix $\boldsymbol{\sigma}$ could be estimated by Eq. (6). Also assume that there exist some positive numbers q_i $(i = 1, 2, 3)$ such that $\boldsymbol{\Xi}$ is negatively definite, then the trivial solution of Eq. (10) is almost surely asymptotically stable, that is, the complete synchronization between noise-perturbed Lorenz and Chen systems could be achieved with probability one in the physical sense.

4. Synchronization between noise-perturbed Lorenz system and Lü system

In this section, our aim is to make synchronization between noise-perturbed Lorenz system and Lü System. For this purpose, the Lü System, as a response system, is given by

$$\begin{cases} \dot{z}_1 = \rho(z_2 - z_1) + u_1(t) + g_1(\boldsymbol{z}, t)\dot{\boldsymbol{W}}, \\ \dot{z}_2 = -z_1 z_3 - \nu z_2 + u_2(t) + g_2(\boldsymbol{z}, t)\dot{\boldsymbol{W}}, \\ \dot{z}_3 = z_1 z_2 - \mu z_3 + u_3(t) + g_3(\boldsymbol{z}, t)\dot{\boldsymbol{W}}, \end{cases} \tag{14}$$

where $\boldsymbol{z} = (z_1, z_2, z_3)$, $\boldsymbol{U} = (u_1, u_2, u_3)^{\mathrm{T}}$ is the active control function to be designed. Then, let $\boldsymbol{e} = \boldsymbol{z} - \boldsymbol{x}$, $\sigma_i(\boldsymbol{e}, t) = g_i(\boldsymbol{z}, t) - g_i(\boldsymbol{x}, t)(i = 1, 2, 3)$, where $g_i(\boldsymbol{x}, t)$ and $\sigma^{\mathrm{T}} = (\sigma_1^{\mathrm{T}}, \sigma_2^{\mathrm{T}}, \sigma_3^{\mathrm{T}})$ satisfy the same conditions as those defined in Section 3.

Subtracting Eq. (4) from Eq. (14) yields the following error dynamical system:

$$\begin{cases} \dot{e_1} = \rho(e_2 - e_1) + (\rho - a)(x_2 - x_1) + u_1(t) + \sigma_1(\boldsymbol{e}, t)\dot{\boldsymbol{W}}, \\ \dot{e_2} = -\nu e_2 - z_3 e_1 - x_1 e_3 - c x_1 - (\nu - 1)x_2 + u_2(t) + \sigma_2(\boldsymbol{e}, t)\dot{\boldsymbol{W}}, \\ \dot{e_3} = -\mu e_3 + z_2 e_1 + x_1 e_2 - (\mu - b)x_3 + u_3(t) + \sigma_3(\boldsymbol{e}, t)\dot{\boldsymbol{W}}. \end{cases} \tag{15}$$

The active control function $\boldsymbol{U}$ is defined as follows:

$$\begin{cases} u_1 = (a-\rho)(x_2 - x_1), \\ u_2 = z_3 e_1 + (\nu - 1)x_2 + c x_1, \\ u_3 = -z_2 e_1 + (\mu - b)x_3. \end{cases} \tag{16}$$

Hence the error system Eq. (15) becomes

$$\begin{cases} \dot{e_1} = -\rho e_1 + \rho e_2 + \sigma_1(\boldsymbol{e}, t)\dot{\boldsymbol{W}}, \\ \dot{e_2} = -\nu e_2 - x_1 e_3 + \sigma_2(\boldsymbol{e}, t)\dot{\boldsymbol{W}}, \\ \dot{e_3} = -\mu e_3 + x_1 e_2 + \sigma_3(\boldsymbol{e}, t)\dot{\boldsymbol{W}}. \end{cases} \tag{17}$$

And the error system can be rewritten in the following matrix form:

$$\begin{aligned} \dot{\boldsymbol{e}} = & \begin{pmatrix} -\rho & \rho & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & -\mu \end{pmatrix} \boldsymbol{e} \\ & + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -x_1 \\ 0 & x_1 & 0 \end{pmatrix} \boldsymbol{e} + \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} \dot{\boldsymbol{W}}. \end{aligned} \tag{18}$$

To this end, construct a function $V = (1/2)\boldsymbol{e}^{\mathrm{T}}\boldsymbol{e}$, then, according to Eq. (A2) in the Appendix, the operator $\mathcal{L}$ onto the function V along with the error system Eq. (18) gives

$$\mathcal{L}V = \boldsymbol{e}^{\mathrm{T}} \begin{pmatrix} -\rho & \rho/2 & 0 \\ \rho/2 & -\nu & 0 \\ 0 & 0 & -\mu \end{pmatrix} \boldsymbol{e} + \frac{1}{2}\mathrm{trace}(\sigma^{\mathrm{T}}\sigma). \tag{19}$$

Therefore, with Eq. (6), one obtains an estimation,

$$\mathcal{L}V \leq \boldsymbol{e}^{\mathrm{T}}\boldsymbol{\Theta}\boldsymbol{e}, \tag{20}$$

where

$$\boldsymbol{\Theta} = \begin{pmatrix} q_1 - \rho & \rho/2 & 0 \\ \rho/2 & q_2 - \nu & 0 \\ 0 & 0 & q_3 - \mu \end{pmatrix}.$$

If we select q_i $(i = 1, 2, 3)$ such that the matrix $\boldsymbol{\Theta}$ is definitely negative, then

$$\mathcal{L}V \leq \lambda \boldsymbol{e}^{\mathrm{T}}\boldsymbol{e}, \tag{21}$$

where $\lambda < 0$ is the largest eigenvalue of the matrix $\boldsymbol{\Theta}$. Then from Lemma 1 in the Appendix, we can obtain

$$\lim_{t\to\infty} \boldsymbol{e}(t) = \boldsymbol{0} \quad \text{almost surely.}$$

We conclude the above argument as Theorem 2:

Theorem 2 Assume that the noise intensity matrix $\boldsymbol{\sigma}$ could be estimated by Eq. (6). Also assume that there exist some positive numbers q_i $(i = 1, 2, 3)$ such that $\boldsymbol{\Theta}$ is negatively definite, then the trivial solution of Eq. (18) is almost surely asymptotically stable, that is, the complete synchronization between noise-perturbed Lorenz and Lü systems could be achieved with probability one in the physical sense.

5. Numerical simulations

In this section, numerical simulations are provided to illustrate the effectiveness of the obtained results. All the noise-perturbed systems are solved with the Euler–Maruyama numerical scheme. In addition, a time step size 0.001 is employed. Meanwhile, assume that $\boldsymbol{W} = [w_1, w_2]^{\mathrm{T}}$ is a two-dimensional Brownian motion and $g_i(\boldsymbol{x}, t) = [0.3x_1 + 0.4x_2, 0.6x_3]$, then the noise intensity matrix is in the form of $\sigma_i(\boldsymbol{e}, t) = [0.3e_1 + 0.4e_2, 0.6e_3]$ for $i = 1, 2, 3$. Thus, $\sigma_i(\boldsymbol{e})\sigma_i^{\mathrm{T}}(\boldsymbol{e}) = (0.3e_1 + 0.4e_2)^2 + 0.36e_3^2 \leq 0.18e_1^2 + 0.32e_2^2 + 0.36e_3^2$, $\mathrm{trace}(\sigma^{\mathrm{T}}\sigma) \leq 0.54e_1^2 + 0.96e_2^2 + 1.08e_3^2$. Hence, $q_1 = 0.27, q_2 = 0.48$ and $q_3 = 0.54$. We select the parameters of Lorenz system as $a = 10$, $b = 8/3$, $c = 28$, the parameters of Chen system as $\alpha = 35$, $\beta = 3$, $\gamma = 28$, and the parameters of Lü system as $\rho = 36$, $\mu = 3$, $\nu = 20$, so that each of the Lorenz, Chen and Lü systems exhibits a chaotic behaviour.

Case I Lorenz system and Chen system.

The initial values of Eqs. (4) and (5) are taken as $x_1(0) = 2, x_2(0) = 6, x_3(0) = -3$, $y_1(0) = 5$, $y_2(0) = -4$, $y_3(0) = 5$, respectively. It is easy to see that

$$\alpha > q_1 + \gamma/2, \gamma/2 > q_2, \beta > q_3.$$

Hence, from Geršgorin disck theorem,[25] the matrix $\boldsymbol{\Xi}$ is negatively definite. Therefore, according to Theorem 1, the complete synchronization between noise-perturbed Lorenz and Chen systems could be achieved in the physical sense. This is clearly shown in Fig. 1.

Case II Lorenz system and Lü system.

The initial values of Eqs. (4) and (14) are taken as $x_1(0) = 2$, $x_2(0) = -6$, $x_3(0) = 3$, $z_1(0) = -10$, $z_2(0) = 11$, $z_3(0) = 13$, respectively. It is easy to see that

$$\rho/2 > q_1, \quad \nu > q_2 + \rho/2, \quad \mu > q_3.$$

Hence from Geršgorin disck theorem,[25] the matrix $\boldsymbol{\Theta}$ is negatively definite. Therefore, according to Theorem 2, the complete synchronization between noise-perturbed Lorenz and Lü systems could be achieved in the physical sense. This is clearly shown by Fig. 2.

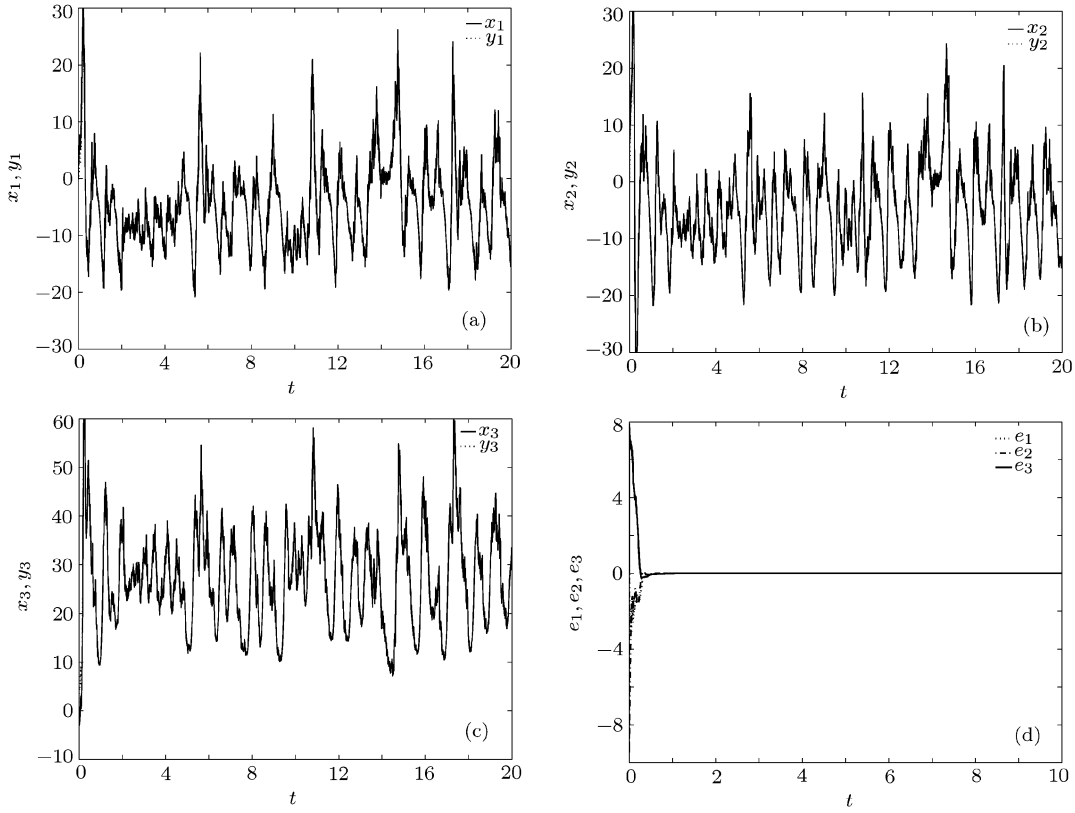


Fig. 1. Synchronization between noise-perturbed Lorenz and Chen systems. Evolution of the driving system state (x_1, x_2, x_3) and response system state (y_1, y_2, y_3) with $t = 20$, evolution of synchronization errors (e_1, e_2, e_3) with $t = 10$: (a) signals x_1 and y_1; (b) signals x_2 and y_2; (c) signals x_3 and y_3; (d) synchronization errors states e_1, e_2 and e_3.

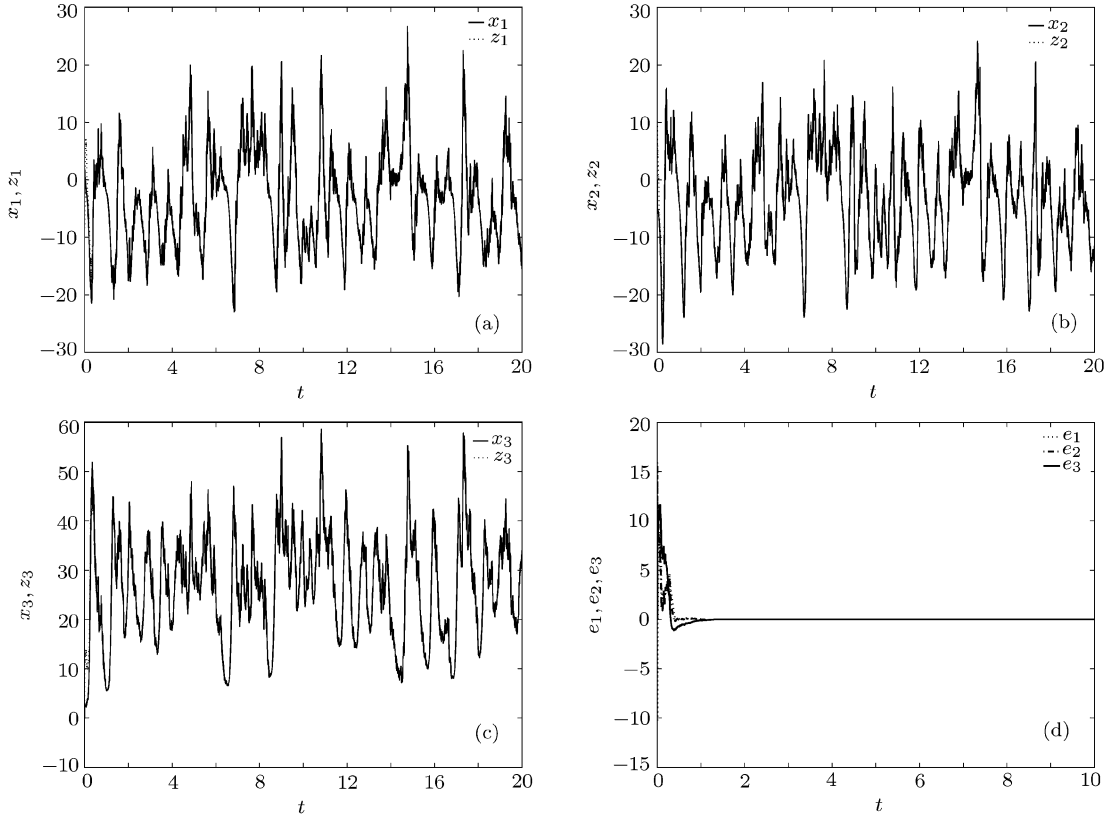


Fig. 2. Synchronization between noise-perturbed Lorenz and Lü systems. Evolution of the driving system state (x_1, x_2, x_3) and response system state (z_1, z_2, z_3) with $t = 20$, evolution of synchronization errors (e_1, e_2, e_3) with $t = 10$: (a) signals x_1 and z_1; (b) signals x_2 and z_2; (c) signals x_3 and z_3; (d) synchronization errors states e_1, e_2 and e_3.

6. Conclusions

This study demonstrates that chaotic synchronization between two different chaotic systems with noise perturbation using active control is achieved. Based on the Lyapunov theory in stochastic differential equations, sufficient conditions for the complete synchronization are derived. Numerical simulations are also given to validate the theoretic results.

Appendix

Consider the following n-dimensional stochastic system:

$$\mathrm{d}\boldsymbol{x} = f(\boldsymbol{x},t)\mathrm{d}t + g(\boldsymbol{x},t)\mathrm{d}\boldsymbol{W}(t), \qquad \text{(A1)}$$

where $f(\boldsymbol{x},t) : \boldsymbol{R}^n \times \boldsymbol{R}_+ \to \boldsymbol{R}^n$ and $g(\boldsymbol{x},t) : \boldsymbol{R}^n \times \boldsymbol{R}_+ \to \boldsymbol{R}^{n\times m}$ are continuous and satisfy the local Lipschitz condition and linear growth condition with $f(\boldsymbol{0},t) = \boldsymbol{0}, g(\boldsymbol{0},t) = \boldsymbol{0}$. So Eq.(A1) admits a trivial solution $x(\boldsymbol{0},t) \equiv \boldsymbol{0}$.

Then, for each $V(\boldsymbol{x},t) \in C^{2,1}(\boldsymbol{R}^n \times \boldsymbol{R})$, define the differential operator $\mathcal{L}$ that is related to system (A1) by

$$\mathcal{L}V = V_t + V_{\boldsymbol{x}} f + \frac{1}{2}\mathrm{trace}[g^{\mathrm{T}} V_{\boldsymbol{x}\boldsymbol{x}} g], \qquad \text{(A2)}$$

where $V_t = \dfrac{\partial V}{\partial t}$, $V_{\boldsymbol{x}} = \left(\dfrac{\partial V}{\partial x_1}, \ldots, \dfrac{\partial V}{\partial x_n}\right)$, $V_{\boldsymbol{x}x} = \left(\dfrac{\partial^2 V}{\partial x_i \partial x_j}\right)_{n\times n}$.

The following result gives conditions on the stability of the trivial solution of Eq. (A1), whose details can be found in Ref. [24].

Lemma 1 Suppose that there exist a nonnegative function $V(\boldsymbol{x},t) \in C^{2,1}(\boldsymbol{R}^n \times [t_0, +\infty], \boldsymbol{R}^+)$ and three positive numbers, i.e. p, α, λ, such that for all $\boldsymbol{x} \neq 0, t \geq t_0$,

$$\alpha|\boldsymbol{x}|^p \leq V(\boldsymbol{x},t), \quad \mathcal{L}V(\boldsymbol{x},t) \leq -\lambda V(\boldsymbol{x},t)$$

holds, then for any $\boldsymbol{x}_0 \in R^n$ the trival solution of Eq. (A1) is almost surely exponentially asymptotically stable, i.e.

$$\lim_{t\to\infty} \sup \frac{1}{t}\lg|\boldsymbol{x}(t,t_0,\boldsymbol{x}_0)| \leq -\frac{\lambda}{p}$$

holds almost surely.

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