Outer synchronization between two different fractional-order general complex dynamical networks*

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Outer synchronization between two different fractional-order general complex dynamical networks is investigated in this paper. Based on the stability theory of the fractional-order system, the sufficient criteria for outer synchronization are derived analytically by applying the nonlinear control and the bidirectional coupling methods. The proposed synchronization method is applicable to almost all kinds of coupled fractional-order general complex dynamical networks. Neither a symmetric nor irreducible coupling configuration matrix is required. In addition, no constraint is imposed on the inner-coupling matrix. Numerical examples are also provided to demonstrate the validity of the presented synchronization scheme. Numeric evidence shows that both the feedback strength k and the fractional order α can be chosen appropriately to adjust the synchronization effect effectively.

Keywords: the fractional-order system, complex networks, synchronization, bidirectional coupling

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1. Introduction

Complex dynamical networks widely exist in various fields of the real world and have attracted tremendous attention in recent years.[1] A complex dynamical network consists of a number of nodes, which are dynamic systems with specific contents, and links between the nodes. Examples of complex networks include the Internet, in which nodes are computers and routers and edges are physical or wireless connections between them; the World Wide Web, in which nodes are web pages and edges are hyperlinks; and various communication networks, biological neural networks, food web, electrical power grids, the transportation networks, etc.[2−5] Especially, the study of nonlinear dynamical behaviours in complex dynamical networks, as well as how network topological structure influences its dynamical behaviours, has currently become a strategic topic of great significance.[2,3,6] The interest has been increasing in the study of synchronization dynamics in large-scale networks composed of coupled chaotic dynamical systems, owing to its tremendous potential applications in secure communication, parallel image processing, neural networks, biological systems, information science, etc.[7−10]

In recent years, the control and synchronization of complex networks has been a focus for many scientists from various fields, such as mathematics, sociology, biology, etc. Most of the studies focused on synchronization inside a coupled network with/without time delays, or called 'inner synchronization' for convenience.[11−16] On the other hand, in reality, synchronization between two coupled networks or 'outer synchronization'[17,18] always indeed exists in our lives. For instance, the relationship between the network of predators and that of preys is important for maintaining balance among different animal species;[19] the infectious diseases, such as SARS, bird flu and H1N1 influenza, originally spread between two communities (or networks). There are a great many examples of relationships between different networks (communities). Therefore, it is necessary and significant to investigate the dynamics between two networks. In the literature, the synchronization between two identical complex dynamical networks and the synchronization between two different complex dynamical networks were widely studied.[17,18,20−27] For example, Li *et al.* studied the outer synchronization between two unidirectionally coupled networks and derived a criterion for the synchronization between two networks with identical topological structures.[18] Hu *et al.* investigated the projective synchronization of the drive-response dynamical networks and adopted the pinning control technique to manipulate the scaling factor of the dynamical networks.[20] Tang *et al.* constructed the adaptive controllers for outer synchro-

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nization between two complex networks with nonidentical topologies.[21] Shortly after, Wu *et al.* employed the nonlinear control technique to achieve the generalized outer synchronization between two completely different complex dynamical networks based on Barbalat's lemma.[27] However, almost all the studies focused on the control and the synchronization of the integer-order complex networks. More recently, Tang *et al.* pioneered in investigating the pining control problem of fractional-order weighted complex dynamical networks.[28] As we know, this is the first paper about the controlling of the fractional-order complex networks.

It is well known that the fractional calculus is a classical mathematical notion, with a history as long as calculus itself. But its applications to physics and engineering are just a recent subject of interest.[29,30] It was found that in interdisciplinary fields, many systems such as the viscoelastic system,[31] dielectric polarization,[32] quantitative finance[33] and quantum evolution of complex system[34] could be elegantly described with the help of fractional derivatives. Recently, the chaotic dynamics of fractional-order systems began to attract much attention. It has been shown that the fractional-order generalizations of many well-known systems can also behave chaotically, such as the fractional-order Rössler system,[35] the fractional-order Chen system,[36] the fractional-order cellular neural networks,[37] and so forth.

Some natural questions arise as follows. (i) Can the outer synchronization between two different fractional-order general complex dynamical networks be achieved? (ii) If question (i) holds, what kinds of control schemes may be designed for two different fractional-order complex networks to realize the outer synchronization? (iii) What factors effectively affect the synchronization effect? To our best knowledge, the outer synchronization between two different fractional-order general complex dynamical networks has not been considered yet to date.

Motivated by the above discussion, in this paper, we study the outer synchronization between two different fractional-order general complex dynamical networks. Based on the stability theory of the fractional-order system, some synchronization criteria are obtained by using the nonlinear control technique and the bidirectional coupling method. In our study, each network can be undirected or directed, connected or disconnected, and nodes in either network may have identical or different dynamics. The coupling configuration matrix need not be symmetric or irreducible. The inner-coupling matrix is not required to be symmetric or diagonal or positive definite. It is noticed that the synchronization effect sensitively depends on the feedback strength (k) and the fractional order (α).

The rest of this paper is organized as follows. In Section 2, the fractional derivative and its approximation are briefly described, and then the fractional-order complex network models and some preliminaries are introduced. In Section 3, using the nonlinear control and the bidirectional coupling schemes, we present some sufficient criteria to synchronize two fractional-order complex networks whose node dynamics and topological structures can be identical or different. Some numerical simulations for verifying the theoretical results are performed in Section 4. Concluding remarks are presented finally in Section 5.

2. Problem description

Notations $\|\boldsymbol{x}\|$ denotes the 2-norm of vector $\boldsymbol{x}$; $\boldsymbol{A}^{\mathrm{T}}$ (or $\boldsymbol{x}^{\mathrm{T}}$) means the transpose of matrix $\boldsymbol{A}$ (or vector $\boldsymbol{x}$); $\boldsymbol{I}_n \in \mathrm{R}^{n\times n}$ represents the identity matrix with dimension n; $\otimes$ denotes the Kronecker product of two matrices; $\lambda_i(\boldsymbol{A})$ represents the i-th eigenvalue of a square matrix; $\arg(\lambda_i(\boldsymbol{A}))$ refers to the argument of the eignvalue λ_i of matrix $\boldsymbol{A}$.

2.1. Fractional derivative and its approximation

Fractional calculus is a generalization of integration and differentiation to a noninteger-order integro-differential operator ${}_aD_t^\alpha$ is defined as

$$ {}_aD_t^\alpha = \begin{cases} \dfrac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}, & R(\alpha) > 0, \\ 1, & R(\alpha) = 0, \\ \int_a^t (\mathrm{d}\tau)^{-\alpha}, & R(\alpha) < 0. \end{cases} \tag{1} $$

There are many definitions for fractional derivatives.[29] Riemann–Liouville definition is commonly used, which is defined as

$$ D^\alpha \boldsymbol{x}(t) = \frac{\mathrm{d}^m}{\mathrm{d}t^m} J^{m-\alpha}\boldsymbol{x}(t), \quad \alpha > 0, \tag{2} $$

where $m = \lceil \alpha \rceil$, i.e. m is the first integer which is not less than α, J^β is the β-order Riemann–Liouville integral operator which is described as

$$ J^\beta \varphi(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1}\phi(\tau)\mathrm{d}\tau \tag{3} $$

for $0 < \beta \leq 1$, where $\Gamma(\cdot)$ is the gamma function.

Throughout our work, the following definition is used:

$$D_*^\alpha \boldsymbol{x}(t) = J^{n-\alpha}\boldsymbol{x}^{(n)}(t), \quad \alpha > 0, \tag{4}$$

where $n = \lceil \alpha \rceil$. It is common practice to call the operator D_*^α the Caputo differential operator of order α because it was used first for solving practical problems by Caputo.[38]

The numerical calculation of a fractional differential equation is not so simple as that of an ordinary differential equation. Here we use a predictor–corrector algorithm for fractional order differential equations,[39−43] which is the generalization of Adams–Bashforth–Moulton one. When $\alpha > 0$, the algorithm is universal. The following is a brief introduction of the algorithm.

Consider the following differential equation:

$$D_*^\alpha \boldsymbol{x}(t) = f(t, \boldsymbol{x}(t)), \quad 0 \le t \le T,$$
$$\boldsymbol{x}^{(i)}(0) = x_0^{(i)}, i = 0, 1, 2, \ldots, n-1,$$

which is equivalent to the following Volterra integral equation:[41]

$$\boldsymbol{x}(t) = \sum_{i=0}^{\lceil \alpha \rceil - 1} \frac{t^i}{i!} x_0^{(i)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, x(\tau)) \mathrm{d}\tau. \tag{5}$$

Set $h = T/N (N \in \mathrm{Z})$, and $t_n = nh$ $(n = 0, 1, 2, \ldots, N)$, then Eq. (5) can be discretized as follows:

$$x_h(t_{n+1}) = \sum_{i=0}^{\lceil \alpha \rceil - 1} \frac{t_{n+1}^i}{i!} x_0^{(i)} + \frac{h^\alpha}{\Gamma(\alpha+2)} f(t_{n+1}, x_h^p(t_{n+1})) + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} f(t_j, x_h(t_j)), \tag{6}$$

where

$$x_h^p(t_{n+1}) = \sum_{i=0}^{n-1} \frac{t_{n+1}^i}{i!} x_0^{(i)} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} f(t_j, x_h(t_j)),$$

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^{\alpha+1}, & j = 0 \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1}, & 1 \le j \le n \\ 1, & j = n+1 \end{cases},$$

$$b_{j,n+1} = \frac{h^\alpha}{\alpha}((n-j+1)^\alpha - (n-j)^\alpha).$$

The estimation error of this approximation is $\boldsymbol{e} = \max_{j=0,1,\ldots,N} |x(t_j) - x_h(t_j)| = \mathrm{O}(h^p)$, where $p = \min(2, 1+\alpha)$.

2.2. Fractional-order network models

Consider a fractional-order general complex dynamical network consisting of N dynamical nodes with linearly couplings, which is characterized by

$$D_*^\alpha \boldsymbol{x}_i(t) = \boldsymbol{A}\boldsymbol{x}_i(t) + f(\boldsymbol{x}_i(t)) + \sum_{j=1}^{N} c_{ij}\boldsymbol{\Gamma}\boldsymbol{x}_j(t), i = 1, 2, \ldots, N, \tag{7}$$

where $0 < \alpha \le 1$ is the fractional order; $\boldsymbol{x}_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^{\mathrm{T}} \in \mathrm{R}^n$ is the state vector of node i; $\boldsymbol{A} \in \mathrm{R}^{n\times n}$ is a constant matrix; $f : \mathrm{R} \times \mathrm{R}^n \to \mathrm{R}^n$ is a smooth nonlinear function; $\boldsymbol{\Gamma} \in \mathrm{R}^{n\times n}$ is an inner-coupling matrix determining the interaction of variables; $\boldsymbol{C} = (c_{ij})_{N\times N} \in \mathrm{R}^{N\times N}$ is the coupling configuration matrix representing the coupling strength and the topological structure of the network, in which c_{ij} is defined as follows: if there is a connection from node i to node j $(j \ne i)$, $c_{ij} > 0$; otherwise, $c_{ij} = 0$ $(j \ne i)$ The diagonal elements of matrix $\boldsymbol{C}$ are defined as

$$c_{ii} = -\sum_{j=1, j\ne i}^{N} c_{ij}, i = 1, 2, \ldots, N.$$

The node dynamical function of network (7) is $D_*^\alpha \boldsymbol{x}(t) = F(\boldsymbol{x}(t)) = \boldsymbol{A}\boldsymbol{x}(t) + f(\boldsymbol{x}(t))$.

Consider another fractional-order complex dynamical network containing N dynamical nodes as follows:

$$D_*^\alpha \boldsymbol{y}_i(t) = \boldsymbol{B}\boldsymbol{y}_i(t) + g(\boldsymbol{y}_i(t))$$

$$+\sum_{j=1}^{N} c_{ij}\boldsymbol{\Gamma}\boldsymbol{y}_j(t)+u_i(t), i=1,2,\ldots,N. \quad (8)$$

Here $\boldsymbol{y}_i(t)=(y_{i1}(t),y_{i2}(t),\ldots,y_{in}(t))^{\mathrm{T}}\in \mathrm{R}^n$ is the state vector of the i-th node. $\boldsymbol{B}\in \mathrm{R}^{n\times n}$ is a constant matrix. $g:\mathrm{R}\times\mathrm{R}^n\to\mathrm{R}^n$ is a smooth nonlinear function. The node dynamical function of network (8) is $D_*^{\alpha}\boldsymbol{y}(t)=G(\boldsymbol{y}(t))=\boldsymbol{B}\boldsymbol{y}(t)+g(\boldsymbol{y}(t))$. $\widehat{\boldsymbol{D}}=(d_{ij})_{N\times N}\in \mathrm{R}^{N\times N}$ is the coupling configuration matrix which has the same meaning as that of matrix $\boldsymbol{C}$. $\boldsymbol{\Gamma}$ has the same meaning as that in Eq. (7). $u_i(t)$ is the controller for node i to be designed according to the specific node dynamics and topological structures of the drive and response networks. Here, the coupling configuration matrices $\boldsymbol{C}$ and $\widehat{\boldsymbol{D}}$ are not necessarily symmetric or irreducible. We do not impose any constraint on the inner-coupling matrix $\boldsymbol{\Gamma}$ either. In what follows we will refer to network (7) as the drive network and network (8) as the response network.

2.3. Preliminaries

In this subsection, a rigorous definition for the outer synchronization between two networks and a useful lemma, which are relevant to the subsequent study, are given.

Definition 1 Suppose that $\boldsymbol{x}_i(t,\boldsymbol{X}_0)$ and $\boldsymbol{y}_i(t,\boldsymbol{Y}_0,u_i)$ $(i=1,2,\ldots,N)$ are the solutions of networks (7) and (8), respectively, where $\boldsymbol{X}_0=(x_1(0),x_2(0),\ldots,x_N(0))^{\mathrm{T}}$, $\boldsymbol{Y}_0=(y_1(0),y_2(0),\ldots,y_N(0))^{\mathrm{T}}$, then $F:\Omega\to\mathrm{R}^n$ and $G:\Theta\to\mathrm{R}^n$ will be the continuously differentiable mappings for all values of t $(\Omega,\Theta\in\mathrm{R}^n)$. If there exists a nonempty open subset $\Delta\subseteq\Omega\cap\Theta$, with $\boldsymbol{x}_i(0)$, $\boldsymbol{y}_i(0)\in\Delta$, such that $\boldsymbol{x}_i(t,\boldsymbol{X}_0)\in\Omega$, $\boldsymbol{y}_i(t,\boldsymbol{Y}_0)\in\Theta$ for all $t\geq 0$ and

$$\lim_{t\to\infty}\|\boldsymbol{y}_i(t,\boldsymbol{Y}_0,u_i)-\boldsymbol{x}_i(t,\boldsymbol{X}_0)\|=0,\quad i=1,2,\ldots,N$$

then the dynamical networks (7) and (8) are said to achieve outer synchronization.

Lemma 1[41] Consider the following autonomous system:

$$D_*^{\alpha}\boldsymbol{Z}(t)=\boldsymbol{W}\boldsymbol{Z}(t),\boldsymbol{Z}(0)=\boldsymbol{Z}_0, \quad (9)$$

where the fractional order $0<\alpha\leq 1$, $\boldsymbol{Z}(t)\in\mathrm{R}^n$ is the state variable and $\boldsymbol{W}\in\mathrm{R}^{n\times n}$ is a constant matrix. According to the fractional-order system stability theory,[44] system (9) is asymptotically stable if and only if $|\arg(\lambda_i(\boldsymbol{W}))|>\alpha\pi/2$ for $i=1,2,\ldots,n$, where $\arg(\lambda_i(\boldsymbol{W}))$ denotes the argument of the eigenvalue λ_i of $\boldsymbol{W}$. Also, system (9) is stable if and only if either it is asymptotically stable or those critical eigenvalues which satisfy $|\arg(\lambda_i(\boldsymbol{W}))|=\alpha\pi/2$ have a geometric multiplicity one each.

3. Outer synchronization criteria

3.1. Outer synchronization between two fractional-order networks with identical topological structures

In this subsection, we will study the outer synchronization between two fractional-order complex dynamical networks with identical topologies, i.e. $\boldsymbol{C}=\widehat{\boldsymbol{D}}$, by using the nonlinear control method.

The fractional-order drive and response networks are given by Eqs. (7) and (8), respectively. Subtracting network (7) from network (8) yields the following error dynamical network:

$$\begin{aligned} & D_*^{\alpha}\boldsymbol{e}_i(t)\\ =\ & \boldsymbol{A}\boldsymbol{e}_i(t)+(\boldsymbol{B}-\boldsymbol{A})\boldsymbol{y}_i(t)+g(\boldsymbol{y}_i(t))-f(\boldsymbol{x}_i(t))\\ & +\sum_{j=1}^{N}c_{ij}\boldsymbol{\Gamma}\boldsymbol{e}_j(t)+u_i(t), \end{aligned} \quad (10)$$

where $\boldsymbol{e}_i(t)=\boldsymbol{y}_i(t)-\boldsymbol{x}_i(t)$ for $i=1,2,\ldots,N$.

To achieve the outer synchronization between two networks, we choose the nonlinear controller $u_i(t)$ as follows:

$$\begin{aligned} u_i(t) =\ & -(\boldsymbol{B}-\boldsymbol{A})\boldsymbol{y}_i(t)-g(\boldsymbol{y}_i(t))\\ & +f(\boldsymbol{x}_i(t))-k\boldsymbol{e}_i(t),\quad i=1,2,\ldots,N, \end{aligned} \quad (11)$$

where the feedback strength k is an arbitrary positive constant. Substituting Eq. (11) into Eq. (10), we can obtain

$$D_*^{\alpha}\boldsymbol{e}_i(t)=\boldsymbol{A}\boldsymbol{e}_i(t)+\sum_{j=1}^{N}c_{ij}\boldsymbol{\Gamma}\boldsymbol{e}_j(t)-k\boldsymbol{e}_i(t). \quad (12)$$

Let $\boldsymbol{e}=(e_1^{\mathrm{T}},e_2^{\mathrm{T}},\ldots,e_N^{\mathrm{T}})^{\mathrm{T}}\in\mathrm{R}^{nN}$ and $\boldsymbol{e}_i=(e_{i1},e_{i2},\ldots,e_{in})^{\mathrm{T}}\in\mathrm{R}^n$, then we will have

$$\begin{aligned} & D_*^{\alpha}\boldsymbol{e}(t)\\ =\ & (\boldsymbol{A}\otimes\boldsymbol{I}_N)\boldsymbol{e}(t)+(\boldsymbol{C}\otimes\boldsymbol{\Gamma})\boldsymbol{e}(t)-(\boldsymbol{K}\otimes\boldsymbol{I}_N)\boldsymbol{e}(t)\\ =\ & \{[(\boldsymbol{A}-\boldsymbol{K})\otimes\boldsymbol{I}_N]+(\boldsymbol{C}\otimes\boldsymbol{\Gamma})\}\boldsymbol{e}(t)\\ =\ & \boldsymbol{H}\boldsymbol{e}(t), \end{aligned} \quad (13)$$

where $\boldsymbol{K}=k\boldsymbol{I}_n\in\mathrm{R}^{n\times n}$ and $\boldsymbol{H}=[(\boldsymbol{A}-\boldsymbol{K})\otimes\boldsymbol{I}_N]+(\boldsymbol{C}\otimes\boldsymbol{\Gamma})$.

Then we can derive the following main theorem.

Theorem 1 For a certain fractional order α, assume that there exists a sufficiently large feedback strength $k = k^*$ such that all the eigenvalues of matrix $\boldsymbol{H}$ satisfy $|\arg(\lambda_i(\boldsymbol{H}))| > \alpha\pi/2 \quad (i = 1, 2, \ldots, Nn)$, error dynamical network (13) can be globally stable, which implies that the outer synchronization between two fractional-order general complex dynamical networks (7) and (8) with identical topological structures and different node dynamics is achieved by using nonlinear controllers (11).

Proof Theorem 1 can be easily proved based on Lemma 1.

Based on Theorem 1, we can easily obtain the following corollary.

Corollary 1 For a certain fractional order α, if drive network (7) and response network (8) have the same topological structures and identical node dynamics, and feedback strength k is sufficiently large, then the two networks can realize the outer synchronization under the following controllers:

$$u_i(t) = -k\boldsymbol{e}_i(t), \quad i = 1, 2, \ldots, N. \tag{14}$$

Proof If the drive and the response networks have identical node dynamics, i.e. $f = g$ and $\boldsymbol{A} = \boldsymbol{B}$, error dynamical network (10) can be rewritten as

$$\begin{aligned} D_*^\alpha \boldsymbol{e}_i(t) =\ & \boldsymbol{A}\boldsymbol{e}_i(t) + f(\boldsymbol{y}_i(t)) - f(\boldsymbol{x}_i(t)) \\ & + \sum_{j=1}^{N} c_{ij}\boldsymbol{\Gamma}\boldsymbol{e}_j(t) + u_i(t). \end{aligned} \tag{15}$$

Suppose that

$$\begin{aligned} & f(\boldsymbol{y}_i(t)) - f(\boldsymbol{x}_i(t)) \\ =\ & m_i(\boldsymbol{y}_i(t) - \boldsymbol{x}_i(t)) \\ =\ & m_i\boldsymbol{e}_i(t), \end{aligned} \tag{16}$$

where m_i depends on the state vectors $\boldsymbol{x}_i(t)$ and $\boldsymbol{y}_i(t)$. Since the states of the chaotic systems are bounded, m_i must be bounded.

Let $\boldsymbol{e} = (e_1^{\mathrm{T}}, e_2^{\mathrm{T}}, \cdots, e_N^{\mathrm{T}})^{\mathrm{T}} \in \mathrm{R}^{nN}$, then we will obtain

$$\begin{aligned} & D_*^\alpha \boldsymbol{e}(t) \\ =\ & (\boldsymbol{A} \otimes \boldsymbol{I}_N)\boldsymbol{e}(t) + \boldsymbol{M}\boldsymbol{e}(t) + (\boldsymbol{C} \otimes \boldsymbol{\Gamma})\boldsymbol{e}(t) \\ & - (\boldsymbol{K} \otimes \boldsymbol{I}_N)\boldsymbol{e}(t) \\ =\ & \{[(\boldsymbol{A} - \boldsymbol{K}) \otimes \boldsymbol{I}_N] + \boldsymbol{M} + (\boldsymbol{C} \otimes \boldsymbol{\Gamma})\}\boldsymbol{e}(t) \\ =\ & \boldsymbol{J}\boldsymbol{e}(t), \end{aligned} \tag{17}$$

where $\boldsymbol{M} = \mathrm{diag}(\boldsymbol{M}_1, \boldsymbol{M}_2, \ldots, \boldsymbol{M}_N) \in \mathrm{R}^{Nn \times Nn}$ (i.e. $\boldsymbol{M}_i = m_i\boldsymbol{I}_N$ is the i-th square block of $\boldsymbol{M}$), $\boldsymbol{K} = k\boldsymbol{I}_n \in \mathrm{R}^{n \times n}$, and $\boldsymbol{J} = [(\boldsymbol{A} - \boldsymbol{K}) \otimes \boldsymbol{I}_N] + \boldsymbol{M} + (\boldsymbol{C} \otimes \boldsymbol{\Gamma})$.

If there is a sufficiently large feedback strength $k = k^*$ which makes all the eigenvalues of matrix $\boldsymbol{J}$ satisfy $|\arg(\lambda_i(\boldsymbol{J}))| > \alpha\pi/2 \quad (i = 1, 2, \ldots, Nn)$, according to Lemma 1, networks (7) and (8) asymptotically achieve the outer synchronization. This completes the proof.

Remark 1 Obviously, Corollary 1 is a special case of Theorem 1. Nonlinear controllers (11) in Theorem 1 are simplified into the rather simple linear controllers (14) for complete outer synchronization in Corollary 1. It is noted that a larger $k = k^*$ to achieve outer synchronization is a sufficient condition but not a necessary one. The numerical simulation in Subsection 4.1.1 shows that a small value of k can also lead to the outer synchronization.

3.2. Outer synchronization between two fractional-order networks with different topological structures

In this subsection, we focus on investigating the outer synchronization between two fractional-order complex dynamical networks with different topologies, i.e. $\boldsymbol{C} \neq \widehat{\boldsymbol{D}}$, by applying a bidirectional coupling scheme.

We consider the following two fractional-order complex networks with bidirectional coupling:

$$\begin{aligned} & D_*^\alpha \boldsymbol{x}_i(t) \\ =\ & \boldsymbol{A}\boldsymbol{x}_i(t) + f(\boldsymbol{x}_i(t)) + \sum_{j=1}^{N} c_{ij}\boldsymbol{\Gamma}\boldsymbol{x}_j(t) + U_1, \end{aligned} \tag{18}$$

$$\begin{aligned} & D_*^\alpha \boldsymbol{y}_i(t) \\ =\ & \boldsymbol{B}\boldsymbol{y}_i(t) + g(\boldsymbol{y}_i(t)) + \sum_{j=1}^{N} d_{ij}\boldsymbol{\Gamma}\boldsymbol{y}_j(t) + U_2, \end{aligned} \tag{19}$$

where

$$\begin{aligned} U_1 &= -\sum\nolimits_{j=1}^{N} c_{ij}\boldsymbol{\Gamma}\boldsymbol{y}_j(t), \\ U_2 &= -\sum\nolimits_{j=1}^{N} d_{ij}\boldsymbol{\Gamma}\boldsymbol{x}_j(t) + u_i(t), \end{aligned}$$

and

$$u_i(t) \ (i = 1, 2, \ldots, N)$$

are the controllers to be constructed.

Denote $\boldsymbol{e}_i(t) = \boldsymbol{y}_i(t) - \boldsymbol{x}_i(t)$. From networks (18) and (19), we obtain the error dynamical network described by

$$\begin{aligned} & D_*^\alpha \boldsymbol{e}_i(t) \\ =\ & \boldsymbol{A}\boldsymbol{e}_i(t) + (\boldsymbol{B} - \boldsymbol{A})\boldsymbol{y}_i(t) + g(\boldsymbol{y}_i(t)) - f(\boldsymbol{x}_i(t)) \end{aligned}$$

$$+\sum_{j=1}^{N}(c_{ij}+d_{ij})\mathbf{\Gamma}\boldsymbol{e}_j(t)+u_i(t). \quad (20)$$

Take $u_i(t)$ as Eq. (11). Substituting Eq. (11) into Eq. (20), then we can derive the following error dynamical network:

$$\begin{aligned} & D_*^{\alpha}\boldsymbol{e}(t) \\ &= (\boldsymbol{A}\otimes\boldsymbol{I}_N)\boldsymbol{e}(t)+((\boldsymbol{C}+\widehat{\boldsymbol{D}})\otimes\mathbf{\Gamma})\boldsymbol{e}(t) \\ &\quad -(\boldsymbol{K}\otimes\boldsymbol{I}_N)\boldsymbol{e}(t) \\ &= \{[(\boldsymbol{A}-\boldsymbol{K})\otimes\boldsymbol{I}_N]+[(\boldsymbol{C}+\widehat{\boldsymbol{D}})\otimes\mathbf{\Gamma}]\}\boldsymbol{e}(t) \\ &= \boldsymbol{P}\boldsymbol{e}(t), \end{aligned} \quad (21)$$

where $\boldsymbol{e}=(e_1^{\mathrm{T}},e_2^{\mathrm{T}},\ldots,e_N^{\mathrm{T}})^{\mathrm{T}}\in \mathrm{R}^{nN}$, $\boldsymbol{K}=k\boldsymbol{I}_n\in \mathrm{R}^{n\times n}$ and $\boldsymbol{P}=[(\boldsymbol{A}-\boldsymbol{K})\otimes\boldsymbol{I}_N]+[(\boldsymbol{C}+\widehat{\boldsymbol{D}})\otimes\mathbf{\Gamma}]$.

Based on Lemma 1, we can obtain the following theorem.

Theorem 2 For a certain fractional order α, suppose that there is a sufficiently large feedback strength $k=k^*$ such that all the eigenvalues of matrix $\boldsymbol{P}$ satisfy $|\arg(\lambda_i(\boldsymbol{P}))|>\alpha\pi/2 \quad (i=1,2,\ldots,Nn)$, error dynamical network (21) can be globally stable. It indicates that the outer synchronization between two fractional-order general complex dynamical networks with nonidentical topological structures and diverse node dynamics is realized by applying a bidirectional coupling with $u_i(t)$ as Eq. (11).

Proof According to Lemma 1, Theorem 2 can be easily proved.

On the other hand, the following corollary can be easily obtained on the basis of Theorem 2.

Corollary 2 For a certain fractional order α, if networks (18) and (19) have different topological structures and uniform node dynamics, and feedback strength k is sufficiently large, then the outer synchronization between two networks can be achieved by using a bidirectional coupling with $u_i(t)$ as Eq. (14).

Proof If the drive and the response networks have the same node dynamics, from Eqs. (14) and (16), error dynamical network (20) can be rewritten as

$$\begin{aligned} & D_*^{\alpha}\boldsymbol{e}_i(t) \\ &= \boldsymbol{A}\boldsymbol{e}_i(t)+m_i\boldsymbol{e}_i(t)+\sum_{j=1}^{N}(c_{ij}+d_{ij})\mathbf{\Gamma}\boldsymbol{e}_j(t)-k\boldsymbol{e}_i(t), \\ & \quad i=1,2,\ldots,N. \end{aligned} \quad (22)$$

Let $\boldsymbol{e}=(e_1^{\mathrm{T}},e_2^{\mathrm{T}},\ldots,e_N^{\mathrm{T}})^{\mathrm{T}}\in \mathrm{R}^{nN}$, then one can obtain

$$\begin{aligned} & D_*^{\alpha}\boldsymbol{e}(t) \\ &= (\boldsymbol{A}\otimes\boldsymbol{I}_N)\boldsymbol{e}(t)+\boldsymbol{M}\boldsymbol{e}(t)+((\boldsymbol{C}+\widehat{\boldsymbol{D}})\otimes\mathbf{\Gamma})\boldsymbol{e}(t) \\ &\quad -(\boldsymbol{K}\otimes\boldsymbol{I}_N)\boldsymbol{e}(t) \\ &= \{[(\boldsymbol{A}-\boldsymbol{K})\otimes\boldsymbol{I}_N]+\boldsymbol{M}+[(\boldsymbol{C}+\widehat{\boldsymbol{D}})\otimes\mathbf{\Gamma}]\}\boldsymbol{e}(t) \\ &= \boldsymbol{Q}\boldsymbol{e}(t), \end{aligned} \quad (23)$$

where the matrices $\boldsymbol{M}$ and $\boldsymbol{K}$ have the same meanings as those in Eq. (17), and $\boldsymbol{Q}=[(\boldsymbol{A}-\boldsymbol{K})\otimes\boldsymbol{I}_N]+\boldsymbol{M}+[(\boldsymbol{C}+\widehat{\boldsymbol{D}})\otimes\mathbf{\Gamma}]$.

If there exists a sufficiently large feedback strength $k=k^*$ such that all the eigenvalues of matrix $\boldsymbol{Q}$ satisfy $|\arg(\lambda_i(\boldsymbol{Q}))|>\alpha\pi/2 \quad (i=1,2,\ldots,Nn)$ according to Lemma 1, drive network (18) and response network (19) asymptotically achieve outer synchronization. This completes the proof.

Remark 2 Clearly, Corollary 2 is also a special case of Theorem 2. Note that a larger $k=k^*$ is a sufficient condition but not a necessary one. Later simulation shows that a small value of k can also achieve the outer synchronization.

Remark 3 In real-life situations, the synchronization of complex networks shows its potential applications in many fields. However, to our best knowledge, the outer synchronization between different fractional-order complex dynamical networks has not been reported yet.

Remark 4 In our synchronization method, coupling configuration matrices $\boldsymbol{C}$ and $\widehat{\boldsymbol{D}}$ are not necessarily symmetric and irreducible. So the drive and the response networks can be undirected or directed, and they may also contain isolated nodes and clusters. In addition, there is not any constraint imposed on inner-coupling matrix $\mathbf{\Gamma}$. Furthermore, each node may have different node dynamics. The presented synchronization scheme is applicable to a great many fractional-order complex dynamical networks.

Remark 5 It is noticed that the synchronization speed can be regulated by feedback strength k and fractional order α. The fractional-order drive and response networks can achieve a fast synchronization when we increase the feedback strength or the fractional order.

4. Numerical simulations

In this section, illustrative examples are given to verify the effectiveness of the theoretical criteria obtained in the preceding section. For this purpose, we consider two fractional-order chaotic systems, i.e. the fractional-order Chen system and the fractional-order financial system.

The well-known fractional-order Chen chaotic system[36] is described by

$$D_*^q \boldsymbol{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \boldsymbol{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + f(\boldsymbol{x}), \qquad (24)$$

where

$$\boldsymbol{A} = \begin{pmatrix} -35 & 35 & 0 \\ -7 & 28 & 0 \\ 0 & 0 & -3 \end{pmatrix}, \quad f(\boldsymbol{x}) = \begin{pmatrix} 0 \\ -x_1 x_3 \\ x_1 x_2 \end{pmatrix}$$

and $0 < q \leq 1$, when $q \in [0.7, 1]$, the fractional-order Chen system behaves chaotically. Figure 1 displays the attractor of the fractional-order Chen chaotic system with $q = 0.96$.

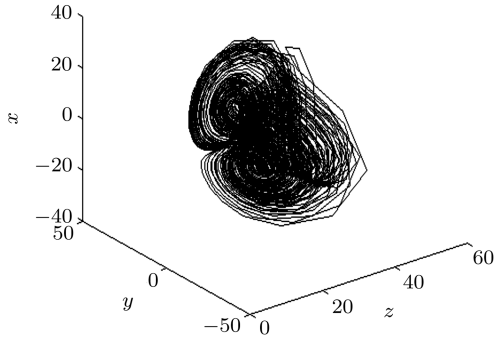


Fig. 1. Chaotic attractor of the fractional-order Chen system with $q = 0.96$.

Recently, a dynamic model of finance was reported.[45] In the model are described the time-variation of three state variables: interest rate y_1, investment demand y_2, and price index y_3. The factors that influence changes in y_1 mainly come from two aspects: one is the contradiction from the investment market, i.e., the surplus between investment and savings, and the other is the structural adjustment from good prices. The changing rate of y_2 is in proportion to the rate of investment, the inversion of the cost of investment, and interest rate. Changes in y_3, on the one hand, are controlled by a contradiction between supply and demand in commercial markets, and on the other hand, are influenced by inflation rates. The fractional-order financial chaotic system is given by

$$D_*^\theta \boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \boldsymbol{B} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + g(\boldsymbol{y}), \qquad (25)$$

where

$$\boldsymbol{B} = \begin{pmatrix} -3 & 0 & 1 \\ 0 & -0.1 & 0 \\ -1 & 0 & -1 \end{pmatrix}, \quad g(\boldsymbol{y}) = \begin{pmatrix} y_1 y_2 \\ 1 - y_1^2 \\ 0 \end{pmatrix}$$

and $0 < \theta \leq 1$.

The fractional-order financial system is chaotic[46] when fractional order θ changes in $[0.85, 1]$. The attractor of the fractional-order financial chaotic system with $\theta = 0.96$ is shown in Fig. 2.

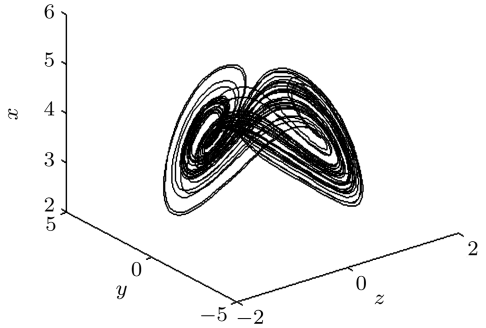


Fig. 2. Chaotic attractor of the fractional-order financial system with $\theta = 0.96$.

In the following, we will take the fractional-order Chen chaotic system and the fractional-order financial chaotic system as node dynamics to illustrate the presented method for the outer synchronization. In all the simulations, let the fractional orders of the drive and the response networks be α, namely, $q = \theta = \alpha$. The network size is taken as $N = 10$. For simplicity, the inner-coupling matrix is chosen as $\boldsymbol{\Gamma} = \boldsymbol{I}_3$ The initial values of the state vectors $\boldsymbol{x}_i(0)$, $\boldsymbol{y}_i(0)$ are chosen arbitrarily in $(-10, 10)$ $(i = 1, 2, \ldots, N)$. The synchronization errors between two networks are defined as $e_{in}(t) = y_{in}(t) - x_{in}(t)$ $(i = 1, 2, \ldots, N; n = 1, 2, 3)$. We introduce the quantity

$$E(t) = \sqrt{\sum_{i=1}^{N} [e_{i1}^2(t) + e_{i2}^2(t) + e_{i3}^2(t)]/N},$$

which is employed to measure the quality of the control process. It is obvious that when $E(t)$ no longer increases, two fractional-order networks achieve the outer synchronization globally.

4.1. Synchronization between fractional-order complex networks with identical topological structures

In this subsection, we will investigate the outer synchronization between two fractional-order complex networks with identical topological structures. Choose the coupling configuration matrices of networks (7) and (8) randomly as follows:

$$\boldsymbol{C} = \widehat{\boldsymbol{D}} = \boldsymbol{T}$$

$$= \begin{pmatrix} -4 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & -2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -5 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & -3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -4 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & -5 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & -3 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & -6 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & -4 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & -6 \end{pmatrix}.$$

4.1.1. Synchronization between two networks with identical node dynamics

We choose the fractional-order Chen chaotic system as node dynamics in the drive and the response networks. According to Corollary 1, we derive the control scheme by Eq. (14). We randomly choose fractional order α to be 0.96, i.e. $\alpha = 0.96$. To obtain a critical value of the feedback strength k to achieve the outer synchronization, we continuously increase feedback strength k, from $k = 0$, in steps of 0.1. When $k < 2.8$, no synchronous phenomenon is observed. When $k = 2.8$, the synchronization errors e_{in} $(i = 1, 2, \ldots, 10; n = 1, 2, 3)$ are shown in Fig. 3, which indicates that the outer synchronization between two complex networks is achieved. This further supports the viewpoint that a sufficiently large $k = k^*$ is only a sufficient condition, and a small value of k can also achieve the outer synchronization. We still choose the fractional order $\alpha = 0.96$ and only adjust the feedback strength k. Figure 4 shows the synchronization errors $E(t)$ for different values of feedback strength k. From Fig. 4, we can see that the synchronization effect turns better with the increase of feedback strength k.

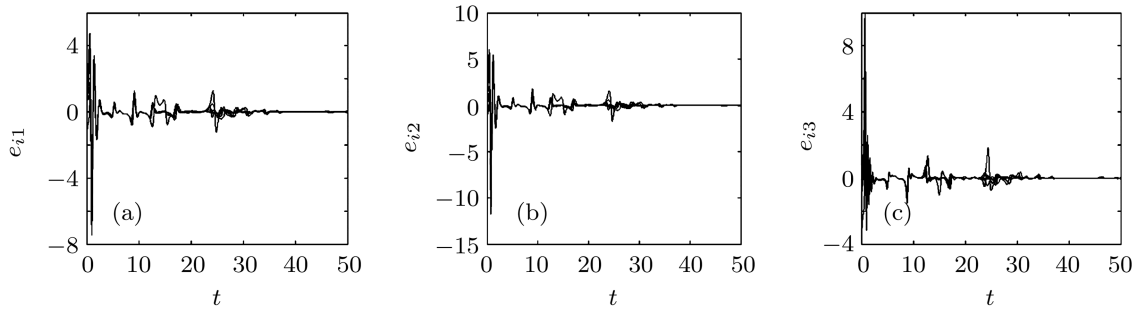


Fig. 3. Time evolutions of synchronization error e_{in} with time at $k = 2.8$.

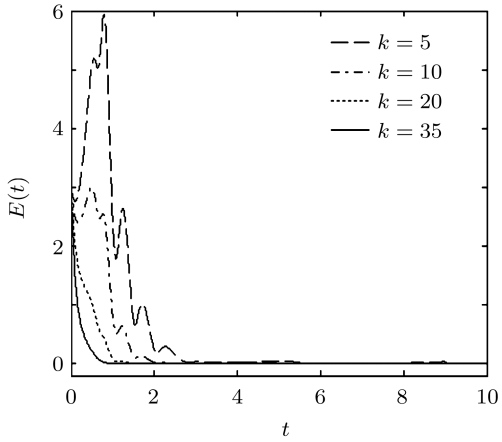


Fig. 4. Time evolutions of synchronization error $E(t)$ with different values of feedback strength k when $\alpha = 0.96$.

Figure 5 shows the curves for synchronization error $E(t)$ versus time t for different fractional orders when $k = 35$. One can easily find that the synchronization effect becomes better with the increase of fractional order α.

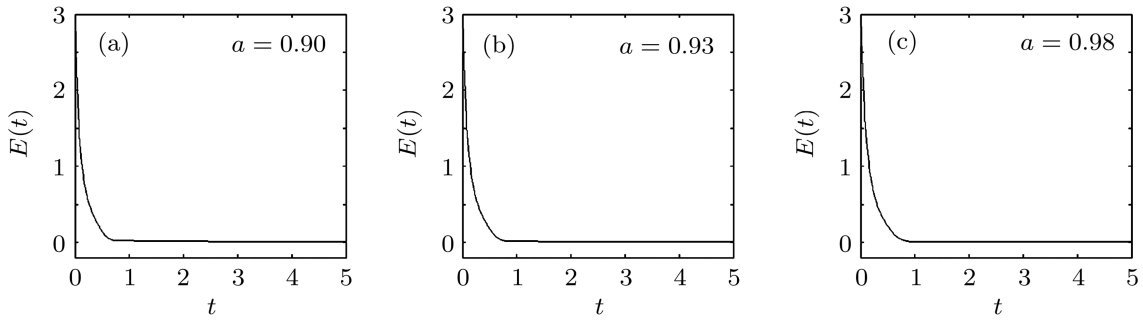


Fig. 5. Time evolutions of synchronization error $E(t)$ with different values of fractional order α when $k = 35$.

4.1.2. Synchronization between two networks with different node dynamics

We consider the fractional-order Chen chaotic system as node dynamics in drive network (7), and the fractional-order financial chaotic system as node dynamics in response network (8). One can realize the outer synchronization between networks (7) and (8) by using nonlinear controllers (11) in Theorem 1. The fractional order α is still taken to be 0.96, i.e. $\alpha = 0.96$. We perform the similar simulations to those in the above example. We find that when $k \geq 24.5$, the outer synchronization between two fractional-order complex networks can only be realized. The values of outer synchronization error $E(t)$ for different values of feedback strength k when $\alpha = 0.96$ are depicted in Fig. 6. Obviously, from Fig. 6, the synchronization effect turns better with the increase of feedback strength k. Let $k = 30$ and only fractional order α be changed. We also use $E(t)$ to evaluate the synchronization effect. The values of synchronization error $E(t)$ with the increase of the fractional order are described in Fig. 7. From Fig. 7, one can easily find that the synchronization effect becomes better with the increase of fractional order α.

Remark 6 From the above two simulations, clearly, increasing the fractional order and the feedback strength will expedite the outer synchronization. The larger the fractional order and the feedback strength are, the faster the outer synchronization will be achieved. On the other hand, between two networks with diverse node dynamics, to reach the same synchronization effect, much larger fractional order and feedback strength are required than between two networks with uniform node dynamics. In addition, with the same fractional order and feedback strength, the synchronization effect in Subsection 4.1.1 turns better than that in Subsection 4.1.2.

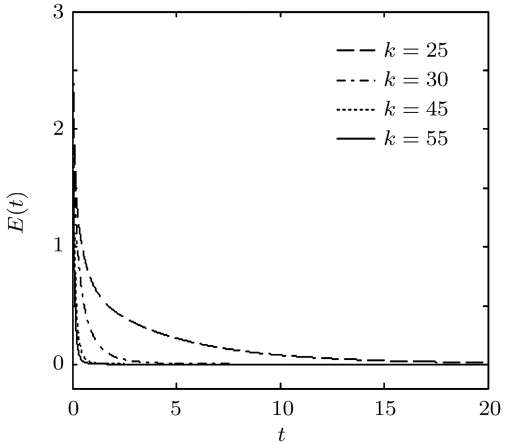


Fig. 6. Time evolutions of synchronization error $E(t)$ with different values of feedback strength k when $\alpha = 0.96$.

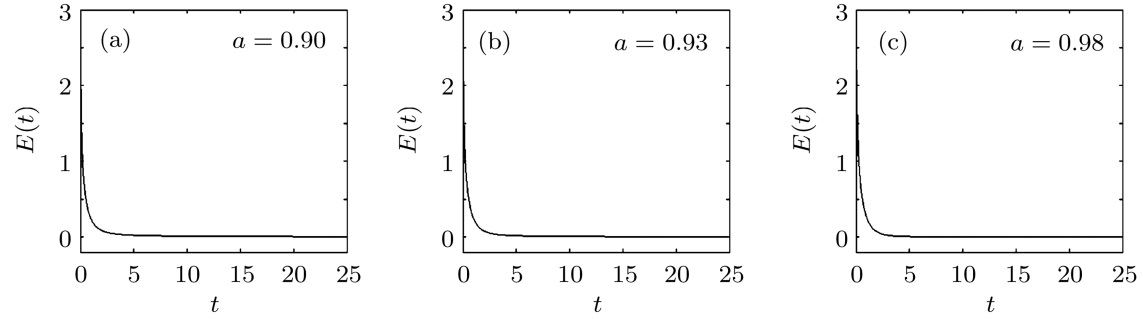


Fig. 7. Time evolutions of synchronization error $E(t)$ with different values of fractional order α when $k = 30$.

4.2. Synchronization between two fractional-order complex networks with different topological structures

In this subsection, we consider the outer synchronization between two fractional-order complex networks with different topological structures. Suppose that the coupling configuration matrix of network (18) is $\boldsymbol{C} = \boldsymbol{T}$ as in the above subsection, and the coupling configuration matrix of network (19) is arbitrarily selected as

$$\widehat{\boldsymbol{D}} = \begin{pmatrix} -2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -4 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -3 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & -5 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & -3 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & -3 \end{pmatrix}.$$

4.2.1. Synchronization between two networks with identical node dynamics

Here, consider the fractional-order Chen chaotic system as node dynamics in the drive and the response networks. Based on Corollary 2, the outer synchronization between drive network (18) and response network (19) can be achieved by using bidirectional coupling. First, let the fractional order α be 0.96, i.e. $\alpha = 0.96$ and only the feedback strength k be changed. It is found that, only when $k \geq 2.6$, can synchronous phenomenon be observed. Figure 8 shows the time evolutions of synchronization error e_{in} $(i = 1, 2, \ldots, 10;\ n = 1, 2, 3)$ between two networks when $k = 2.6$. It is seen from the figure that the outer synchronization is achieved after quite a short transient period when the feedback strength is small. It indicates that a larger $k = k^*$ is a sufficient condition but not a necessary one. Figure 9 describes the time evolutions of outer synchronization error $E(t)$ with the fractional order $\alpha = 0.96$ while the feedback strength k is increasing, from which we can see that

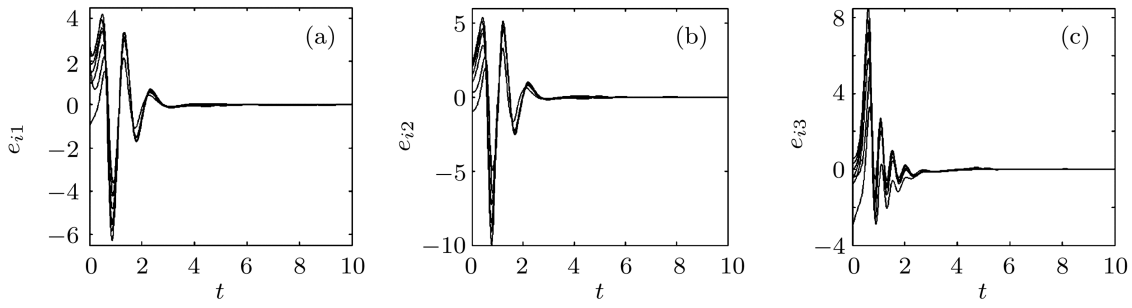


Fig. 8. Time evolutions of synchronization error e_{in} with $k = 2.6$.

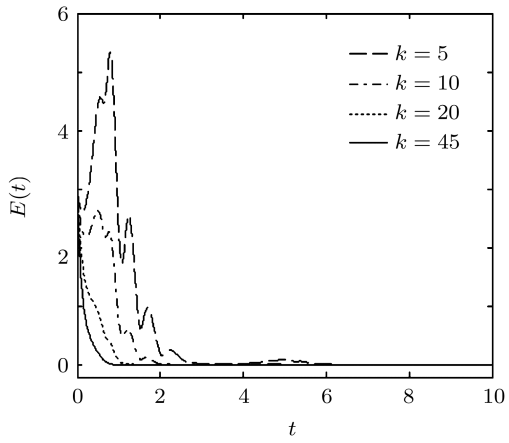


Fig. 9. Time evolutions of synchronization error $E(t)$ with different values of k.

the synchronization effect turns better with the increase of the feedback strength. Then we choose $k = 20$ and tune only fractional order α. Figure 10 displays the time evolutions of the outer synchronization error $E(t)$ when fractional order α increases. The stimulation shows that the synchronization effect becomes better with the increase of fractional order α.

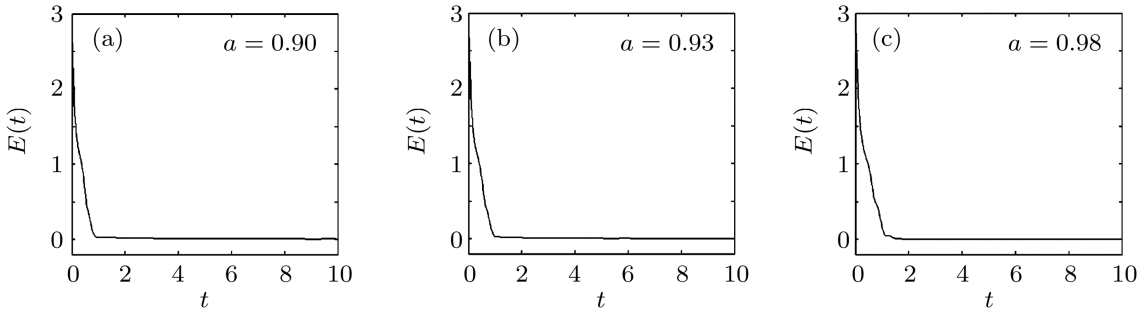


Fig. 10. Time evolutions of synchronization error $E(t)$ with different values of fractional order α when $k = 20$.

4.2.2. Synchronization between two networks with different node dynamics

We take the fractional-order Chen chaotic system as node dynamics in drive network (18), fractional-order financial chaotic system as node dynamics in response network (19). The outer synchronization between the drive and response networks are studied based on Theorem 2. Let $\alpha = 0.96$ and only the feedback strength k be adjusted. Numeric evidence show that the outer synchronization can be realized when $k \geq 28$. Figure 11 shows the time evolution of outer synchronization error $E(t)$ with fractional order $\alpha = 0.96$ with feedback strength k increasing. From Fig. 11, we can easily conclude that the synchronization effect becomes better with the increase of feedback strength k. Then let $k = 50$ and keep it constant. Figure 12 shows the time evolutions of outer synchronization error $E(t)$ with fractional order α increasing. We can easily see that the synchronization effect turns better with the increase of fractional order α.

Remark 7 Obviously, different fractional orders and feedback strengths will lead to different rates towards synchronization. The appropriately large fractional order and feedback strength will lead to a fast

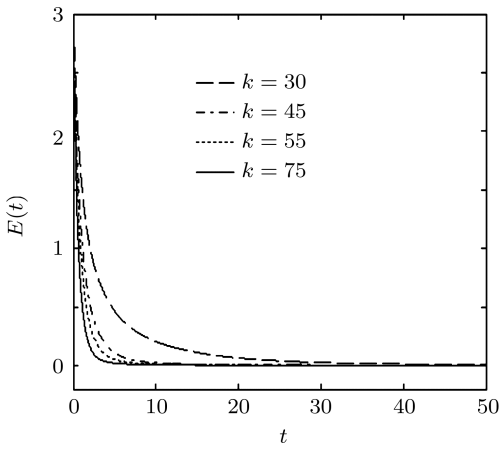


Fig. 11. Time evolutions of synchronization error $E(t)$ with different values of k.

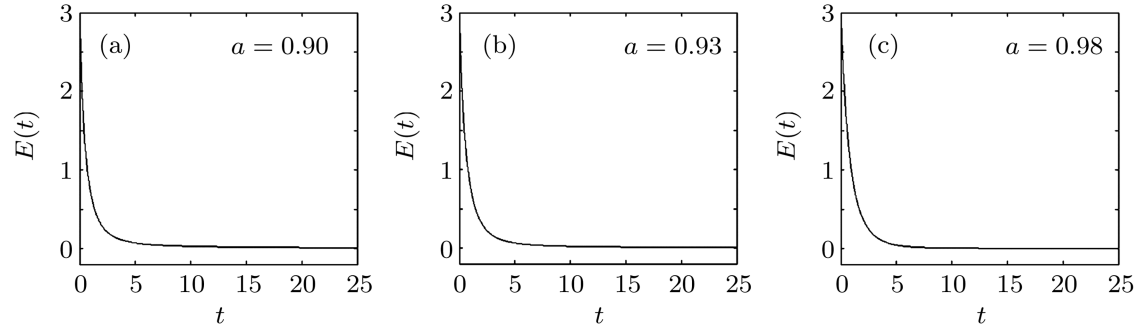


Fig. 12. Time evolutions of synchronization error $E(t)$ with different values of fractional orders α when $k = 50$.

outer synchronization. Note that to achieve the same synchronization effect, a much larger fractional order and feedback strength are required between two networks with different node dynamics than those between two networks with identical node dynamics. Furthermore, with a uniform fractional order and feedback strength, the synchronization effect in Subsection 4.2.1 is better than that in Subsection 4.2.2.

5. Conclusion

Synchronization of the integer-order complex networks has been extensively studied in the past decade. But few studies concern the fractional-order complex networks. Particularly, to the best of our knowledge, the outer synchronization between two different fractional-order complex dynamical networks has not been reported in the literature to date. In this paper, the outer synchronization between two fractional-order complex dynamical networks with different topologies and diverse node dynamics is studied. Based on the stability theory of the fractional-order system and using the nonlinear control and the bidirectional coupling schemes, some sufficient synchronization criteria are obtained and the suitable controllers are designed. The presented method can be applied to a large variety of fractional-order complex dynamical networks. It is worth noting that feedback strength k and fractional order α can effectively affect the synchronization performance. By choosing appropriate feedback strength k and fractional order α, one can achieve the good synchronization effect. Finally, numerical simulations are performed to verify the effectiveness of the proposed synchronization method.

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