A specific state variable for a class of 3D continuous fractional-order chaotic systems

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A specific state variable in a class of 3D continuous fractional-order chaotic systems is presented. All state variables of fractional-order chaotic systems of this class can be obtained via a specific state variable and its (q-order and $2q$-order) time derivatives. This idea is demonstrated by using several well-known fractional-order chaotic systems. Finally, a synchronization scheme is investigated for this fractional-order chaotic system via a specific state variable and its (q-order and $2q$-order) time derivatives. Some examples are used to illustrate the effectiveness of the proposed synchronization method.

Keywords: fractional-order chaotic systems, state variable, q-order and $2q$-order time derivatives, chaotic synchronization

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1. Introduction

In 1990, Pecora and Carroll[1] presented the concept of 'chaotic synchronization' for the first time and introduced a method to synchronize two identical chaotic systems. Chaos synchronization between two chaotic systems or two fractional-order chaotic systems has gained a lot of attention among scientists from a variety of research fields in recent years. It has become an active research subject in nonlinear science because of its potential applications in secure communication,[2] neuron systems,[3] the study of laser dynamics[4] and control processing.[5−22] So, the research on chaos synchronization is of significance both in theoretical study and in practical application. Up to now, different types[5−16] of synchronization phenomena have been observed and studied in a variety of chaotic systems and fractional order chaotic systems, such as complete synchronization, anti-synchronization, phase synchronization, generalized synchronization, projective synchronization as well as generalized projective synchronization.

Most of the above-mentioned methods of synchronizing two chaotic systems or two fractional-order chaotic systems are the state feedback controlling methods. All or several of the state variables are transmitted to the response system. However,[23] it is apparent that it is not always possible to detect all state variables in a physical system, especially in many complicated or large systems. Furthermore, only the observable or measured variables of physical significance are worthy to be investigated.[23] For example, we are interested just in frequency in an oscillator, or temperature (pressure) in a thermodynamically system, etc.

This work, inspired by the above discussion, addresses one observable state variable for a class of 3D continuous fractional-order chaotic systems. It should be pointed out that all the state variables for a class of 3D continuous fractional-order chaotic systems will be yielded by using only a specific state variable and its (q-order and $2q$-order) time derivatives. To demonstrate this view in our paper, several well-known fractional-order chaotic systems are taken for example. To our best knowledge, there is little information about this view, available from the literature. Finally, to show its application, a synchronization scheme is investigated for a fractional-order chaotic system via a specific state variable and its (q-order and $2q$-order) time derivatives. The synchronization technique, based on the stability theory of fractional-order systems, is simple and rigorous theoretically. Some examples are used to illustrate the effectiveness of the proposed synchronization method. Numerical simulations coincide with the theoretical analyses.

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2. A specific state variable for a class of 3D continuous fractional-order chaotic systems

Several definitions of fractional derivatives were reported. In this paper, we use the most common one of them:

$$\frac{\mathrm{d}^q f(t)}{\mathrm{d}t^q} \equiv D^q f(t) = J^{m-q} f^{(m)}(t), \quad q > 0,$$

where D^q is generally called 'q-order Caputo differential operator',[24] and m is the first integer which is not less than q, and $f^{(m)}(t)$ is the m-order derivative in the usual sense, and $J^q(q > 0)$ is the q-order Riemann–Liouville integral operator with expression:

$$J^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f(\tau) \mathrm{d}\tau.$$

Here $\Gamma(\cdot)$ denotes the gamma function.

First, we review the result presented in Ref. [25]. Consider the chaotic system (1) with scalar output z,

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{f}(\boldsymbol{x}), \quad z = h(\boldsymbol{x}), \tag{1}$$

where $\boldsymbol{x} \in R^n$, $\boldsymbol{f} : R^n \to R^n$, and z is the scalar output. Let $k(\boldsymbol{x})$ be a scalar differentiable function and $\boldsymbol{V}(\boldsymbol{x})$ be a vector field and $\boldsymbol{V} : R^n \to R^n$. The Lie derivative of $k(\boldsymbol{x})$ along $\boldsymbol{V}(\boldsymbol{x})$ is defined as $L_V k =< \boldsymbol{V}(\boldsymbol{x}), \mathrm{grad}\, k(\boldsymbol{x}) >$, and the ith Lie derivative of $k(\boldsymbol{x})$ along $\boldsymbol{V}(\boldsymbol{x})$ is $L_V^{(i)} k =< \boldsymbol{V}(\boldsymbol{x}), \mathrm{grad}\, L_V^{(i-1)} k(\boldsymbol{x}) >$, where $\langle\cdot\rangle$ is the Euclidean inner product. It is easy to obtain Theorem 1.

Theorem 1[25] If map $\Phi = [h(\boldsymbol{x}), L_{\boldsymbol{f}} h(\boldsymbol{x}), L_{\boldsymbol{f}}^{(2)} h(\boldsymbol{x}), \ldots L_{\boldsymbol{f}}^{(n-1)} h(\boldsymbol{x})]^{\mathrm{T}}$ is a diffeomorphism, then Eq. (1) is observable through scalar output z and its ith derivative for $i \leq n-1$. In addition,

$$\boldsymbol{x} = \Phi^{-1}\left[z(t), \frac{\mathrm{d}z(t)}{\mathrm{d}t}, \frac{\mathrm{d}^2 z(t)}{\mathrm{d}t^2}, \ldots, \frac{\mathrm{d}^{n-1} z(t)}{\mathrm{d}t^{n-1}}\right]^{\mathrm{T}}.$$

Second, according to **Theorem 1**, we may assume that there exists a class of 3D continuous fractional-order chaotic systems described as

$$D^q \boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x}), \tag{2}$$

where $\boldsymbol{x} \in R^3$, and $\boldsymbol{f} : R^3 \to R^3$ is the continuous vector differentiable function, and system (2) has the following property: if a specific state variable $x_i(t)$ is chosen as scalar output, then $x_j(t) = g_j(x_i(t), D^q x_i(t), D^{2q} x_i(t))$, $j \neq i$ $(i, j = 1, 2, 3)$. It implies that all state variables of chaotic system (2) will be yielded via a specific state variable (a scalar output) and its (q-order and $2q$-order) time derivatives. So, this specific state variable $x_i(t)$ can be chosen as the observable or measured variable of physical significance. In this paper, we observe that many 3D fractional-order chaotic systems have all this property, such as, 3D fractional-order unified chaotic system,[26] 3D fractional-order Liu chaotic system,[27] 3D fractional-order Rössler chaotic system,[22] 3D fractional-order Sprott's chaotic system,[19] etc.

To illustrate the previous idea, some well-known fractional-order chaotic systems are taken for example.

Example 1 Consider 3D fractional-order unified chaotic[26] system (3),

$$\begin{aligned} D^q x_1 &= -(25\alpha + 10)(x_1 - x_2), \\ D^q x_2 &= (28 - 35\alpha) x_1 + (29\alpha - 1) x_2 - x_1 x_3, \\ D^q x_3 &= x_1 x_2 - (\alpha + 8) x_3/3, \end{aligned} \tag{3}$$

where $q = 0.9$, and $0.4 \leq \alpha \leq 1$. If state variable $x_1(t)$ is selected as the specific state variable and as the scalar output, then state variables $x_2(t)$ and $x_3(t)$ are expressed as

$$\begin{aligned} x_2 &= (D^q x_1)/(25\alpha + 10) + x_1 = g_2(x_1, D^q x_1, D^{2q} x_1), \\ x_3 &= [-D^q x_2 + (28 - 35\alpha) x_1 + (29\alpha - 1) x_2]/x_1 \\ &= [-D^{2q} x_1/(25\alpha + 10) - D^q x_1]/x_1 \\ &\quad + \{(29\alpha - 1)[D^q x_1/(25\alpha + 10) + x_1] \\ &\quad + (28 - 35\alpha) x_1\}/x_1 \\ &= g_3(x_1, D^q x_1, D^{2q} x_1), x_1 \neq 0. \end{aligned}$$

Example 2 Consider 3D fractional-order Liu chaotic[27] system (4),

$$\begin{aligned} D^q x_1 &= 10(x_2 - x_1), \\ D^q x_2 &= 40 x_1 - 10 x_1 x_3, \\ D^q x_3 &= 4 x_1^2 - 2.5 x_3, \end{aligned} \tag{4}$$

where order $q = 0.9$. If state variable $x_1(t)$ is selected as the specific state variable and as the scalar output, then state variables $x_2(t)$ and $x_3(t)$ are expressed as

$$\begin{aligned} x_2 &= D^q x_1/10 + x_1 = g_2(x_1, D^q x_1, D^{2q} x_1), \\ x_3 &= (-D^q x_2 + 40 x_1)/10 x_1 \\ &= (-D^{2q} x_1/10 - D^q x_1 + 40 x_1)/10 x_1 \\ &= g_3(x_1, D^q x_1, D^{2q} x_1), x_1 \neq 0. \end{aligned}$$

Example 3 Consider 3D fractional-order Rössler chaotic[22] system (5),

$$\begin{aligned} D^q x_1 &= -x_2 - x_3, \\ D^q x_2 &= x_1 + 0.6x_2, \\ D^q x_3 &= x_3(x_1 - 10) + 0.2, \end{aligned} \tag{5}$$

where $q = 0.9$. If state variable $x_2(t)$ is selected as the specific state variable and as the scalar output, then state variables $x_1(t)$ and $x_3(t)$ are given as

$$\begin{aligned} x_1 &= D^q x_2 - 0.6x_2 = g_1(x_2, D^q x_2, D^{2q} x_2), \\ x_3 &= -D^q x_1 - x_2 = -D^{2q} x_2 + 0.6 D^q x_2 - x_2 \\ &= g_3(x_2, D^q x_2, D^{2q} x_2). \end{aligned}$$

Example 4 Consider 3D fractional-order Sprott's chaotic[19] system (6),

$$\begin{aligned} D^q x_1 &= -x_1 - 4x_2, \\ D^q x_2 &= x_1 + x_3^2, \\ D^q x_3 &= x_1 + 1, \end{aligned} \tag{6}$$

where $q = 0.99$. If state variable $x_3(t)$ is selected as the specific state variable and as the scalar output, then state variables $x_1(t)$ and $x_2(t)$ are written as

$$\begin{aligned} x_1 &= D^q x_3 - 1 = g_1(x_3, D^q x_3, D^{2q} x_3), \\ x_2 &= -(D^q x_1 + x_1)/4 = -(D^{2q} x_3 + D^q x_3 - 1)/4 \\ &= g_2(x_3, D^q x_3, D^{2q} x_3). \end{aligned}$$

The four examples seem to suggest a positive answer to the previous idea, i.e. all the state variables in a class of 3D continuous fractional-order chaotic systems can be yielded via a specific state variable and its (q-order and $2q$-order) time derivatives.

3. Synchronization for fractional-order chaotic system

Now, we discuss the fractional-order chaotic system synchronization via a specific state variable and its (q-order and $2q$-order) time derivatives.

Without loss of generality, we take $x_1(t)$ as a specific state variable for fractional-order chaotic system (2). It can be rewritten as system (7), and as a drive system,

$$D^q \boldsymbol{x} = \boldsymbol{A}x + \boldsymbol{G}(x_1, D^q x_1, D^{2q} x_1), \tag{7}$$

where $\boldsymbol{A}$ is the constant real matrix, $\boldsymbol{G}(x_1, D^q x_1, D^{2q} x_1)$ is the 3×1 matrix including all nonlinear terms of fractional-order chaotic system (2). Now, let

$$\begin{aligned} \boldsymbol{S}(\boldsymbol{x}) &= \boldsymbol{G}(x_1, D^q x_1, D^{2q} x_1) \\ &\quad + \boldsymbol{M} \begin{pmatrix} x_1 \\ g_2(x_1, D^q x_1, D^{2q} x_1) \\ g_3(x_1, D^q x_1, D^{2q} x_1) \end{pmatrix} \end{aligned}$$

be the output of fractional-order chaotic system (2), where $\boldsymbol{G}(x_1, D^q x_1, D^{2q} x_1)$ is the nonlinear vector of fractional-order chaotic system (7), and $g_j(x_1(t), D^q x_1(t),\ D^{2q} x_1(t)) = x_j(t) (j = 2, 3)$, and $\boldsymbol{M}$ is the 3×3 real constant matrix.

Now, take fractional-order chaotic system (2) as a drive system, and construct the following response system :

$$D^q \boldsymbol{y} = \boldsymbol{f}(\boldsymbol{y}) + \boldsymbol{S}(\boldsymbol{x}) - \boldsymbol{S}(\boldsymbol{y}). \tag{8}$$

Theorem 2 Given a feedback gain matrix $\boldsymbol{M}$. If all the eigenvalues of $\boldsymbol{A} - \boldsymbol{M}$ satisfy $|\arg \lambda_i| > 0.5\pi q$ $(i = 1, 2, 3)$, then the chaotic synchronization between fractional order drive system (2) and response system (8) can be achieved.

Proof

Let $\boldsymbol{e} = \boldsymbol{y} - \boldsymbol{x}$ be the synchronization error between drive system (2) and response system (8), then we will have

$$\begin{aligned} D^q \boldsymbol{e} &= D^q \boldsymbol{y} - D^q \boldsymbol{x} \\ &= \boldsymbol{f}(\boldsymbol{y}) - \boldsymbol{f}(\boldsymbol{x}) - (\boldsymbol{S}(\boldsymbol{y}) - \boldsymbol{S}(\boldsymbol{x})). \end{aligned} \tag{9}$$

Because

$$\begin{aligned} \boldsymbol{S}(\boldsymbol{x}) &= \boldsymbol{G}(x_1, D^q x_1, D^{2q} x_1) \\ &\quad + \boldsymbol{M} \begin{pmatrix} x_1 \\ g_2(x_1, D^q x_1, D^{2q} x_1) \\ g_3(x_1, D^q x_1, D^{2q} x_1) \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} \boldsymbol{S}(\boldsymbol{y}) &= \boldsymbol{G}(y_1, D^q y_1, D^{2q} y_1) \\ &\quad + \boldsymbol{M} \begin{pmatrix} y_1 \\ g_2(y_1, D^q y_1, D^{2q} y_1) \\ g_3(y_1, D^q y_1, D^{2q} y_1) \end{pmatrix}. \end{aligned}$$

Thus, error system (9) is described as

$$D^q \boldsymbol{e} = \boldsymbol{A}(\boldsymbol{y}-\boldsymbol{x}) - \boldsymbol{M}\begin{pmatrix} y_1 - x_1 \\ g_2(y_1, D^q y_1, D^{2q} y_1) - g_2(x_1, D^q x_1, D^{2q} x_1) \\ g_3(y_1, D^q y_1, D^{2q} y_1) - g_3(x_1, D^q x_1, D^{2q} x_1) \end{pmatrix}. \tag{10}$$

Because

$$x_j = g_j(x_1, D^q x_1, D^{2q} x_1), \quad (j = 2, 3),$$
$$y_j = g_j(y_1, D^q y_1, D^{2q} y_1), \quad (j = 2, 3).$$

Therefore, we can obtain

$$D^q \boldsymbol{e} = (\boldsymbol{A} - \boldsymbol{M})\boldsymbol{e}. \tag{11}$$

Because all the eigenvalues of $\boldsymbol{A} - \boldsymbol{M}$ satisfy $|\arg \lambda_i| > 0.5\pi q$ $(i = 1, 2, 3)$, according to the stability theory of fractional-order systems,[28] the equilibrium point $e_i = 0$ $(i = 1, 2, 3)$ in Eq. (11) is asymptotically stable. So, the chaotic synchronization between drive system (2) and response system (8) can be achieved.

The above results manifest that the fractional-order chaotic systems (2) and (8) are synchronized under suitable gain matrix $\boldsymbol{M}$. The proof is completed here. ■

Remark 1 Obviously, only the state variable $x_1(t)$ in driving system (2) and its (q-order and $2q$-order) time derivatives are transmitted to response system (8). So, this synchronization method in the paper is different from one mentioned above.

Remark 2 In a real physical system, in order to obtain a response system, we only observe or measure the specific state variable $x_1(t)$. Therefore, our synchronization scheme and the previous idea mentioned above are of significance both in physics and in application.

To illustrate the effectiveness of the proposed synchronization scheme, some examples are considered and the numerical simulation is performed. Now, we take 3D fractional-order Liu chaotic[27] system (4) for numerical simulation.

The chaotic attractor of 3D fractional-order Liu chaotic system (4) is shown in Fig. 1.

Because $\boldsymbol{A} = \begin{pmatrix} -10 & 10 & 0 \\ 40 & 0 & 0 \\ 0 & 0 & -2.5 \end{pmatrix}$ is a constant matrix, one can easily find a matrix $\boldsymbol{M}$ that satisfies $|\arg \lambda_i| > 0.5\pi q$ $(i = 1, 2, 3)$.

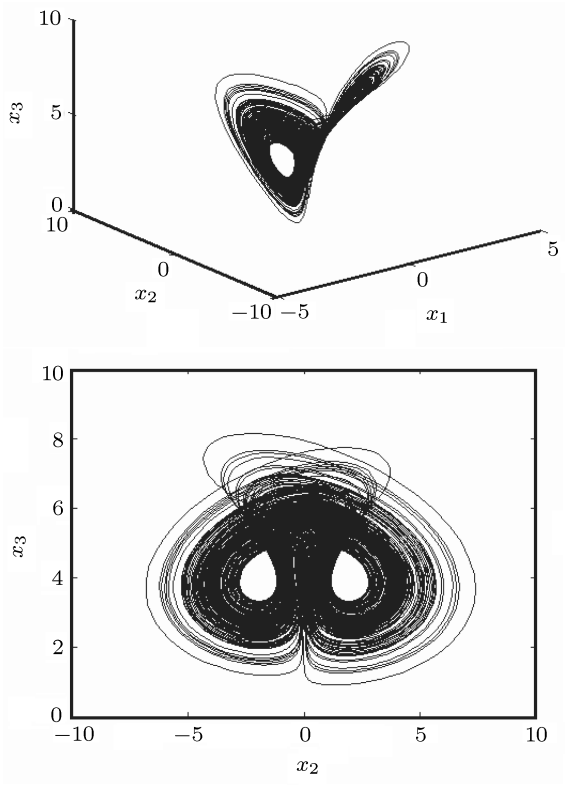


Fig. 1. Chaotic attractor of fractional-order Liu chaotic system (4).

When $\boldsymbol{M} = \begin{pmatrix} -11 & 11 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then all the eigenvalues of $\boldsymbol{A} - \boldsymbol{M}$ are $0.5 \pm 6.3048\,\mathrm{i}$, and -2.5, i.e. $|\arg \lambda_i| > 0.5\pi q$ $(i = 1, 2, 3)$. So, the chaotic synchronization for fractional order Liu chaotic system (4) can be achieved. The simulation result is shown in Fig. 2. In this paper $\varepsilon = \left(\sum_{i=1}^{3} e_i^2\right)^{1/2}$, initial conditions are $(x_{10},\ x_{20},\ x_{30}) = (1, 2, 3)$, and $(y_{10},\ y_{20},\ y_{30}) = (0.1, 0.2, 0.3)$.

When $\boldsymbol{M} = \begin{pmatrix} -1 & 11 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then all the eigenvalues of $\boldsymbol{A} - \boldsymbol{M}$ are $-4.5 \pm 4.4441\,\mathrm{i}$, and -2.5, i.e. $|\arg \lambda_i| > 0.5\pi q$ $(i = 1, 2, 3)$. So, the chaotic synchronization for fractional order Liu chaotic system (4) can be achieved. The simulation result is shown in Fig. 3.

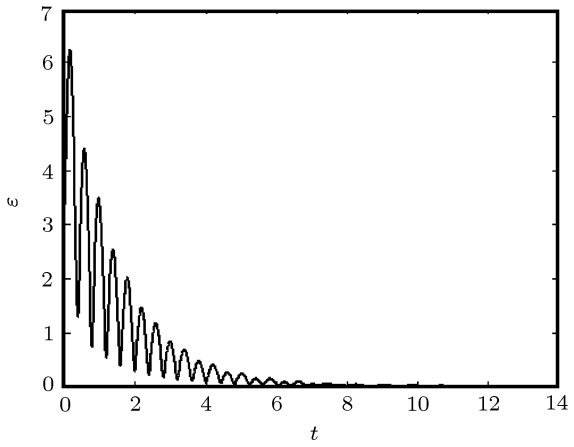


Fig. 2. Synchronization error versus time t.

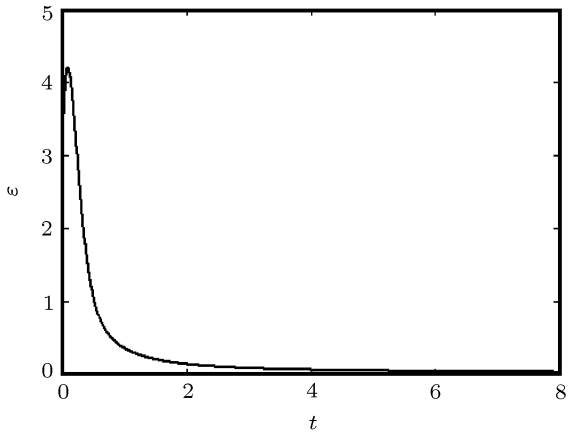


Fig. 3. Synchronization error versus time t.

According to the simulation results, we can obtain that numerical simulations accord well with the theoretical analyses.

4. Conclusion

In this paper, we obtain the observable or measured state variables for the 3D fractional-order unified chaotic system, the 3D fractional-order Liu chaotic system, the 3D fractional-order Rössler chaotic system, and the 3D fractional-order Sprott's chaotic system. Our results indicate that all state variables of a class of 3D continuous fractional-order chaotic systems can be yielded via a specific state variable and its (q-order and $2q$-order) time derivatives. To show its application, a synchronization scheme is proposed by using a specific state variable and its (q-order and $2q$-order) time derivatives. Numerical simulations show the effectiveness of the proposed approach.

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