Stochastic resonance in a time-delayed bistable system subject to multiplicative and additive noise*

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This paper investigates the stochastic resonance in a time-delayed bistable system subjected to multiplicative and additive white noise and asymmetric dichotomous noise. Under the adiabatic approximation condition, the expression of the signal-to-noise ratio (SNR) is obtained. It finds that the SNR is a non-monotonic function of the delayed times, of the amplitude of the driving square-wave signal, as well as of the asymmetry of the dichotomous noise. In addition, the SNR varies non-monotonously with the intensities of the multiplicative and additive noise as well as the system parameters. Moreover, the SNR depends non-monotonically on the correlate rate of the dichotomous noise.

Keywords: stochastic resonance, time-delayed bistable system, signal-to-noise ratio
PACC: 0540, 0250

1. Introduction

The concept of stochastic resonance (SR) was originally put forward in the seminal papers by Benzi and collaborators, [1−7] wherein they address the problem of the periodically recurrent ice ages. There are lots of stochastic systems with time-delayed feedback. By means of such feedback loops, signals or other key quantities (such as information, matter, energy, and so on) of the systems are fed back from output to input interfaces. The SR phenomenon in time-delayed systems has been studied by lots of researchers.[3−9] Steve *et al.*[3] considered the effect of external fluctuations (noise) on delay differential equations. For small delays, they yielded a steady-state probability density and a conditional average drift. Frank[4] studied nonlinear stochastic systems with time-delayed feedback using the concept of delay Fokker–Planck equations and derived an analytical expression for stationary distributions using first-order perturbation theory. Zeng *et al.*[5] studied the SR phenomenon in a time-delayed gene transcriptional regulatory system with a multiplicative noise and an additive noise and a weak periodic signal. It is found that the delay time can enhance the SR of the system.

In this paper, we investigate the SR phenomenon in a bistable system with time delay driven by square-wave signal and multiplicative and additive noise. According to the expression of the signal-to-noise ratio (SNR), we find that the SNR behaves non-monotonically with the increase of the delayed times and the driving signal as well as the parameters of the noises.

2. The bistable system with time delay and its output SNR

We consider a bistable system with time delay driven by an external force $f(t)$ and multiplicative noise $\xi(t)$ and additive noise $\eta(t)$, described by the following Langevin equation:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = ax(t-\tau) - bx(t)^3 + x(t-\theta)\xi(t) + \eta(t) + f(t), \tag{1}$$

where τ denotes the delay time in the deterministic force and θ denotes the delay time in the random force, a and b are the system parameters, $\xi(t)$ and $\eta(t)$ are uncorrelated Gaussian white noises with zero mean and correlations

$$\langle \xi(t_1)\xi(t_2)\rangle = 2D\delta(t_1 - t_2),$$
$$\langle \eta(t_1)\eta(t_2)\rangle = 2P\delta(t_1 - t_2), \tag{2}$$

in which D and P are the strength of the multiplicative and additive noises, respectively. The external

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force has the form

$$f(t) = q(t) + As(t), \tag{3}$$

where $s(t)$ is a periodic square-wave signal with period T,

$$s(t) = \begin{cases} -1, & 0 < t \leq T/2, \\ 1, & T/2 < t \leq T, \end{cases} \tag{4}$$

where $q(t)$ is an asymmetric dichotomous noise with zero mean, $q(t) = \{J_1, J_2\}, J_{1,2} > 0$. Its correlation function is

$$\begin{aligned} &\langle q(t_1)q(t_2)\rangle = E\lambda \exp\left(-\lambda\left|t_1 - t_2\right|\right), \\ &E = \sigma/\lambda, \quad \sigma = J_1 J_2. \end{aligned} \tag{5}$$

Here σ and λ denote the intensity and correlation rate of the dichotomous noise $q(t)$, respectively. The asymmetric dichotomous noise $q(t)$ can be decomposed into a symmetric dichotomous noise and a constant, i.e.,

$$q(t) = \varepsilon p(t) + r, \tag{6}$$

the constant $r = \kappa/2$, the asymmetry $\kappa = J_1 - J_2$ and $p(t)$ is a symmetric dichotomous noise with unit amplitude, $p(t) = \pm 1$.

$$\varepsilon = (J_1 + J_2)/2 = \sqrt{\sigma + \kappa^2/4}. \tag{7}$$

From Eqs. (3) and (6), $f(t)$ can be rewritten as

$$f(t) = r + \varepsilon p(t) + As(t). \tag{8}$$

From Eq. (1), one can see that, there is one unstable state $x_0 = 0$ and two stable states $x_\pm = \pm\sqrt{a/b}$ of the non-delayed bistable system. According to Eqs. (2)–(5), using the small time delay approximation,[3,4] the approximate Fokker–Planck equation corresponding to Eq. (1) is derived through the Novikov theorem[10,11] and the Fox approach:[12]

$$\frac{\partial \rho(x,t)}{\partial t} = -\frac{\partial}{\partial x}F(x,\tau)\rho(x,t) + \frac{\partial^2}{\partial x^2}G(x,\tau)\rho(x,t), \tag{9}$$

where

$$F(x) = gx - hx^3 + kf(t), \tag{10}$$

$$g = a(1 + a\tau) + D(1+\theta)^2,$$

$$h = b(1 + a\tau), \quad k = 1 + a\tau, \tag{11}$$

$$G(x) = nx^2 + P, \quad n = D(1+\theta)^2. \tag{12}$$

Assuming that the frequency $\Omega = 2\pi/T$ of the driving square-wave signal is very small, i.e., frequency smaller than the rate of the intrawell relaxation, thus there is enough time for the system to reach the equilibrium during the period of T, i.e., we make the assumption that the system satisfies the adiabatic approximation condition.[13] The solution of the corresponding Fokker–Planck equation may then be approximated as a steady-state distribution with a slow time variation, which can be written as

$$\begin{aligned} \rho_{\rm st}(x) &= \frac{N}{\sqrt{G(x)}} \exp\left[\int^x \frac{F(x)}{G(x)}\right] \\ &= \frac{N}{\sqrt{G(x)}} \exp\left[-\frac{\Phi(x,\tau)}{D}\right], \end{aligned} \tag{13}$$

where N is the normalization constant, $\Phi(x,t)$ is the rectified potential function

$$\begin{aligned} \Phi(x) &= \frac{Dh}{2n}x^2 - \frac{Dr}{2n}\ln(nx^2 + P) \\ &\quad - \frac{kDf(t)}{\sqrt{nP}} \arctan\left(\sqrt{n/P}x\right). \end{aligned} \tag{14}$$

Under the adiabatic limit condition, the transition rates out of $x_\pm$ can be given by

$$\begin{aligned} W_\pm\left(f(t)\right) &= \frac{\sqrt{|U''(x_0)U''(x_\pm)|}}{2\pi} \\ &\quad \times \exp\left[-\frac{\Phi(x_0) - \Phi(x_\pm)}{D}\right] \\ &= W_0 \exp\left[\mp\gamma f(t)\right], \end{aligned} \tag{15}$$

with

$$\begin{aligned} &U'(x) = \frac{\mathrm{d}U}{\mathrm{d}x} = -ax + bx^3, \\ &\gamma = \frac{k}{\sqrt{nP}} \arctan\left(\sqrt{na/(Pb)}\right). \end{aligned} \tag{16}$$

Here W_0 denotes the characteristic switching frequency of the bistable system when it is only driven by the multiplicative and additive Gaussian white noise, which is given by

$$W_0 = \frac{a}{\sqrt{2\pi}} \exp\left[\frac{ha}{2nb} - \frac{r}{2n}\left(\frac{na}{bP} + 1\right)\right]. \tag{17}$$

Applying the method in Ref. [14], we obtain the expression of the correlation function as

$$\begin{aligned} K(t_0) &= \left[B_1(r,\varepsilon,A) + B_3(r,\varepsilon,A)\right] \exp\left(-\lambda\left|t_0\right|\right) \\ &\quad + B_2(r,\varepsilon,A)\varphi(t_0) + C(r,\varepsilon,A)\delta(t_0), \end{aligned} \tag{18}$$

where

$$\varphi(t_0) = \frac{4}{\pi^2}\sum_{j=0}^{\infty}(2j+1)^{-2}\exp[-\mathrm{i}(2j+1)\Omega t_0], \tag{19}$$

$$B_1(r,\varepsilon,A)=\frac{1}{16}\left[w(r+\varepsilon+A)-w(r-\varepsilon-A)+w(r+\varepsilon-A)-w(r-\varepsilon+A)\right]^2, \tag{20}$$

$$B_2(r,\varepsilon,A)=\frac{1}{16}\left[w(r+\varepsilon+A)-w(r-\varepsilon-A)\ -w(r+\varepsilon-A)+w(r-\varepsilon+A)\right]^2, \tag{21}$$

$$B_3(r,\varepsilon,A)=\frac{1}{16}\left[w(r+\varepsilon+A)+w(r-\varepsilon-A)\ -w(r+\varepsilon-A)-w(r-\varepsilon+A)\right]^2, \tag{22}$$

$$C(r,\varepsilon,A)=\frac{1}{4}\left\{C_0(r+\varepsilon+A)+C_0(r-\varepsilon-A)\ +C_0(r+\varepsilon-A)+C_0(r-\varepsilon+A)\right\}, \tag{23}$$

$$w(\mu)=\sqrt{\frac{a}{b}}\frac{W_-(\mu)-W_+(\mu)}{W_-(\mu)+W_+(\mu)}=\sqrt{\frac{a}{b}}\frac{\exp(2\gamma\mu)-1}{\exp(2\gamma\mu)+1}, \tag{24}$$

$$C_0(\mu)=\frac{8W_0^2}{[W_-(\mu)+W_+(\mu)]^3}. \tag{25}$$

The power spectrum, defined as the Fourier transform of the correlation function, can be given by

$$S(\omega)=S_1(0)+S_2(\omega), \tag{26}$$

where

$$S_1(0)=\frac{2}{\lambda}\left[B_1(r,\varepsilon,A)+B_3(r,\varepsilon,A)\right]+C(r,\varepsilon,A), \tag{27}$$

$$S_2(\omega)=B_2(r,\varepsilon,A)\varphi(\omega), \tag{28}$$

$$\varphi(\omega)=\frac{8}{\pi}\sum_{j=0}^{\infty}(2j+1)^{-2}\delta[\omega-(2j+1)\Omega]. \tag{29}$$

Here $S_1(0)$ denotes the power density at the zero frequency associated with the noise background, $S_2(\omega)$ is the power spectrum of the output signal. The output SNR is the ratio between the power density of the signal and the noise at the signal frequency, which has the form

$$\mathrm{SNR}=\frac{8}{\pi}\frac{B_2(r,\varepsilon,A)}{C(r,\varepsilon,A)+2[B_1(r,\varepsilon,A)+B_3(r,\varepsilon,A)]/\lambda}. \tag{30}$$

3. Discussion and conclusion

By virtue of Eq. (30), the effect of the external force and the noise as well as the system parameter on the SNR are discussed through Figs. 1–7.

Square-wave signal is widely used in various systems, such as digital electronic circuit and communication systems. Here we consider a bistable system with two time-delayed parameters subjected to square-wave signal and noises. As seen from Fig. 1, the SNR is a non-monotonic function of the amplitude of the square-wave signal. With the increase of the amplitude of the square-wave, the SNR can take two extreme values. At first the SNR reaches a maximum value and then decreases to a minimum point, at last it increases monotonically with the increase of the amplitude. This is an interesting result, which is not mentioned in Refs. [14] and [15].

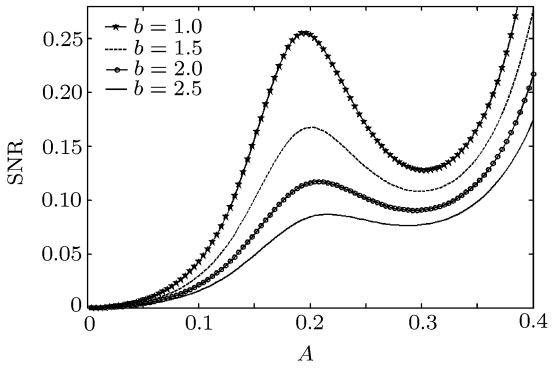


Fig. 1. The SNR versus the amplitude A of the square-wave for $a=0.5$, $\tau=0.01$, $\theta=0.3$, $\kappa=0.3$, $D=0.3$, $P=0.04$, $\lambda=3$, $\sigma=0.01$ with different values of the system parameter b.

Dichotomous noise can by generated by a two-state Poisson process. Because its relatively simple calculation and well-defined limiting procedures lead to both Gaussian white noise and white shot noise, coloured dichotomous noise has been paid more and more attentions.[16,17] In Fig. 2, we plot the curve of the SNR as a function of the dichotomous noise intensity with different values of the system parameter a. One can clearly conclude that the SNR varies non-monotonically with the increase of the intensity of the dichotomous noise, i.e., the SR phenomenon takes place. In Fig.3, the SNR is shown as a function of the asymmetry of the dichotomous noise with different values of the correlation rate λ of the dichotomous noise. As shown in the figure, the SNR also behaves non-monotonically as the asymmetry κ varies. For

small value of the correlation rate ($\lambda = 1$), we can observe two peaks on the SNR curve, i.e., stochastic multi-resonance. As the increase of the correlation rate, the height of the first peak decreases ($\lambda = 10$), and finally, there is only one peak on the SNR curve.

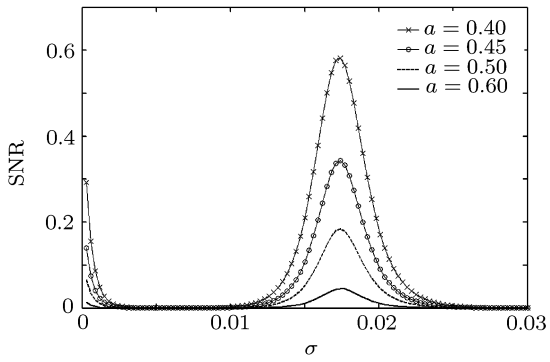


Fig. 2. The SNR versus the intensity σ of the dichotomous noise for $b = 0.6$, $A = 0.14$, $\tau = 0.015$, $\kappa = 0.1$, $D = 0.15$, $P = 0.002$, $\lambda = 3$, $\theta = 0.1$ with different values of the system parameter a.

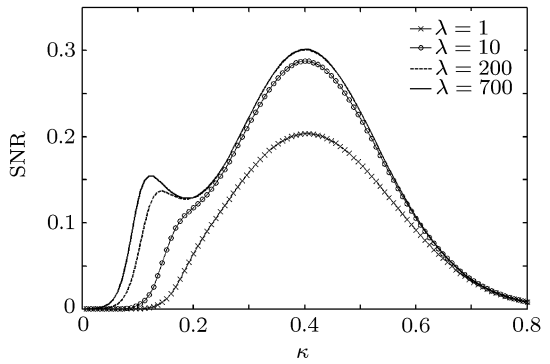


Fig. 3. The SNR versus the asymmetry κ of the dichotomous noise for $a = 1$, $b = 1$, $A = 0.2$, $\tau = 0.1$, $\theta = 0.01$, $D = 0.1$, $P = 0.01$, $\sigma = 0.08$ with different values of the correlate rate λ of the dichotomous noise.

We analyse the effect of the system parameters on the SNR from Figs. 4 and 5. One can see that, the SNR has non-monotonic behaviour with the increase of the system parameter a and b, i.e., the SR in a broad sense occurs. This can be explained by using the potential function $\Phi(x)$.[17] From Eq. (14), one can see that the potential function $\Phi(x)$ is a function of the system parameters a and b, so the height $\Delta V = V(x_0) - V(x_\pm)$ of the potential barrier is also a function of the parameters. Therefore, the system SNR can be maximized at some mediate system parameters. Furthermore, by analysing the curves in Figs. 5 and 6, we find that, with the increase of the delayed-time θ in the random force and τ in the deterministic force, the SNR varies non-monotonically, too.

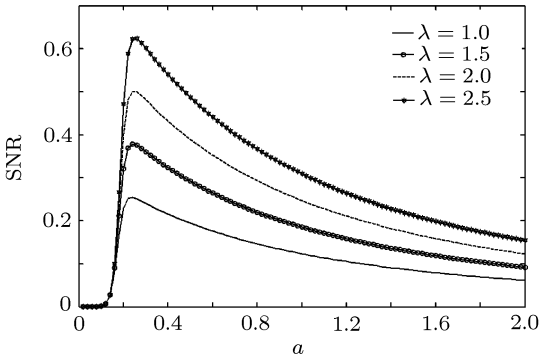


Fig. 4. The SNR versus the system parameter a for $b = 1.5$, $A = 0.15$, $\tau = 0.01$, $D = 0.01$, $P = 0.2$, $\theta = 0.01$, $\kappa = 0.1$, $\sigma = 0.08$ with different values of the correlation rate λ of the dichotomous noise.

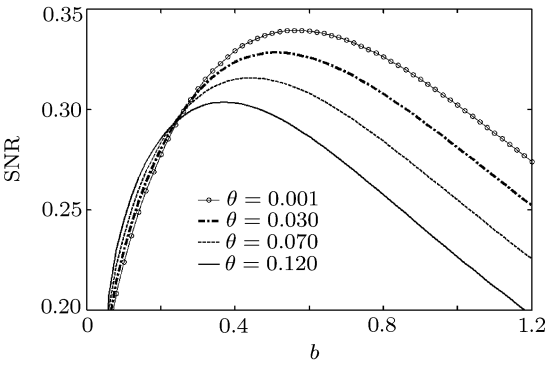


Fig. 5. The SNR versus the system parameter b for $a = 1$, $A = 0.15$, $\tau = 0.01$, $D = 0.2$, $P = 0.3$, $\lambda = 5$, $\kappa = 0.15$, $\sigma = 0.1$ with different values of the delayed-time θ in the random force.

The influence of the additive noise and multiplicative noise on the output SNR is studied from Figs. 6 and 7. We see that the SNR curve also presents a peak as a function of the intensity of the additive and multiplicative noise, as seen in these two figures.

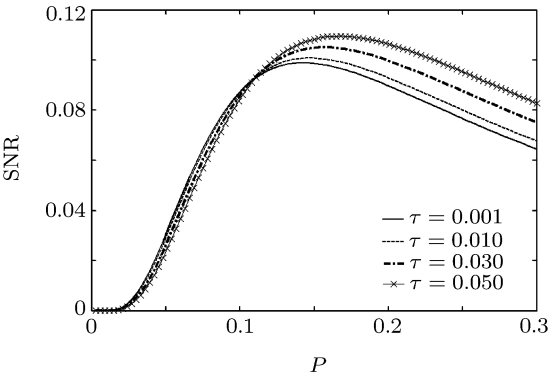


Fig. 6. The SNR versus the strength P of the additive noise for $a = 1.5$, $b = 1$, $A = 0.02$, $\theta = 0.01$, $D = 0.25$, $\lambda = 15$, $\kappa = 0.01$, $\sigma = 0.02$ with different values of the delayed-time τ in the deterministic force.

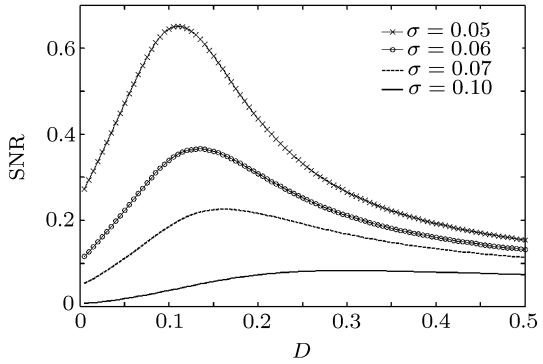


Fig. 7. The SNR versus the intensity D of the multiplicative noise for $a = 1$, $b = 1$, $A = 0.1$, $\tau = 0.01$, $\theta = 0.01$, $\kappa = 0.01$, $P = 0.1$, $\lambda = 30$ $\sigma = 0.05$ with different values of the intensity σ of the dichotomous noise.

In conclusion, in this paper, we investigate the SR phenomenon in a time-delayed bistable system subjected simultaneously to square wave periodic signal, asymmetry dichotomous noise as well as multiplicative noise and additive white noise. We have found that the SNR exhibits three kinds of SR phenomenon. The first is as a function of the noises, i.e., the asymmetric dichotomous noise, the multiplicative noise and additive noise. The second is as a function of the intensity of the system parameters. The thrid is as a function of the delayed-time.

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