Bright and dark soliton solutions in growing Bose–Einstein condensates*

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This paper develops the Hirota method carefully for applying into the growing model of quasi-one-dimensional Bose–Einstein condensations with attractive and repulsive interaction, respectively. After a tedious calculation it obtains the exact bright and dark soliton solutions analytically. It shows that the growing model has the important effect on the soliton amplitude and the time-dependent potential only contributes to the phase and phase velocity. A detailed analysis for the asymptotic behaviour of two-soliton solutions shows that the collision of two soliton is elastic.

Keywords: bright soliton, dark soliton, growing Bose–Einstein condensates
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1. Introduction

The concept of soliton was introduced first by Zabusky and Kruskal[1] to characterize the nonlinear solitary waves which preserves completely its localized form and speed during propagation and after collision. It means that the energy can propagate in the localized form and can be used to describe elementary particles produced to stress the character of these solitary waves. This intrinsic favourable property of soliton has motived a great attention on the nonlinear systems in many fields of physics, especially in high-rate telecommunications with optical fibres[2,3] and Bose–Einstein condensations (BECs).[4] The realization of BECs in dilute atomic gases has offered a new fecund ground to investigate the nonlinear properties of matter waves, such as vortices[5−7] and solitons.[8−19] For an individual atomic BECs the macroscopic wave function obeys the so-called Gross–Pitaevskii equation (GPE),[4] whose nonlinearity is a result of the interatomic interactions. Depending on the attractive or repulsive nature of the interatomic interactions, the bright or dark soliton solutions are admitted for GPE. In BECs the bright soliton[20−26] is expected for the balance between the dispersion and the attractive mean-field energy. However, large condensates are necessarily associated with repulsive interaction, for which bright soliton might seem impossible because the nonlinearity cannot compensate for the kinetic energy part in the atomic dynamics. For the case of repulsive interatomic interactions there are dark solitons[27−32] in BECs[10−19,33−35] characterized by a local density minimum and a phase gradient of the wave function at the position of the minimum. It means that dark soliton can only exist within the condensate itself.

When the longitudinal dimension of BECs is much longer than its transverse dimension which has the order of its healing length, the GPE can be reduced to the quasi-one-dimensional regime. This trapped quasi-one-dimensional[36] condensates offer a useful tool to investigate the nonlinear excitations which are more stable than the three-dimensional case, where the solitons suffer from the transverse instability and the vortices can bend. So the studies of both theory and experiment are very important for the soliton excitation in quasi-one-dimensional BECs. Recently, it is reported that the interatomic interactions denoted by the s-wave scattering length can be tuned experimentally by utilizing the Feshbach resonance,[37−39] even including its sign. A sinusoidal variation of the scattering length has also been used to form patterns such as Faraday waves.[40,41] When the scattering length is switched from positive value

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to negative one, the BECs abruptly loss atoms[37] and the growing modes of BECs[42−44] realized by means of Feshbach resonance technique become important. Theoretically, the corresponding GPE takes the form

$$\begin{aligned} \mathrm{i}\hbar\Psi_{\mathrm{T}} = & -\frac{\hbar^2}{2m}\Psi_{XX} + \frac{4\pi\hbar^2 a_{\mathrm{s}}}{m}|\Psi|^2\Psi \\ & + V\Psi + \mathrm{i}\Gamma\Psi, \end{aligned} \tag{1}$$

where m is the atomic mass, a_{s} is the time-dependent scattering length, $V = 2\cos^2(\omega_\perp T/2)$ is time-dependent potential, and the coefficient Γ is the gain (loss) term for $\Gamma > 0$ ($\Gamma < 0$). For $\Gamma > 0$, Eq. (1) represents the mechanism of loading the thermal cloud into the BECs by optically pumping atoms from the external source. When Γ is negative, Eq. (1) describes the dynamics of BECs depleted continuously due to losing atoms.

Now, the growing BEC is a hot topic for matter waves and the centre of mass oscillating violently in the trap[45] have been predicted. However, the dynamic properties of soliton solutions in growing BECs has not well explored and it is our aim in the present article. By introducing the dimensionless variables, namely $x = \sqrt{2}X/a_\perp$, $t = \omega_\perp T$, $\psi = \sqrt{2\pi a_{\mathrm{B}}}a_\perp\Psi$, and $k = \Gamma/(\hbar\omega_\perp)$ with the transverse oscillator frequency $\omega_\perp$, the transverse oscillator length $a_\perp = \sqrt{\hbar/(m\omega_\perp)}$, and the Bohr radius a_{B}, we can reduce Eq. (1) to the dimensionless form

$$\begin{aligned} \mathrm{i}\psi_t = & -\psi_{xx} - 2\mu\mathrm{e}^{-2kt}|\psi|^2\psi \\ & + 2\cos^2\left(\frac{t}{2}\right)\psi + \mathrm{i}k\psi, \end{aligned} \tag{2}$$

where we have taken the form of the scattering length as $a_{\mathrm{s}} = -a_{\mathrm{B}}\mathrm{e}^{-2kt}$, and the parameter $\mu = \pm 1$ denotes the attractive (repulsive) case, respectively.

In this article, in terms of the developed Hirota method we obtain the analytical bright and dark soliton solutions of Eq. (2). With the analytical solutions we mainly discussed the corresponding properties and two solitons interaction in detail.

2. Developed hirota method and exact soliton solutions

Among the method for solving some nonlinear equations, Hirota method[25,46,47] is an effective straightforward technique. It first applies a direct transformation to the nonlinear equation. Then in terms of some skillful bilinear operators the nonlinear equation can be decoupled into a series of equations. At last, with some reasonable assumption the exact soliton solutions can be obtained effectively. Taking the normal procedure of Hirota method we consider the transformation

$$\psi(x,t) = \frac{G(x,t)}{F(x,t)}, \tag{3}$$

where $G(x,t)$ is a complex function and $F(x,t)$ is a real function. Substituting this transformation into Eq. (2) we have

$$0 = F\left[\left(\mathrm{i}D_t + D_x^2\right) - 2\cos^2\left(\frac{t}{2}\right) - \mathrm{i}k\right]G\cdot F - G[D_x^2F\cdot F - 2\mathrm{e}^{-2kt}\mu|G|^2], \tag{4}$$

where D_x^2 and D_t are called the Hirota bilinear operators which is defined as

$$D_x^m D_t^n G(x,t)\cdot F(x,t) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n G(x,t)\cdot F(x',t')|_{x=x',t=t'}. \tag{5}$$

In the following section, we will construct the exact bright and dark soliton solutions of Eq. (2) with the help of the developed Hirota method.

2.1. Exact bright soliton solution

In order to get the exact bright soliton solution of Eq. (2) in the case of $\mu = +1$, we can separate Eq. (4) into two equations

$$\mathcal{A}_1 G\cdot F = 0, \quad \mathcal{A}_2 F\cdot F = 2\mathrm{e}^{-2kt}|G|^2, \tag{6}$$

where

$$\mathcal{A}_1 = \mathrm{i}D_t + D_x^2 - 2\cos^2\left(\frac{t}{2}\right) - \mathrm{i}k. \tag{7}$$

2.1.1. One-soliton solution

For the bright one-soliton solution of Eq. (2), we can assume that

$$G = \chi g_1, \quad F = 1 + \chi^2 f_1, \tag{8}$$

where χ is an arbitrary auxiliary parameter which will be absorbed in the expression of soliton solution. Substituting Eq. (8) into Eq. (6), and then collecting the terms with same power in χ, we obtain

1) for the coefficient of χ

$$\mathcal{A}_1 g_1 \cdot 1 = 0; \tag{9}$$

2) for the coefficient of χ^2

$$\mathcal{A}_2 \left(f_1 \cdot 1 + 1 \cdot f_1\right) - 2\mathrm{e}^{-2kt} g_1^* g_1 = 0; \tag{10}$$

3) for the coefficient of χ^3

$$\mathcal{A}_1 g_1 \cdot f_1 = 0; \tag{11}$$

4) for the coefficient of χ^4

$$\mathcal{A}_2 f_1 \cdot f_1 = 0. \tag{12}$$

It is obvious to see that Eqs. (9)–(12) can be solved analytically in turn. With the Hirota bilinear operators in Eq. (5) and after a tedious calculation we have

$$g_1 = \mathrm{e}^{\eta_1}, \quad f_1 = A_{11}\mathrm{e}^{\eta_1+\eta_1^*}, \tag{13}$$

where $\eta_1 = P_1\left(x + \mathrm{i}\, P_1 t\right) - \mathrm{i}\left(\sin t + t\right)$, $A_{11} = \mathrm{e}^{-2kt}/(P_1 + P_1^*)^2$ with P_1 being complex and k being real in general. The superscript '$*$' indicates the complex conjugate. Taking $P_1 \equiv \alpha_1 + \mathrm{i}\,\beta_1$ and combining Eqs. (3), (8), and (13) we get the exact bright one-soliton solution of growing BECs

$$\psi = \frac{\alpha_1 \mathrm{e}^{kt}}{\cosh\theta_1}\mathrm{e}^{\mathrm{i}\,\varphi_1}, \tag{14}$$

where the parameter $\theta_1 = \alpha_1 x - 2\alpha_1\beta_1 t - \ln\left(2\alpha_1\right)$ and $\varphi_1 = x\beta_1 - \left(\beta_1^2 - \alpha_1^2 + 1\right)t - \sin t$. The solution in Eq. (14) represents the exact bright one soliton of growing BECs characterized with the amplitude $\alpha_1\mathrm{e}^{kt}$, soliton velocity $2\beta_1$, and the phase φ_1. From Eq. (14) we find that the time-dependent potential only contributes to the phase and the phase velocity which reads $\alpha_1^2 - \beta_1^2 - 1 + \cos t$. The dynamic evolution of soliton solution in Eq. (14) shows that the amplitude increases with time, while the soliton width and velocity is constant, respectively.

2.1.2. Two-soliton solution

To get the exact bright two-soliton solution of Eq. (2) we let G and F have the expressions:

$$G = \chi g_1 + \chi^3 g_3, \quad F = 1 + \chi^2 f_2 + \chi^4 f_4. \tag{15}$$

Substituting the above equation into Eq. (6) and collecting the coefficients with same power of χ, we will obtain a set of equations as follows:

1) for the coefficient of χ

$$\mathcal{A}_1 g_1 \cdot 1 = 0; \tag{16}$$

2) for the coefficient of χ^2

$$\mathcal{A}_2 \left(f_2 \cdot 1 + 1 \cdot f_2\right) = 2\mathrm{e}^{-2kt} g_1 g_1^*; \tag{17}$$

3) for the coefficient of χ^3

$$\mathcal{A}_1 \left(g_1 \cdot f_2 + g_3 \cdot 1\right) = 0; \tag{18}$$

4) for the coefficient of χ^4

$$\mathcal{A}_2 \left(f_2 \cdot f_2 + 2 f_4 \cdot 1\right) = 2\mathrm{e}^{-2kt}\left(g_1 g_3^* + g_3 g_1^*\right); \tag{19}$$

5) for the coefficient of χ^5

$$\mathcal{A}_1 \left(g_1 \cdot f_4 + g_3 \cdot f_2\right) = 0; \tag{20}$$

6) for the coefficient of χ^6

$$\mathcal{A}_2 \left(f_2 \cdot f_4 + f_4 \cdot f_2\right) = 2\mathrm{e}^{-2kt} g_3 g_3^*; \tag{21}$$

7) for the coefficient of χ^7

$$\mathcal{A}_1 \left(g_3 \cdot f_4\right) = 0. \tag{22}$$

8) for the coefficient of χ^8

$$\mathcal{A}_2 \left(f_4 \cdot f_4\right) = 0. \tag{23}$$

As discussed in the section before, we can solve Eqs. (16)–(23) in turn for getting the expressions of G and F. First, we get

$$g_1 = \mathrm{e}^{kt}\left(\mathrm{e}^{\eta_1} + \mathrm{e}^{\eta_2}\right), \tag{24}$$

where $\eta_j = P_j\left(x + \mathrm{i}\, P_j t\right) - \mathrm{i}(\sin t + t)$, $j = 1, 2$, with P_j being complex constant, respectively. Substituting Eq. (24) into Eq. (17), we obtain the expression of f_2

$$f_2 = \sum_{m,n} A_{mn}\mathrm{e}^{\eta_m + \eta_n^*}, \tag{25}$$

where $A_{mn} = \mathrm{e}^{-2kt}/\left(P_m + P_n^*\right)^2$, $m, n = 1, 2$, and P_m (P_n) being complex. With the help of Eqs. (24) and (25), we can simplify Eq. (18) as

$$\begin{aligned} &\mathrm{i}\, g_{3t} + g_{3xx} - 2\cos^2\left(\frac{t}{2}\right) g_3 - \mathrm{i}\, k g_3 \\ &= \mathrm{e}^{kt}\frac{2\left(P_1 - P_2\right)^2 \mathrm{e}^{\eta_1 + \eta_1^* + \eta_2}}{(P_2 + P_1^*)(P_1 + P_1^*)} \\ &\quad + \mathrm{e}^{kt}\frac{2\left(P_1 - P_2\right)^2 \mathrm{e}^{\eta_1 + \eta_2^* + \eta_2}}{(P_1 + P_2^*)(P_2 + P_2^*)}. \end{aligned}$$

The above equation implies that g_3 has the solution

$$g_3 = \mathrm{e}^{kt}\left(\mathrm{e}^{\eta_1+\eta_1^*+\eta_2+\delta_1} + \mathrm{e}^{\eta_1+\eta_2^*+\eta_2+\delta_2}\right), \quad (26)$$

where $\delta_1 = \ln[(1/(P_1+P_1^*) - 1/(P_2+P_1^*))^2]$ and $\delta_2 = \ln[(1/(P_1+P_2^*) - 1/(P_2+P_2^*))^2]$. Substituting Eqs. (24)–(26) into Eq. (19), we obtain

$$f_4 = \mathrm{e}^{\eta_1+\eta_1^*+\eta_2+\eta_2^*+\kappa}, \quad (27)$$

where

$$\kappa = \ln\frac{|P_2-P_1|^4}{(P_1+P_1^*)^2(P_2+P_2^*)^2\,|P_1+P_2^*|^4}. \quad (28)$$

Combining Eqs. (24)–(27) one can find that Eqs. (20) and (23) are satisfied to the moment. Taking $P_j \equiv \alpha_j + \mathrm{i}\,\beta_j$, $j = 1, 2$, the exact bright two-soliton solution of Eq. (2) has the form

$$\psi = \mathrm{e}^{kt}\frac{\mathrm{e}^{\eta_1} + \mathrm{e}^{\eta_2} + \mathrm{e}^{\eta_1+\eta_1^*+\eta_2+\delta_1} + \mathrm{e}^{\eta_1+\eta_2^*+\eta_2+\delta_2}}{1 + \sum\limits_{m,n} A_{mn}\mathrm{e}^{\eta_m+\eta_n^*} + \mathrm{e}^{\eta_1+\eta_1^*+\eta_2+\eta_2^*+\kappa}}, \quad (29)$$

where $m, n = 1, 2$, $\eta_j = \theta_j + \mathrm{i}\,\varphi_j$ with $\theta_j = \alpha_j x - 2\alpha_j\beta_j t - \ln(2\alpha_j)$ and $\varphi_j = x\beta_j - \left(\beta_j^2 - \alpha_j^2 + 1\right)t - \sin t$, $j = 1, 2$. The solution Eq. (29) describes a general scattering process of two bright solitons with different central velocity V_1 and V_2 with $V_j = 2\beta_j$, $j = 1, 2$, respectively. The two solitons can move toward each other if we choose the proper parameters. For better understand the nature of two solitons interaction, we can analyse the asymptotic behaviour of the solution Eq. (29). Asymptotically, the solution Eq. (29) can be written as a combination of two one-soliton solution Eq. (14). The asymptotic form of two soliton solutions in limits $t \to -\infty$ and $t \to \infty$ is similar to that of the one soliton solution Eq. (14).

(i) Before collision (limit $t \to -\infty$)

(a) soliton 1 ($\theta_1 \approx 0$, $\theta_2 \to -\infty$)

$$\psi \to \frac{\alpha_1\mathrm{e}^{kt}}{\cosh\theta_1}\mathrm{e}^{\mathrm{i}\,\varphi_1}; \quad (30)$$

(b) soliton 2 ($\theta_2 \approx 0$, $\theta_1 \to \infty$)

$$\psi \to \frac{\alpha_2\mathrm{e}^{kt}}{\cosh(\theta_2 + \kappa/2 - x_{01})}\mathrm{e}^{\mathrm{i}(\varphi_2+\varphi_0)}. \quad (31)$$

(ii) After collision (limit $t \to \infty$):

(a) soliton 1 ($\theta_1 \approx 0$, $\theta_2 \to \infty$)

$$\psi_2 \to \frac{\alpha_1\mathrm{e}^{kt}}{\cosh(\theta_1 + \kappa/2 - x_{02})}\mathrm{e}^{\mathrm{i}(\varphi_1+\varphi_0)}; \quad (32)$$

(b) soliton 2 ($\theta_2 \approx 0$, $\theta_1 \to -\infty$)

$$\psi_2 \to \frac{\alpha_2\mathrm{e}^{kt}}{\cosh\theta_2}\mathrm{e}^{\mathrm{i}\,\varphi_2}. \quad (33)$$

Where

$$x_{0j} = \ln\sqrt{2\alpha_j}, \quad (34)$$

$$\varphi_0 = 2\left(\arctan\frac{\beta_1-\beta_2}{\alpha_1-\alpha_2} + \arctan\frac{\beta_1-\beta_2}{\alpha_1+\alpha_2}\right). \quad (35)$$

The above asymptotic behaviour shows that there is a centre and phase shift in the process of two solitons collision which denotes the elastic interaction of two solitons.

2.2. Dark soliton solution

In this section we consider the case of $\mu = -1$. In order to get the exact dark one-soliton solution of Eq. (2), a real parameter λ to be determined should be added to Eq. (4) and we have

$$\mathcal{G}_1 G\cdot F = 0, \quad \mathcal{G}_2 F\cdot F = -2\mathrm{e}^{-2kt}\,|G|^2, \quad (36)$$

where

$$\mathcal{G}_1 = \mathrm{i}\,D_t + D_x^2 - 2\cos^2\left(\frac{t}{2}\right) - \mathrm{i}\,k - \lambda, \quad (37)$$

$$\mathcal{G}_2 = D_x^2 - \lambda. \quad (38)$$

Repeating the same normal procedure in getting the exact bright soliton solutions, we will obtain analytically the expressions of G and F and construct the exact dark soliton solutions of Eq. (2).

2.2.1. Exact dark one-soliton solution

In order to construct the dark one-soliton solution for Eq. (2), we assume that

$$G = g_0(1 + \chi g_1), \quad F = 1 + \chi f_1. \quad (39)$$

Substituting the above expressions into Eq. (36) and collecting the coefficients with same power in χ, we have:

1) for the coefficient of χ^0

$$\mathcal{G}_1 g_0\cdot 1 = 0, \quad (40)$$

$$\mathcal{G}_2 1\cdot 1 + 2\mathrm{e}^{-2kt} g_0 g_0^* = 0; \quad (41)$$

2) for the coefficient of χ^1

$$\mathcal{G}_1 g_0\,(1\cdot f_1 + g_1\cdot 1) = 0, \quad (42)$$

$$\mathcal{G}_2\,(f_1\cdot 1 + 1\cdot f_1) + 2\mathrm{e}^{-2kt}|g_0|^2\,(g_1 + g_1^*) = 0; \quad (43)$$

3) for the coefficient of χ^2

$$\mathcal{G}_1 g_0\,(g_1\cdot f_1) = 0, \quad (44)$$

$$\mathcal{G}_2\,(f_1\cdot f_1) + 2\mathrm{e}^{-2kt}|g_0|^2|g_1|^2 = 0. \quad (45)$$

Solving Eqs. (40) and (41), we obtain

$$g_0 = \gamma e^{i\,\eta_0}, \tag{46}$$

where $\eta_0 = Bx - \left(\lambda + B^2\right)t - (\sin t + t)$, in which B is real, and $\gamma = e^{2kt}\sqrt{\lambda/2}$ which shows that the existence of dark soliton solution demands the parameter $\lambda > 0$. Substituting (46) into Eqs. (42) and (43), we obtain the expressions of g_1 and f_1 as

$$g_1 = ze^{\eta_1}, \quad f_1 = e^{\eta_1}, \tag{47}$$

where $\eta_1 = Rx - \Omega t + \xi_0$, with the real R, Ω and ξ_0, respectively. The complex parameter z is given by

$$z = \frac{\sqrt{2\lambda - R^2} + i\,R}{\sqrt{2\lambda - R^2} - i\,R}, \tag{48}$$

which implies that $|z|^2 = 1$ and $\lambda \geq R^2/2$. With Eqs. (3), (39), (46) and (47), after absorbing the auxiliary parameter χ we obtain the exact dark one-soliton solution of Eq. (2) as

$$\psi = \frac{1}{2}\gamma e^{i\,\eta_0}\left[(1+z) - (1-z)\tanh\frac{\eta_1}{2}\right]. \tag{49}$$

From the above solution we can see two special cases, i.e., $z = \pm 1$. When $z = 1$, the solution Eq. (49) reduces to plane-wave solution $\psi = e^{2kt + i\,\eta_0}\sqrt{\lambda/2}$. When $z = -1$ the solution Eq. (49) reduces to the black soliton, i.e., $\psi = -e^{2kt + i\,\eta_0}\sqrt{\lambda/2}\tanh(\eta_1/2)$, where $\eta_1 = Rx - \Omega t + \xi_0$. The dynamic dark soliton solution shows that with the increasing time the dark solitary wave has an increase in peak value and the soliton width shows invariability. We also obtain the soliton peak value $|\psi|_{\min} = e^{2kt}\left(\sqrt{2\lambda} - R^2/\sqrt{2\lambda}\right)$ which is affected by the soliton width defined by $1/R$.

2.2.2. Dark two-soliton solution

In this section, we will give the analytical expression of two dark two-soliton solution of Eq. (2). To this purpose the expressions of G and F can be assumed as

$$G = g_0\left(1 + \chi g_1 + \chi^2 g_2\right), \quad F = 1 + \chi f_1 + \chi^2 f_2, \tag{50}$$

where the expression of g_0 has been obtained before. Repeating the same normal procedure and collecting the coefficients with the same power in χ we obtain:

1) for the coefficient of χ

$$\mathcal{L}_1(1 \cdot f_1 + g_1 \cdot 1) = 0, \tag{51}$$

$$\mathcal{G}_2(1 \cdot f_1 + f_1 \cdot 1) = \lambda(g_1 + g_1^*); \tag{52}$$

2) for the coefficient of χ^2

$$\mathcal{L}_1(g_1 \cdot f_1 + 1 \cdot f_2 + g_2 \cdot 1) = 0, \tag{53}$$

$$\mathcal{G}_2(1 \cdot f_2 + f_1 \cdot f_1 + f_2 \cdot 1) = -\lambda(|g_1|^2 + g_2 + g_2^*). \tag{54}$$

3) for the coefficient of χ^3

$$\mathcal{L}_1(g_1 \cdot f_2 + g_2 \cdot f_1) = 0, \tag{55}$$

$$\mathcal{G}_2(f_1 \cdot f_2 + f_2 \cdot f_1) = -\lambda(g_1 g_2^* + g_2 g_1^*); \tag{56}$$

4) for the coefficient of χ^4

$$\mathcal{L}_1(g_2 \cdot f_2) = 0, \tag{57}$$

$$\mathcal{G}_2 f_2 \cdot f_2 = -\lambda|g_2|^2. \tag{58}$$

Where $\mathcal{L}_1 = i\,D_t + 2\,i\,BD_x + D_x^2$ and $\mathcal{G}_2 = D_x^2 - \lambda$. We solve Eqs. (51)–(58) in turn, and can easily find the following solutions:

$$g_1 = z_1 e^{\eta_1} + z_2 e^{\eta_2}, \quad f_1 = e^{\eta_1} + e^{\eta_2}, \tag{59}$$

$$g_2 = A_{12} z_1 z_2 e^{(\eta_1 + \eta_2)}, \quad f_2 = A_{12} e^{(\eta_1 + \eta_2)}, \tag{60}$$

where $\eta_j = R_j x - (2B - \sqrt{2\lambda - R_j^2})R_j t + \zeta_j, j = 1, 2$. The other parameters are given by

$$z_j = \frac{\sqrt{2\lambda - R_j^2} + i\,R_j}{\sqrt{2\lambda - R_j^2} - i\,R_j}, \tag{61}$$

$$A_{12} = \frac{2\lambda - R_1 R_2 - \sqrt{2\lambda - R_1^2}\sqrt{2\lambda - R_2^2}}{2\lambda + R_1 R_2 - \sqrt{2\lambda - R_1^2}\sqrt{2\lambda - R_2^2}}. \tag{62}$$

With the forms of g_0, g_1, g_2, f_1, f_2 and Eq. (3), we obtain the analytical form of exact dark two-soliton solution for growing BECs

$$\psi = \frac{1}{2}\gamma e^{i\,\eta_0}\frac{1 + z_1 e^{\eta_1} + z_2 e^{\eta_2} + A_{12} z_1 z_2 e^{\eta_1 + \eta_2}}{1 + e^{\eta_1} + e^{\eta_2} + A_{12} e^{\eta_1 + \eta_2}}. \tag{63}$$

The solution Eq. (63) describes a general scattering process of two dark soliton solutions with different centre velocity V_1 and V_2 with $V_j = 2B - \sqrt{2\lambda - R_j^2}, j = 1, 2$. Under the proper parameters two soliton solution can move toward each other. To clearly demonstrate the nature of two solitons interaction, we analyse the asymptotic behaviour of the solution Eq. (63). Asymptotically, the solution Eq. (63) can be written as a combination of two soliton solutions expressed by Eq. (49). The asymptotic form of two soliton solutions in limits $t \to -\infty$ and $t \to \infty$ is similar to that of one soliton solution Eq. (49).

(i) Before collision (limit $t \to -\infty$):

(a) soliton 1 ($\eta_1 \approx 0$, $\eta_2 \to -\infty$)

$$\psi = \frac{1}{2}\gamma e^{i\,\eta_0}\left[(1 + z_1) - (1 - z_1)\tanh\frac{\eta_1}{2}\right]; \tag{64}$$

(b) soliton 2 ($\eta_2 \approx 0$, $\eta_1 \to \infty$)

$$\psi = \frac{1}{2}\gamma z_1 \mathrm{e}^{\mathrm{i}\,\eta_0}\Bigg[(1+z_2)-(1-z_2) \times \tanh\left(\frac{1}{2}(\eta_2+\delta_0)\right)\Bigg]. \quad (65)$$

(ii) After collision (limit $t \to \infty$):
(a) soliton 1 ($\eta_1 \approx 0$, $\eta_2 \to \infty$)

$$\psi = \frac{1}{2}\gamma z_2 \mathrm{e}^{\mathrm{i}\,\eta_0}\Bigg[(1+z_1)-(1-z_1) \times \tanh\left(\frac{1}{2}(\eta_1+\delta_0)\right)\Bigg]; \quad (66)$$

(b) soliton 2 ($\eta_2 \approx 0$, $\eta_1 \to -\infty$).

$$\psi = \frac{1}{2}\gamma \mathrm{e}^{\mathrm{i}\,\eta_0}\left[(1+z_2)-(1-z_2)\tanh\frac{\eta_1}{2}\right]. \quad (67)$$

Where the central shift is given by $\delta_0 = \ln A_{12}$. By analysing the asymptotic behaviour in detail we know that there is no change of the amplitude for each soliton solution during a collision, while one should notice that the factor $|z_j| = 1$, $j = 1, 2$, respectively. However, from Eqs. (64)–(67) we find a phase exchange δ_0 for soliton 1 and soliton 2 during a collision. These results show that the collision of two soliton solutions is elastic.

3. Conclusion

In conclusion, we have developed a systematic analytical approach to study the nonlinear dynamic properties of quasi-one dimensional growing BECs. The Hirota method is developed carefully for solving such a nonlinear Schrödinger equation. With the skillful assumption the exact bright and dark soliton solutions are constructed effectively. Our results show that the growing model has the important effect on the amplitude and the time-dependent potential only contributes to the phase and phase velocity. A detailed analysis shows that the amplitude of one-soliton solution changes with time, while the collision of two soliton solutions is elastic.

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