Stabilised bright solitons in Bose–Einstein condensates in an expulsive parabolic and complex potential*

Zhang Tao(张 涛)[a)], Yang Zhan-Ying(杨战营)[a)],
Zhao Li-Chen(赵立臣)[a)], and Yue Rui-Hong(岳瑞宏)[a)b)]

[a)] *Department of Physics, Northwest University, Xi'an* 710069, *China*
[b)] *Department of Physics, Ningbo University, Ningbo* 315211, *China*

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The exact solitonic solutions of the one-dimensional nonlinear Schrödinger equation, which describes the dynamics of bright soliton in Bose–Einstein condensates with the time-dependent interaction in an expulsive parabolic and complex potential, are obtained by Darboux transformation. The results show that one can compress a bright soliton into an assumed peak of matter wave density by adusting the experimental parameter of the ratio of axial oscillation to radial oscillation or feeding parameter. Especially, when parameters satisfy the relation $\lambda = 2\gamma$, the soliton is stable with time evolution without changing its shape and amplitude.

Keywords: Bose–Einstein condensates, one-dimensional Gross–Pitaevskii equation, bright soliton
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1. Introduction

Since their appearance in shallow water, solitons have intrigued great interest in experimental and theoretical investigation. Due to the dynamical balance between nonlinearity and dispersion, solitons can propagate stably as pulses or wave packets in a nonlinear medium, which can be described in terms of nonlinear differential equations.[1−3] Afterwards, physicists extended the classical solitons to describe elementary particles, and investigated nonautonomous solitons in different physical system.[4−6] Actually, due to the presence of dissipation, such as impurity influence, the soliton inevitably fades away. So we want to construct a 'pure' system eliminating this influence to study the dynamics of soliton, and the realisation of Bose–Einstein condensation (BEC) of alkali atoms can provide us such a system to study solitons and thus opens a new way to realise an ideal soliton transmission.

The wave function of BECs at zero temperature can be described by multi-dimensional nonlinear Schrödinger equation with a trapped potential, the so-called Gross–Pitaevskii (GP) equation. The dynamical properties of BECs in different systems have been studied[7] and opened a new way to investigate the nonlinear phenomena in matter wave. Because of nonlinear interaction between atoms, the exact solution of the system is scarce and most are focused on numerical solution,[8,9] so the construction of the exact solution is most important. In the case of reduced dimensionality, one can derive effective one-dimensional (1D) GP-equation which describes cigar-shaped condensates from 3D GP-equation.[10,11] To exactly solve the 1D GP-equation, one also can use inverse scattering method or Bethe ansatz method to obtain the analytic solitary solution directly from the GP-equation. The advent of solitons in BECs has paved the way for new development in manipulating the soliton, such as atom lasers and atom interferometry, so it is interesting to develop a new technique which allows us to construct a particular soliton with the assumed peak. Liang *et al.*[12] have presented a family of exact soliton solutions by Darboux transformation and analysed the dynamics of bright soliton. Their results showed that the bright solitons can be compressed into very high matter densities by adjusting the atom scattering length in a safe range of parameters, while the exact number of the atoms in bright soliton keeps invariant during the process of compression, so the peak of matter wave can be squeezed between the minimum and maximum values. Recently, experimental research has reported the observation of different types of solitons in 1D potential, such as dark solitons for the repul-

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†Corresponding author. E-mail: tzhang505@eyou.com
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sive atomic interaction,[13−18] and bright solitons for the attractive atomic interaction,[19−23] which can be controlled via Feshbach resonance, some theoretical soliton results are also obtained.[24−26] However, the presence of dissipation effect makes the solitons depicted above unstable with time evolution, so that they will inevitably decay and eventually disappear in background cold atom. By feeding atoms from thermal cloud, which can compensate the dissipation effect, we can realise an almost ideal soliton evolution, which is discussed in the present paper.

In this paper, we present an analytic bright soliton solution of BECs in an expulsive parabolic and complex potential by Darboux transformation[27] from the trivial seed solution. In a safe range of parameters, our results show that the bright soliton in BECs can be compressed into an assumed peak of matter wave by adjusting the time-dependent scattering length, which is dependent on the experimental parameter of the ratio of axial oscillation to radial oscillation(λ) or feeding parameter(γ). The peak of matter wave increases with the increase of the ratio, while the width is compressed, but the width cannot be compressed by feeding atom from thermal cloud. Especially, when $\lambda = 2\gamma$, the soliton is stable with time evolution without changing its shape and amplitude.

2. The soliton solutions

The mean-field GP equation governs the evolution of the macroscopic wave function of BECs. Here we consider the 1D nonlinear Schrödinger equation with time-dependent atomic scattering length in an expulsive parabolic and complex potential[24−26,28,29]

$$\mathrm{i}\,\partial\Psi/\partial t + \partial^2\Psi/\partial x^2 + a(t)|\Psi|^2\Psi + \frac{1}{4}\lambda^2 x^2\Psi + \mathrm{i}\,\gamma\Psi = 0, \quad (1)$$

where time t and coordinate x are measured in units $2/\omega_\perp$ and $a_\perp$, with $a_\perp = \sqrt{\hbar/m\omega_\perp}$ and $a_0 = \sqrt{\hbar/m\omega_0}$ being linear oscillator lengths in the transverse and cigar-axis directions respectively; the parameter $\omega_\perp$ and ω_0 are the radial and axial oscillation frequency respectively, and $|\lambda| = 2|\omega_0|/\omega_\perp \ll 1$. The parameter m is the atomic mass and the last term of Eq. (1) is the complex potential which is a small parameter related to the feeding of condensates from the thermal cloud (when $\gamma < 0$).[24,25] The function $a(t) = 2g_0\mathrm{e}^{\lambda t+2\gamma t}$ is the time-dependent scattering length depending on the feeding parameter(γ), and $g_0 = \pi a_s m\omega_\perp N/\sqrt{2\pi\eta k}$, here k is the parameter reflecting the feeding parameter to the scattering length at $t = 0$. Liang *et al.* presented a family of exact soliton solution of Eq. (1) with $\gamma = 0$, their result showed that the bright soliton can be compressed into high matter densities by increasing the scattering length. Kengne and Talla[26] studied Eq. (1) with $a(t) = g_0$ and $\gamma = -\lambda/2$, they found that the amplitude of bright soliton and the number of its peaks increase with increasing time t when feeding atom from thermal cloud.

We assume that the solution of Eq. (1) is

$$\Psi(x,t) = Q(x,\ t)\exp[-\mathrm{i}\,\lambda x^2/4 - \lambda t/2 - \gamma t], \quad (2)$$

where $Q(x,\ t)$ satisfies the equation

$$\mathrm{i}\,Q_t + Q_{xx} - \mathrm{i}\,\lambda x Q_x - \mathrm{i}\,\lambda Q + 2g_0|Q|^2Q = 0.$$

From this ansatz, we construct a new exact solution of Eq. (1) by the Lax-pair method where the pair function should admit $\Phi_x = U\Phi$, $\Phi_t = V\Phi$, $\Phi = (\Phi_1,\ \Phi_2)^{\mathrm{T}}$, with

$$U = \begin{pmatrix} \zeta & \sqrt{g_0}Q \\ -\sqrt{g_0}Q^* & \zeta \end{pmatrix},$$
$$V = \begin{pmatrix} 2\,\mathrm{i}\,\zeta^2 + \lambda\zeta x + \mathrm{i}\,g_0|Q|^2 & \sqrt{g_0}(2\,\mathrm{i}\,\zeta Q + \mathrm{i}\,Q_x + \lambda x Q) \\ \sqrt{g_0}(-2\,\mathrm{i}\,\zeta Q^* + \mathrm{i}\,Q_x^* - \lambda x Q^*) & -2\,\mathrm{i}\,\zeta^2 - \lambda\zeta x - \mathrm{i}\,g_0|Q|^2 \end{pmatrix}, \quad (3)$$

Q^* is the complex conjugate of Q, $\zeta = (\alpha + \mathrm{i}\,\beta)\mathrm{e}^{\lambda t}$ is a time-dependent complex function, U and V satisfy the relation $U_t - V_x + UV - VU = 0$. Performing the Darboux transformation from the trivial seed solution, we can obtain the exact solitonic solution of Eq. (1)

$$\Psi = \frac{4\alpha A_{\mathrm{c}}\exp\theta}{\sqrt{g_0}[1 + A_{\mathrm{c}}^2\exp\varphi]}, \quad (4)$$

where

$$\theta = 2\,\mathrm{i}(\alpha - \mathrm{i}\,\beta)^2 \mathrm{e}^{2\lambda t}/\lambda - 2(\alpha - \mathrm{i}\,\beta)x\mathrm{e}^{\lambda t} + \lambda t/2 - \mathrm{i}\,\lambda x^2/4 - \gamma t,$$
$$\varphi = -4\alpha x \mathrm{e}^{\lambda t} + 8\alpha\beta \mathrm{e}^{2\lambda t}/\lambda.$$

Here A_c, α and β are arbitrary real constants. If $\alpha = 0$ or $A_\mathrm{c} = 0$, the solution $\Psi = 0$ is trivial, thus we discuss the equations at α, $A_\mathrm{c} \neq 0$. Since λ is related to the oscillation frequency, A_c, α and β can be adjusted. If $\gamma = -\lambda/2$, $A_\mathrm{c} = 0.5$, $2\alpha = \sqrt{g_0}A_\mathrm{s}$ and $2\beta = k_\mathrm{s}$, then Eq. (4) becomes the well-known solitary solution $\Psi = (A_\mathrm{s}/\cosh\theta_\mathrm{s})\exp(\lambda t + \mathrm{i}\,\phi_\mathrm{s})$, with $\phi_\mathrm{s} = [k^2 - g_0 A_\mathrm{s}^2][\exp(2\lambda t) - 1]/2\lambda - k_\mathrm{s} x \exp(\lambda t) - (\lambda/4)x^2$ and $\theta_\mathrm{s} = \sqrt{g_0}A_\mathrm{s} x \exp(\lambda t) - g_0 A_\mathrm{s} k_\mathrm{s}[\exp(2\lambda t) - 1]/\lambda$. From Eq. (4), we know that when $x \to \pm\infty$, $\phi \to 0$, the solution given by Eq. (4) is the bright soliton embedded in the background. The evolution of the soliton depends on the time-dependent atomic scattering length and the feeding parameter.

From the solution of Eq. (4), one can obtain

$$|\,\Psi|^2 = \frac{16\alpha^2 A_\mathrm{c}^2 \exp[\varphi + \lambda t - 2\gamma t]}{g_0[1 + A_\mathrm{c}^2 \exp\varphi]^2}. \tag{5}$$

Because the matter wave density is dependent on λ and γ, so we plot the dependence of density distribution on λ and γ in Figs. 1 and 2 respectively. Here we choose the experimental parameter as that in Ref. [16], $\omega_\perp = 2\pi \times 710$ Hz, so a unit scale in coordinate $\Delta x = 1$ in the dimensionless variables corresponds to 1.4 μm. We take the constants $g_0 = 0.25$, $\alpha = 2$, $\beta = 0.2$ and $A_\mathrm{c} = 100$. Figure 1 shows the evolution of matter wave density when $\lambda = -0.012$, from these values and $\lambda = -2|\omega_0|/\omega_\perp$, we can obtain $\omega_0 = 2\,\mathrm{i}\,\pi \times 8.5$ Hz. We define the width of half wave density as the width of soliton, that is $W_\mathrm{s} = (\mathrm{e}^{-\lambda t}/4\alpha)\ln[(3 + 2\sqrt{2})/(3 - 2\sqrt{2})]$. The result shows that the peak of matter wave density increases while its width is compressed(see the inset of Fig. 1) with the increase of the ratio of oscillation frequencies. So we can adjust the oscillation frequency to compress a bright soliton in BECs into an assumed peak of matter density. By comparing the matter waves with different λ, the soliton contrail with time evolution moves to $-x$ direction when λ increases, as shown in the inset of Fig. 1, where the offset of x is about 13.7 when $t = 20$. Actually, this offset is due to coordinate x in Eq. (5), so one can add a coordinate-dependent field to manipulate the soliton evolution, this is worth to consider further. Figure 2 shows the matter wave density for to the feeding parameter $\gamma = -0.02$, the peak of wave density increases as its width is broaden with the time evolution. If the feeding parameter decreases (feeding more atom from thermal cloud per second), the peak of matter wave density increases, but the width cannot be compressed at the same time, which are shown in inset of Fig. 2, because the potential cannot compensate the dispersion of adding more atoms, this conclusion is different from that in Ref. [26].

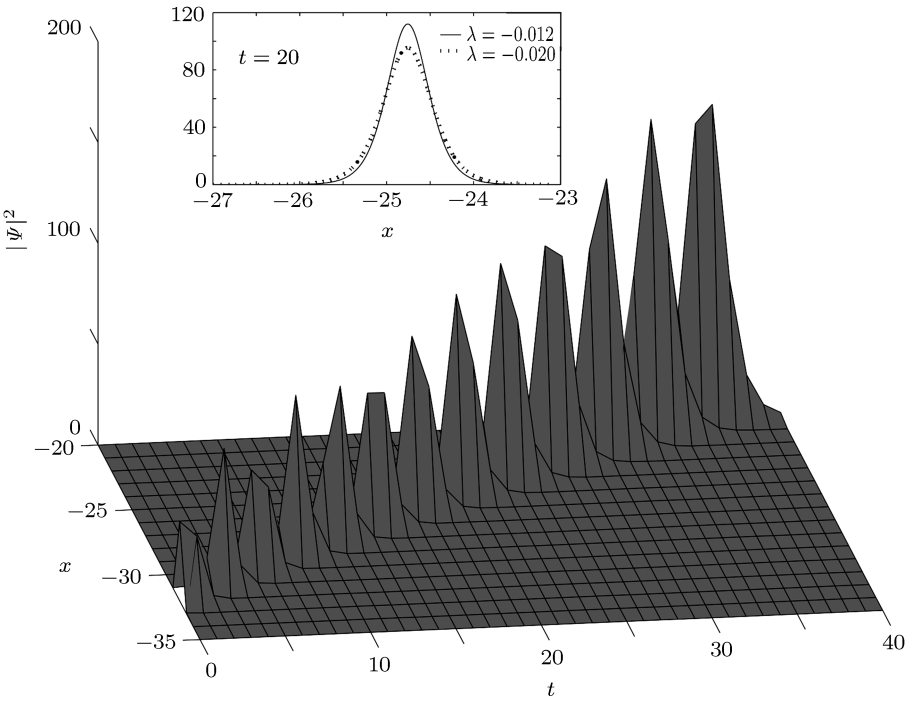


Fig. 1. The evolution of matter wave density with $\lambda = -0.012$, $\alpha = 2$, $\beta = 0.2$, $A_\mathrm{c} = 100$, $\gamma = -0.018$, and $g_0 = 0.25$. The inset shows the matter wave density for different λ at $t = 20$.

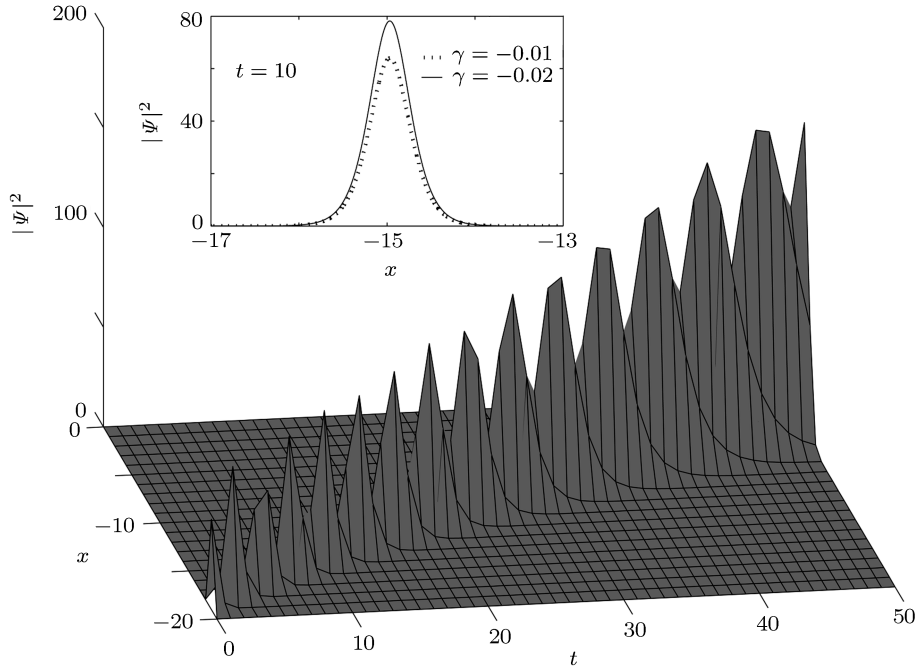


Fig. 2. The evolution of matter wave density with $\gamma = -0.02$, where $\alpha = 2$, $\beta = 0.2$, $A_c = 100$, $\lambda = -0.02$, and $g_0 = 0.25$. The inset shows the matter wave density for different γ at $t = 10$.

We further investigate this by calculating the exact number of atoms in the bright soliton against the background and obtain

$$\lim_{L\to\infty}\int_{-L}^{+L}[|\Psi(x,t)|^2 - |\Psi(\pm L,t)|^2]\mathrm{d}x = \frac{4\alpha}{g_0}\exp(-2\gamma t), \tag{6}$$

where $2L$ is the length of the spatial background and must be very large compared to the width of the soliton.[12,24] From Eq. (6) one can see that the scattering length decreases when feeding more cold atom from thermal cloud, but the matter wave possesses an increase in atomic number, so the peak increases. One can also observe that the soliton evolution with time has the same contrail for different γ, this result is different from that of changing λ. From the above results, one can control the experimental parameters, through changing λ or γ, to compress a bright soliton in BECs into an assumed peak of matter density. In Ref. [12], they have reported that the peak of soliton decreases and its width is broadened with decreasing scattering length when $\gamma = 0$. If one feeds cold atom from thermal cloud to background, the dissipative effect can be suppressed. When $\lambda = -0.04$ and $\gamma = -0.005$, the density decreases with time (shown in Fig. 3(a)), and the peak of soliton increased by feeding atoms cannot compensate that decreased by the ratio of oscillation frequency. However, when $\lambda = -0.03$ and $\gamma = -0.02$, the density increases with time (shown in Fig. 3(b)). So one can adjust the parameters to keep the bright soliton stabilised with time evolution. We found that when $\lambda = 2\gamma$, the maximum of wave density should satisfy the relation $A_c^2 \exp\varphi - 1 = 0$, and

$$|\Psi|^2_{\max} = 4\alpha^2/g_0, \tag{7}$$

which is a constant related to α and g_0. From Eq. (7), we know that the maximum of bright soliton is unchanged with time (shown in Figs. 3(c) and 3(d) when $\lambda = -0.02$ and $\gamma = -0.01$), and the number of atoms in bright soliton increases, while the width increases slightly. The stabilised soliton has been confirmed by many experiments in other papers,[30,31] and its applications have been extensively explored.[32,33] We expect the stabilized soliton in BECs will be investigated experimentally.

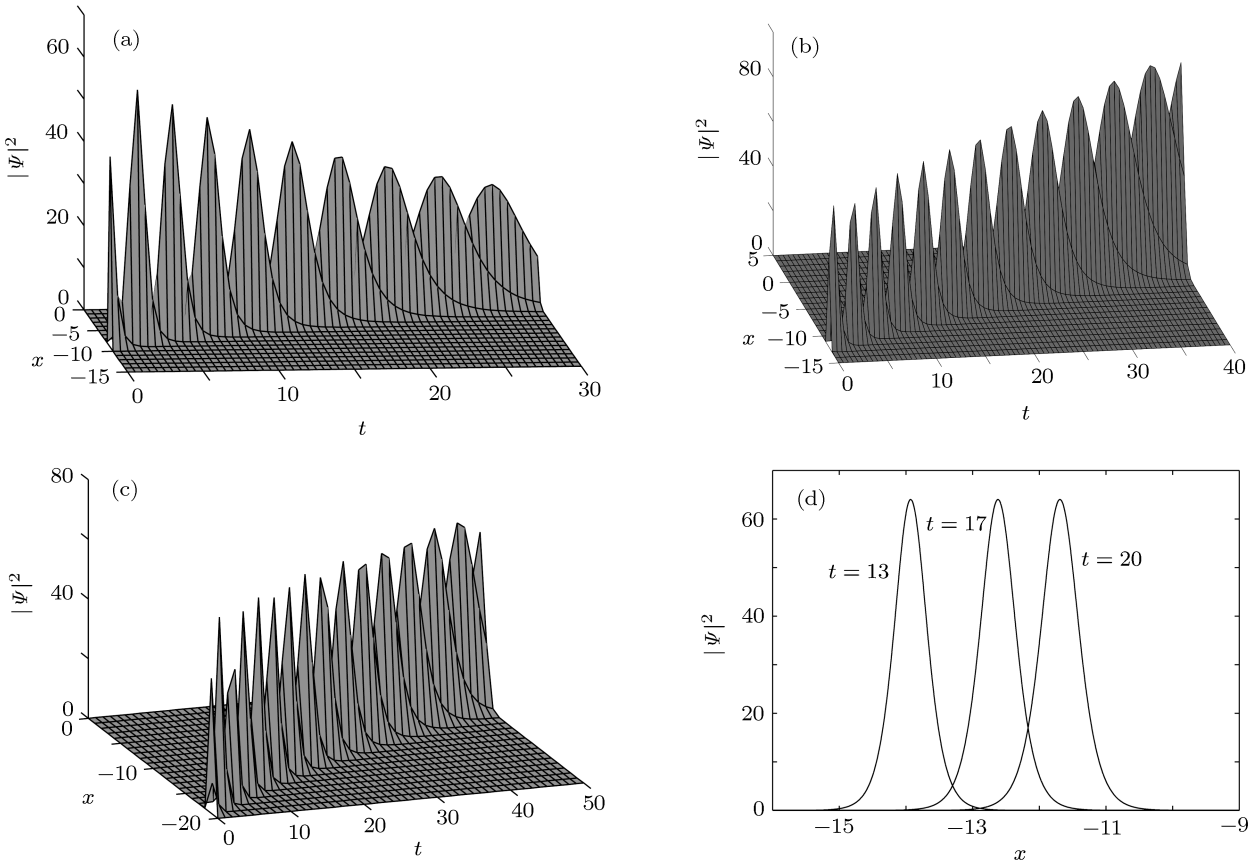


Fig. 3. The evolution of matter wave density for different γ and λ, for $\alpha = 2$, $\beta = 0.2$, $A_c = 100$, and $g_0 = 0.25$. (a) $\lambda = -0.04$, $\gamma = -0.005$; (b) $\lambda = -0.03$, $\gamma = -0.02$; (c) $\lambda = -0.02$, $\gamma = -0.01$; and (d) 2D plot of (c) for different time.

3. Conclusion

In conclusion, we have studied the bright solitonic solution in BECs by solving the 1D time-dependent nonlinear Schrödinger equation with an expulsive parabolic and complex potential by Darboux transformation. From our results, one can observe that the matter wave density of bright soliton is influenced greatly by scattering length, which can be adjusted by the ratio of oscillation frequency ($|\lambda|$) and the feeding parameter (γ). The bright soliton with time evolution in BECs can be compressed into an assumed peak by adjusting the scattering length, but its width cannot be compressed by adjusting feeding parameter. Especially, when $\lambda = 2\gamma$, the soliton is stable with time evolution without changing its shape and amplitude.

References

[1] Zabusky N J and Kruskal M D 1965 *Phys. Rev. Lett.* **15** 240

[2] Hirota R and Suzuki K 1973 *Proc. IEEE* **61** 1483

[3] Lonngren K and Scott A (editor) 1978 *Solitons in Action* (New York: Academic)

[4] Serkin V N, Hasegawa A and Belyaeva T L 2007 *Phys. Rev. Lett.* **98** 074102

[5] Serkin V N and Hasegawa A 2000 *Phys. Rev. Lett.* **85** 4502

[6] Wu L, Zhang J F, Li L, Finot C and Porsezian K 2008 *Phys. Rev.* A **78** 053807

[7] Eğrifes H, Demirhan D and Buyclkkilic F 2000 *Phys. Lett.* A **275** 229

[8] Dalfovo F and Stringari S 1996 *Phys. Rev.* A **53** 2477

[9] Dodd R J, Edwards M, Williams C J and Clark C W 1996 *Phys. Rev.* A **54** 661

[10] Salasnich L, Parola A and Reatto L 2002 *Phys. Rev.* A **65** 043614

[11] Wang S J Jia C L, Zhao D, Luo H G and An J H 2003 *Phys. Rev.* A **68** 015601

[12] Liang Z X, Zhang Z D and Liu W M 2005 *Phys. Rev. Lett.* **94** 050402

[13] Burger S, Bongs K, Dettmer S, Ertmer W and Sengstack K 1998 *Phys. Rev. Lett.* **83** 5198

[14] Busch T and Anglin J R 2000 *Phys. Rev. Lett.* **84** 2298

[15] Chen R X, Yang R S and Yao C M 2009 *Chin. Phys.* B **18** 3736
[16] Hu X H, Zhang X F, Zhao D, Luo H G and Liu W M 2009 *Phys. Rev.* A **79** 023619
[17] Anderson B P, Haljan P C, Regal C A, Feder D L, Collins L A, Clark C W and Cornell E A 2001 *Phys. Rev. Lett.* **86** 2926
[18] Wu B, Liu J and Niu Q 2002 *Phys. Rev. Lett.* **88** 034101
[19] Strecker K E, Partridge G B, Truscott A G and Hulet R G 2002 *Nature* **417** 150
[20] Wang Y Y and Zhang J F 2009 *Chin. Phys.* B **18** 1168
[21] Kevrekidis P G, Theocharis G, Frantzesskakis D J and Malomed B A 2003 *Phys. Rev. Lett.* **90** 230401
[22] Khaykovich L, Schreck F, Ferrari G, Bourdel T, Cubizolles J, Carr L D, Castin Y and Salomon C 2002 *Science* **296** 1290
[23] Eiermann B, Anker T, Albiez M, Taglieber M, Treutlein P, Marzlin K P and Oberthaler M K 2004 *Phys. Rev. Lett.* **92** 230401
[24] Li B, Zhang X F, Li Y Q, Chen Y and Liu W M 2008 *Phys. Rev.* A **78** 023608
[25] Radha R and Kumar V R 2007 *Phys. Lett.* A **370** 46
[26] Kengne E and Talla P K 2006 *J. Phys.* B **39** 3679
[27] Matveev V B and Salli M A 1991 *Darboux Transformations and Solitions Springer Series in Nonlinear Dynimics* (Berlin: Springer)
[28] Kevrekidis P G and Frantzeskakis D J 2004 *Mod. Phys. Lett.* B **18** 173
[29] Brazhnyi V A and Konotop V V 2004 *Mod. Phys. Lett.* B **18** 627
[30] Streed E W, Mun J, Boyd M, Campbell G K, Medley P and Ketterle W 2006 *Phys. Rev. Lett.* **97** 260402
[31] Itano W M, Heinzem D J, Bollinger J J and Wineland D J 1990 *Phys. Rev.* A **41** 2295
[32] Hosten O, Rakher M T, Barreiro J T, Peters N A and Kwiat P G 2006 *Nature* **439** 949
[33] Erez N, Gordon G, Nest M and Kurizki B 2008 *Nature* **452** 1724